The averaging theorem in Electrostatics

25th January 2007

A very important property of the potential is that given any sphere that is entirely contained in a region where $\nabla^2\phi = 0$, the average of potential on that sphere is equal to the potential at the centre.

There are important consequences to this theorem, the most important being that potential reaches its maximum or minimum values only at boundaries or at charges. That immediately means that you cannot trap charges via an applied Electric Field, since there will always be some direction or other along which the field will allow them to escape.

Proof

Let us shift our coordinates so that the centre of the sphere is at the origin. Start with the surface integral

$$\begin{aligned}
\int \phi(r,\theta,\psi)\, a d\theta\, a\sin\theta d\psi &= \int \left(\phi(\text{origin}) - \int_0^r E_r dr\right) a d\theta\, a \sin\theta d\psi \\
&= 4\pi a^2 \phi(\text{origin}) - \int E_r dr\, a d\theta\, a\sin\theta d\psi \\
&= 4\pi a^2 \phi(\text{origin}) - \int \left(\int E_r a d\theta\, a \sin\theta d\psi\right) dr \\
&= 4\pi a^2 \phi(\text{origin})
\end{aligned}$$

since the normal flux of Electric field must be zero over any closed surface containing no charge. This immediately implies

$$\phi\,(\text{origin}) = \langle\phi\rangle_{\text{surface}}$$

since $4\pi a^2$ is nothing but the surface area of the sphere. The crucial magic lies in exchanging the integrals over r and over θ and ψ.