The Cross Product, Volume and Determinants

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We all know the formula

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

But what underlies this peculiar definition? The answer lies in how we define differential volume, dV. In three dimensions with coordinates u, v and w, dV is the volume enclosed by three vectors du, dv and dw. For instance, in a crystal lattice, these could be defined by $u = x$, $v = x + y$, $w = x + y + z$. We "know" several things about this volume dV:

- The volume is given by base times height.
- If the dz or dw were in the negative direction, the dV would be negative. i.e., we have a "right hand rule" for volume that distinguishes positive volume from negative volume.
- If the three vectors lie in a plane, or any two vectors lie along a line, we end up with zero dV.
- If I scaled any of the three axes by an amount α, the volume scales by that α. More generally, the volume is linear in each of its coordinates.

So volume subtended by three vectors can be written as the sum of volumes subtended by the various components of the vectors (linearity):

$$\begin{aligned} V(\vec{u},\vec{v},\vec{w}) &= V(u_x\hat{x}, v_x\hat{x}, w_x\hat{x}) + V(u_y\hat{y}, v_x\hat{x}, w_x\hat{x}) + \ldots \\ &= u_x v_x w_x V(\hat{x},\hat{x},\hat{x}) + u_y v_x w_x V(\hat{y},\hat{x},\hat{x}) + \ldots \\ &= u_1 v_1 w_1 \varepsilon_{111} + u_2 v_1 w_1 \varepsilon_{211} + \ldots \end{aligned}$$

where $\varepsilon_{ijk} = V(\hat{x}_i, \hat{x}_j, \hat{x}_k)$ is third rank tensor that defines what we mean by volume in this space. The volume is then a scalar function of three vectors (see the article on vector identities `http://www.ee.iitm.ac.in/~hsr/ec301/vectoridentities.pdf` for the algebraic notation I am using here)

$$u_i v_j w_k \varepsilon_{ijk}$$

This operation *is nothing more or less than taking the determinant of the matrix composed of $\vec{u}$, $\vec{v}$ and $\vec{w}$ as row vectors.*

Think about it: the determinant is linear in each row and goes to zero if any set of rows (or columns) are linearly dependent. It flips sign when rows or columns are exchanged (i.e., right hand rule for volume changed to left hand rule).

What is more interesting is that the determinant is the unique function satisfying these properties (unique to within a scaling factor). Thus,

$$dV = \det \begin{vmatrix} du_x & du_y & du_z \\ dv_x & dv_y & dv_z \\ dw_x & dw_y & dw_z \end{vmatrix} = \varepsilon_{ijk} du_i dv_j dw_k$$

The volume is also geometrically the dot product of the third dimension with the vector normal to the plane defined by the other two vectors (with magnitude equal to the area of the base). So there is something which looks like a vector in that it can enter into a dot product with another vector and give a scalar. How can we define such a vector? Just replace the u_i coordinates with place holders:

$$\det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dv_x & dv_y & dv_z \\ dw_x & dw_y & dw_z \end{vmatrix} \equiv \varepsilon_{ijk} \hat{x}_i v_j w_k$$

Then, dot producting this "vector" with another vector $\vec{dw}$ yields the the familiar expression for volume dV.

$$\varepsilon_{ijk} \hat{x}_i v_j w_k \cdot \hat{x}_l u_l = \varepsilon_{ijk} u_i v_j w_k$$

since $\hat{x}_i \cdot \hat{x}_l$ is non-zero only when $l = i$.

This is where the expression for cross product comes from. The cross product is not really a vector actually. Rather it is what is called a form, something that acts on a vector and produces a scalar. It lives in the dual space to vectors. But for all practical purposes it looks like and behaves like a vector and we treat it as such.