Class # 6

Nov. 02, 2025

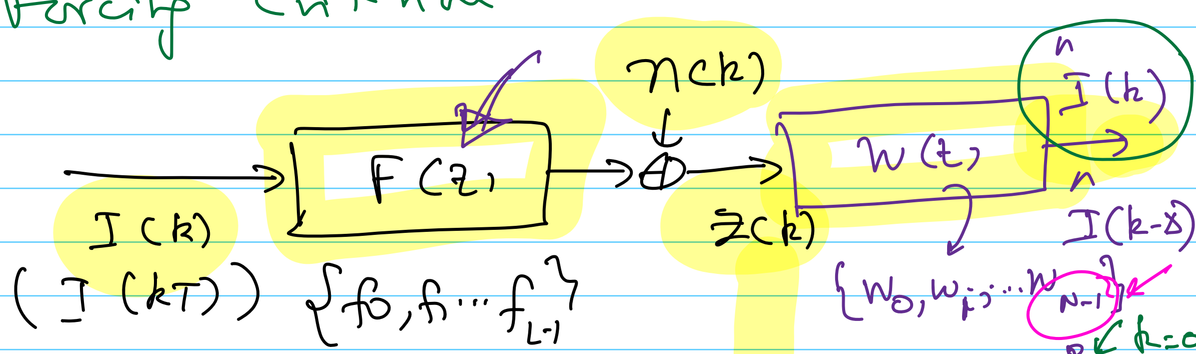
EE-4140

Equalisation for ISI channels.

- Linear Equaliser
  - Zero Forcing Criterion (ZF)
  - Linear MMSE Criterion
- Non Linear Eq.
  - DFE
- Symbol-Rate Eq.
- Fractionally-spaced Eq.

1. Zero Forcing Criterion:

Wold's Decomposition Thm



$F(z) \cdot W(z) = 1$, $N \to \infty$

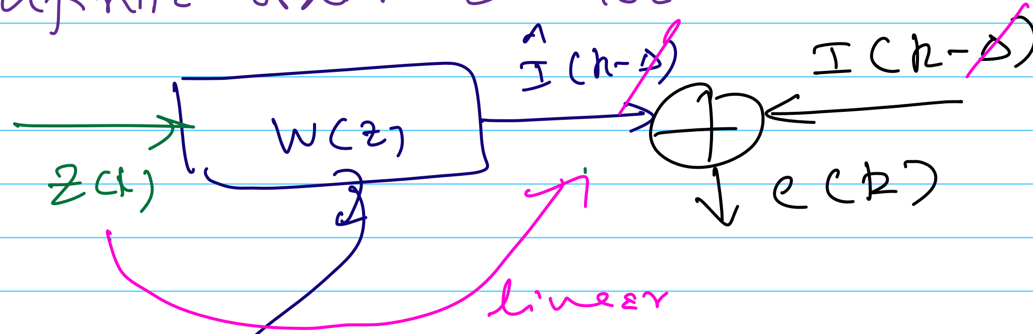
(*) Infinite order ZF

$$\{f_0, f_1 \cdots f_{L-1}\} * \{w_0 \cdots w_{N-1}\} = \delta(k)$$

$$\sum_l f_l \, w_{k-l} = \delta(k)$$

(A) $\Rightarrow$ $\boxed{W(z) = \dfrac{1}{F(z)}}$ $\longrightarrow$ ZF $(F(z))^{-1}$

(*) Infinite order LMMSE



Minimum Mean Square Error (MMSE) Criterion

Pick $f(\cdot)$ s.t. $\min E\left[\,|e(k)|^2\,\right]$ ← Bayesian MMSE / Unconstrained MMSE

$\hat{I}(k) = f\left(z(k)\right)$

$$\min_{\{w_\ell\}} E\left[\,|e(k)|^2\,\right] \leftarrow \text{L. MMSE}$$

$e(k)\, e^*(k)$

Complex Differentiation:

$x \to$ real

$f(x)$

$\to f(w_\ell)$

$$\boxed{w_\ell = w_{I,\ell} + j\, w_{Q,\ell}}$$

I-phase   Q-phase

$\nabla \longrightarrow$ Complex Gradient operation

$\bar{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$

$$\frac{\partial}{\partial w_\ell} \cancel{f(w_\ell)}\; f(w_\ell) =$$

$$(1) \leftarrow \boxed{\frac{\partial}{\partial w_\ell} f(w_\ell) \triangleq \frac{\partial}{\partial w_{I,\ell}} f(w_\ell) + j \frac{\partial}{\partial w_{Q,\ell}} f(w_\ell)}$$

$(*)$ $\quad r\, w_\ell \longrightarrow \frac{\partial}{\partial w_\ell} \longrightarrow 0$; $\quad r \to$ Complex $\;(r_a + j r_b)$

