Boundary conditions at an Interface for Wave Problems.

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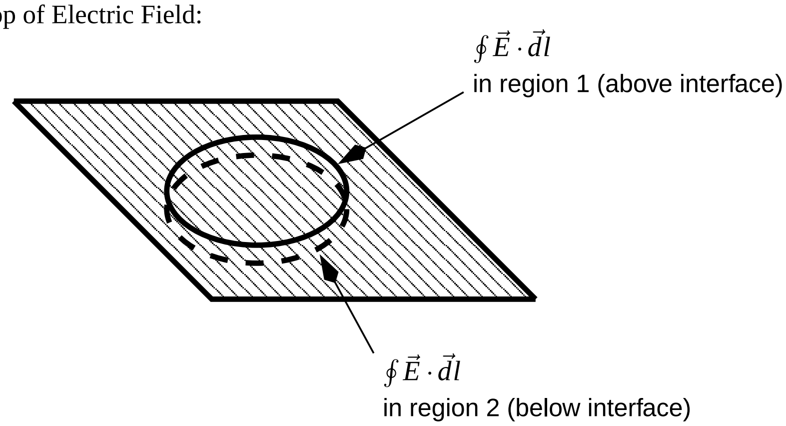
Dielectric Interface

We know from Maxwell's Equations that at a dielectric interface, E_t, H_t, D_n and B_n are continuous. But the solution of the oblique wave problem works as follows:

- Incoming wave has only forward wave component, and that is specified by user (i.e., the brightness and direction of his torch).
- Reflected wave has only backward wave component, whose amplitude and phase are the only unknowns (one complex number).
- Transmitted wave has only forward wave component, whose amplitude and phase are the only unknowns (one complex number).

So to fix two numbers ($r = E_r/E_i$ and $t = E_t/E_i$) we have four conditions. Obviously overdetermined! How can we resolve this?

Suppose we have continuity of E_t and H_t, the tangential components of the field. Now, from Faraday's Law, the normal component of $\vec{B}$ is obtained from a tangential loop of Electric Field:



$$\oint \vec{E}_t \cdot \vec{dl} = -j\omega B_n A$$

where A is the area of the loop. Since the tangential Electric Field is continuous, the loop integral is the same on both sides of the interface. Which means that B_n must also be continuous.

Now consider Ampere's Law, apply the same thinking to a loop integral of $\vec{H}$ that is tangential to the boundary.

$$\oint \vec{H}_t \cdot \vec{d}l = I_{\text{encl}} + j\omega D_n A = j\omega D_n A$$

Since $\vec{H}_t$ is continuous, it follows that D_n must be continuous as well.

This is why we need only worry about the continuity of tangential components. The normal components are automatically taken care of.

Conducting Boundaries

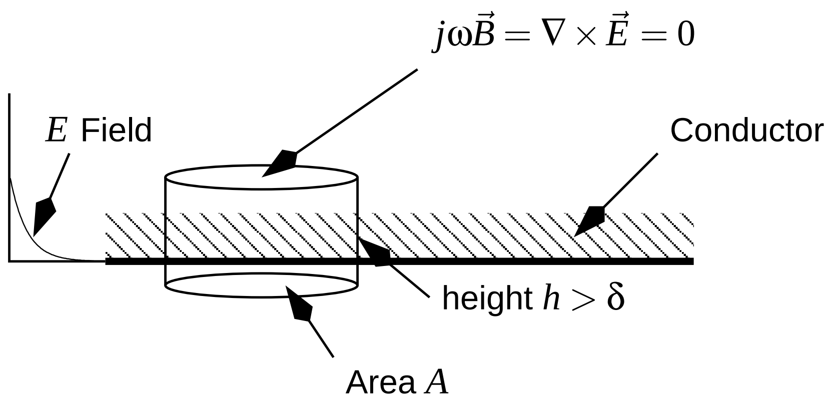
We can take two approaches here. Firstly, we can treat the conduction currents in full detail. Then the dielectric boundary conditions above apply. The surface charge and surface current density are solved for as a skin depth problem.

This approach is used when we want to solve for "Eddy currents" in devices. The skin currents need to be solved for in detail, and we cannot approximate them as living entirely on the surface.

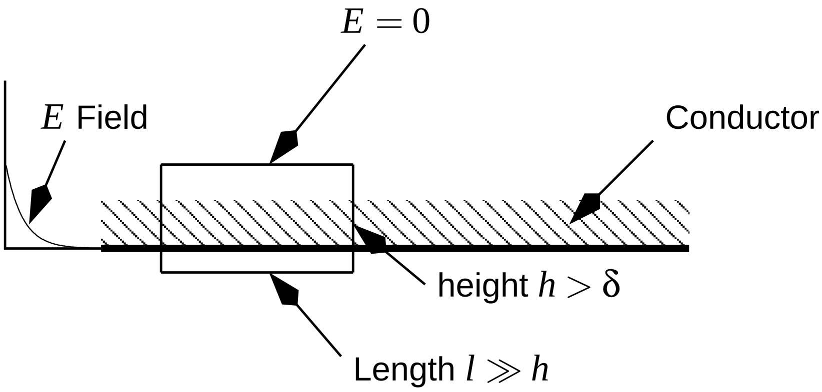
The second approach is to assume that the conducting boundary is "thick" and that scale lengths of interest are much larger than a skin depth. Then, we use the standard arguments about taking narrow loops or "pill boxes" and obtaining the jump conditions from the equations. The difference here is that we choose the loops or boxes to include the entire skin current region.

1. Consider $\nabla \cdot \vec{B} = 0$. The magnetic field inside the conductor is related to the electric field via Faraday's Law, which means that it is non-zero only in the skin depth region. So we choose a pillbox that penetrates more than a skin depth and obtain

$$B_n = 0 \qquad \text{at the surface}$$



Similarly, taking a loop and applying Faraday's Law to it, we obtain



$$\vec{E}_t = \vec{0}$$

The other two boundary conditions become, following the same arguments

$$\hat{n} \times \vec{H}_t = \vec{j}_{\text{surface}}$$

and

$$\vec{D}_n = \sigma_{\text{surface}}$$

where σ_{surface} is the surface charge, and $\vec{j}_{\text{surface}}$ is the current integrated over several skin depths.

How accurate are these boundary conditions? To determine that, we assume that the fields are varying as $e^{-jk_y y}$ along the traverse direction. Then, applying $\nabla \cdot \vec{B} = 0$ to a small cuboid oriented along $\hat{y}$, we get a contribution from the sloping sides as well. Specifically, the sides separated in y give fluxes that do not add up to zero. Instead we get

$$B_{\text{n,surface}} A + B_y A_{\text{sloping}} \left(1 - e^{jk_y \Delta y}\right) = 0$$

Now,

$$\frac{A_{\text{sloping}}}{A} \sim \frac{\delta}{\Delta y}$$

So, the condition becomes

$$B_{n,\text{surface}} = -\left(jk_y \delta\right) B_y$$

So the condition is exact when $k_y \delta$ is zero, i.e., when $\delta \ll \lambda_\perp / 2\pi$, it is a good approximation to assume

$$B_{n,\text{surface}} = 0$$

Exactly the same analysis applies in all four conditions. Since in practical waveguides $\delta \ll \lambda$, these "ideal conducting wall" boundary conditions are commonly used in practice.