Ampere's Law

24th January 2007

Consider

$$\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}\int \vec{j}(\vec{r}') \times \nabla \frac{1}{R_{12}} dV'$$

Taking the curl, we obtain

$$\nabla \times \vec{B}(\vec{r}) \quad = \quad -\frac{\mu_0}{4\pi}\int \nabla \times \left(\vec{j}(\vec{r}') \times \nabla \frac{1}{R_{12}}\right) dV'$$

Now the term in brackets is given by

$$\vec{j}(\vec{r}') \times \nabla \frac{1}{R_{12}} = \varepsilon_{ikl}\hat{x}_i j_k \partial_l \frac{1}{R_{12}} = -\varepsilon_{ilk}\hat{x}_i \partial_l \frac{j_k}{R_{12}} = -\nabla \times \frac{\vec{j}(\vec{r}')}{R_{12}}$$

Thus,

$$\begin{aligned}
\nabla \times \vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi}\int \nabla \times \left(\nabla \times \frac{\vec{j}(\vec{r}')}{R_{12}}\right) dV' \\
&= \frac{\mu_0}{4\pi}\int \left[\nabla \left(\nabla \cdot \frac{\vec{j}(\vec{r}')}{R_{12}}\right) - \vec{j}(\vec{r}')\nabla^2 \frac{1}{R_{12}}\right] dV' \\
&= \mu_0 \vec{j}(\vec{r}) + \mu_0 \nabla \left(\int \nabla \cdot \frac{\vec{j}(\vec{r}')}{R_{12}} dV'\right) \qquad (1)
\end{aligned}$$

Now, if the second term is zero we are done. The integral in the second term becomes

$$\begin{aligned}
\int \nabla \cdot \frac{\vec{j}(\vec{r}')}{R_{12}} dV' &= \int \vec{j}(\vec{r}') \cdot \nabla \frac{1}{R_{12}} dV' \\
&= -\int \vec{j}(\vec{r}') \cdot \nabla' \frac{1}{R_{12}} dV' \\
&= -\int \nabla' \cdot \frac{\vec{j}(\vec{r}')}{R_{12}} dV' + \int \frac{1}{R_{12}} \nabla' \cdot \vec{j}(\vec{r}') dV' \\
&= -\oint \frac{\vec{j}(\vec{r}')}{R_{12}} \cdot d\vec{S}' + \int \frac{1}{R_{12}} \nabla' \cdot \vec{j}(\vec{r}') dV'
\end{aligned}$$

Assuming that the currents are localized, the surface integral vanishes. For static fields, the divergence of current is zero. Equation 1 therefore becomes

$$\nabla \times \vec{B} \quad = \quad \mu_0 \vec{j}$$

Had the field not been static, the divergence of current would be related to $\partial\rho/\partial t$. However, this does not lead to the generalized Ampere's Law, since the Biot Savart Law itself is invalid for time varying fields. That law assumes that a current at $\vec{r}'$ at time t creates a magnetic field at another point $\vec{r}$ *at the same time*. Just like Coulomb's law, this violates relativity and we need to modify things to take care of time varying fields.