What is the Curl?

1st February 2007

We have the definition of curl from our earlier Physics course

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix}$$

Where does this definition come from? And what kinds of fields have non-zero curl?

The Definition and Basic Properties

The curl of a field is defined as

$$\nabla \times \vec{V}(\vec{r}) = \lim_{A \to 0} \frac{1}{A} \oint_{C_A} \vec{V} \cdot \vec{dl}$$

where C_A is a closed contour around a surface A containing $\vec{r}$ whose area is shrunk to zero. What is so special about this quantity?

- We already know that path independence of integrals requires $\oint \vec{V} \cdot \vec{dl} = 0$ for all closed loops. Thus, the ability to associate vector fields with a scalar field representing an integral of the field depends on the field being "curl-free".

  $$\nabla \times \nabla \phi = 0$$

- Clearly, by the argument of tiling any surface with little loops, we have Stokes' Law:

  $$\oint_{C_S} \vec{V} \cdot \vec{dl} = \int_S \nabla \times \vec{V} \cdot \vec{dS}$$

- Equally clearly, since any two surfaces that link the same curve C_S are equal to the same line integral, we must have

  $$\oint_{S_V} \nabla \times \vec{B} \cdot \vec{dS} = \int_V \nabla \cdot \nabla \times \vec{B} dV = 0$$

  So the curl of a field is divergence free.

Examples

1. The Coulomb Electric Field

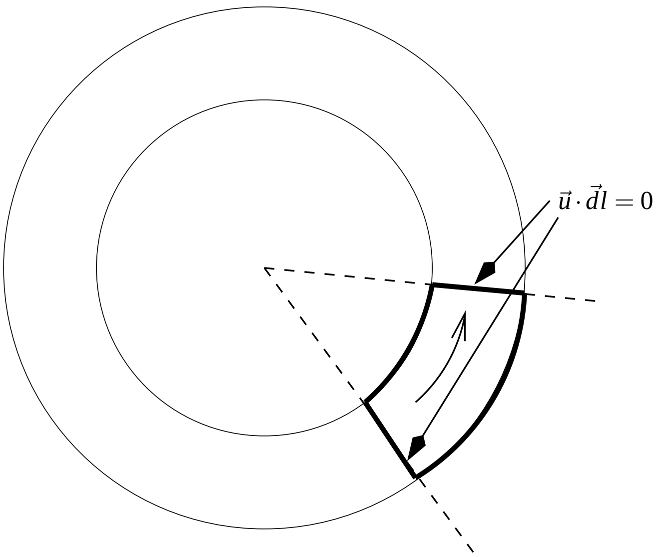
$$\vec{E} = -\frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\nabla\frac{1}{R_{12}}dV'$$

Taking the curl,

$$\nabla\times\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\nabla\times\nabla\frac{1}{R_{12}}dV' = 0$$

2. A rigidly rotating fluid

$$\vec{u}(\vec{r}) = \omega\hat{\theta}$$



The loop integral now becomes for the loop shown in the figure

$$\oint \vec{u}\cdot\vec{dl} = u_\theta(r+dr)\left[(r+dr)d\theta\right] - u_\theta(r)\left[rd\theta\right] = \omega dr d\theta$$

The area of the small region is $rdrd\theta$, and so

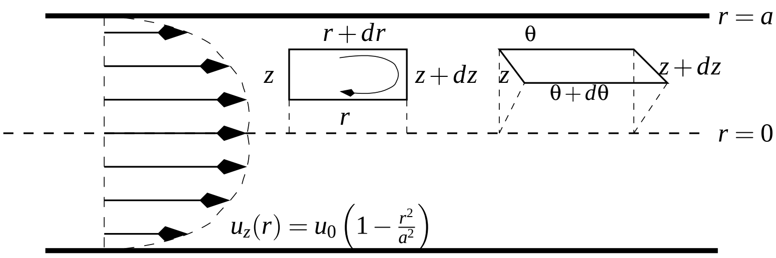
$$\nabla\times\vec{u} = \frac{\omega}{r}$$

A rigidly rotating fluid has curl!

3. Viscus flow in a pipe. A fluid flows in a pipe of radius a with velocity

$$\vec{u}(r) = u_0\left(1-\frac{r^2}{a^2}\right)\hat{z}$$

Now we take a small square loop in the r-z plane, bounded by r, $r+dr$, z and $z+dz$.



Since the flow is along z, only the sides parallel to the z-axis contribute to the integral. We get

$$\nabla\times\vec{u}\cdot\vec{dS}=\oint\vec{u}\cdot\vec{dl}=[u_z(r+dr)dz-u_z(r)dz]=-\frac{2u_0r}{a^2}drdz$$

If you draw out the loop, it will be clear that the normal points along $-\hat{\theta}$ (i.e., into the page). We could also draw a loop whose sides were z, $z+dz$, θ and $\theta+d\theta$. Only the sides parallel to the z-axis contribute to the integral. But now,

$$\nabla\times\vec{u}\cdot\vec{dS}=u_z(r,\theta+d\theta)dz-u_z(r,\theta)dz=0$$

since u_z does not depend on θ. Finally, if we created a loop with sides r, $r+dr$, θ, $\theta+d\theta$, we would get zero, since $\vec{u}\cdot\vec{dl}=0$ on all four sides. So,

$$\nabla\times\vec{u}=\frac{2u_0r}{a^2}\hat{\theta}$$

The Curl Formula

Clearly the definition of curl given above is a valid vector field:

- It has a well defined magnitude and direction.
- The values of magnitude and direction are defined in ways that do not change if we rotate, translate or even scale our measuring instrument.

But it is an inconvenient definition for quick calculation (on the other hand it is perfect for gaining intuition about what curl is). We need an algebraic formula.

Let us look at a vector field along $\hat{x}$ only. Suppose it varies along y. We imagine our right hand rule creating a loop in the x-y plane. Then, $\oint\vec{u}\cdot\vec{dl}=(-u(y+dy)+u(x))\,dx$ which means

$$\nabla\times\vec{u}\cdot\hat{z}dxdy=-\partial_yu_xdxdy$$

On the other hand, if the vector field were along $\hat{y}$ and depended on x. Then the same argument yields

$$\nabla\times\vec{u}\cdot\hat{z}dxdy=+\partial_xu_ydxdy$$

If the field were along $\hat{x}$ and depended on x, we would cancellation of the two arms of the loop. Similarly for the $\hat{y}$ component that depended on y.

What does this teach us?

- The curl points in the "third direction"

- if component varied along its own direction (i.e., identical directions for field and variation of field), zero curl.

- if component varied along orthogonal direction, magnitude depends on right hand rule.

Clearly all these rules are nothing but what we had for the cross product. This is why we can write the curl as

$$\nabla \times \vec{B} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix}$$

To verify the last example above,

$$\nabla \times \vec{u} = \frac{1}{r} \det \begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \partial_r & \partial_\theta & \partial_z \\ 0 & 0 & u_z(r) \end{vmatrix} = -\hat{\theta}\partial_r u_z = \frac{2u_0 r}{a^2}\hat{\theta}$$

The extra factors of r come from the fact that $\vec{\theta}$ is not a distance but an angle. The scale factors convert $d\theta$ to a distance are added in to give a generalized formula (see the text book for details).