$(*)$ $\quad w_\ell^*\, r \longrightarrow \frac{\partial}{\partial w_\ell} \longrightarrow 2r$; $\quad (w_{I,\ell} + j w_{Q,\ell})$

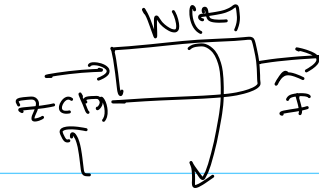
$(*)$ $\quad \underset{N\times N}{R}\, \bar{w}_{N\times 1} \longrightarrow \frac{\partial}{\partial \bar{w}} \longrightarrow \bar{0}_{N\times 1}$;

← Vector Diff.

$(*)$ $\quad \bar{w}^H R \longrightarrow \frac{\partial}{\partial \bar{w}} \longrightarrow 2R_{N\times N}$;

$(*)$ $\quad \bar{w}^H R\, \bar{w} \longrightarrow \frac{\partial}{\partial \bar{w}} \longrightarrow 2R_{N\times N}$;

$$\min_{W_a} E\left[\,|e(n)|^2\,\right] \qquad e(n)\,e^*(n)$$



$$e(n) = \left(I(n) - \hat{I}(n)\right) e^*(n) \qquad w^*$$

$$\hat{I}(n) = \sum_{j=-\infty}^{+\infty} w_j z_{k-j}$$

$$\boxed{\;E\left[\Big(\underbrace{I(k) - \sum w_j z_{k-j}}_{e(n)}\Big)\cdot z^*_{k-l}\right] = 0\;}$$

orthogonality condition

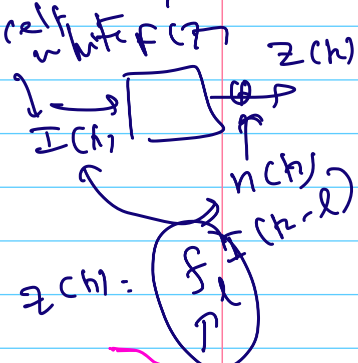
error.

data → $\hat{I}$

not random

$0,\dots,L-1$

$$\boxed{\;E\left[I_k z^*_{k-l}\right] = \sum w_j E\left[z_{k-j} z^*_{k-l}\right]\;}$$



$$\begin{cases} f^*_{-l}, & -L \le l \le 0 \\ 0, & \text{o.w.} \end{cases}$$

$\{f_l * f_l\}$

$$\begin{cases} x_{l-j} + N_0\,\delta_{ij}, & (|l-j| < L) \\ 0, & \text{o.w.} \end{cases}$$

$\frac{N_0}{2}, \frac{N_0}{2}$

↓ Z.T.

$$F^*(z^{-1}) = W(z)\cdot\left[F(z)\,F^*(z^{-1}) + N_0\right]$$

B →

$$\boxed{\;W(z) = \frac{F^*(z^{-1})}{F^*(z^{-1})\,F(z) + N_0}\;}$$

LMMSE

A →

$$W(z) = \frac{1}{F(z)};$$

ZF

infinite SNR

$N_0 \to 0$

ZF

Finite Order Case

$I(k) \rightarrow \boxed{f_0 \ f_1 \ f_2}_{L=3} \rightarrow \oplus \xleftarrow{} n(k)$, then $\rightarrow \boxed{w_0 \ w_1 \ w_2 \ w_3 \ w_4}_{M=5} \rightarrow \hat{I}(k-\Delta)$

$f_0 \ f_1 \ f_2$: known. Lengths: 3 and 5.

$3+5-1=7$

$$\{f_l\} * \{w_l\} \longrightarrow \delta(k)$$

$$\begin{bmatrix} f_0 & 0 & 0 & 0 & 0 \\ f_1 & f_0 & 0 & 0 & 0 \\ f_2 & f_1 & f_0 & 0 & 0 \\ 0 & f_2 & f_1 & f_0 & 0 \\ 0 & 0 & f_2 & f_1 & f_0 \\ 0 & 0 & 0 & f_2 & f_1 \\ 0 & 0 & 0 & 0 & f_2 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}_{5\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{7\times 1}$$

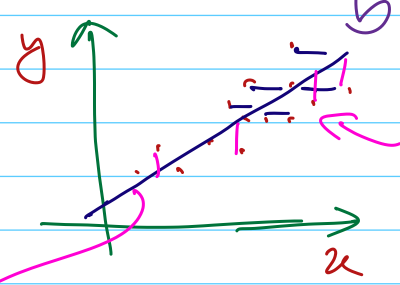
A: 7×5 $\qquad$ $\bar{b}$

Overdetermined system

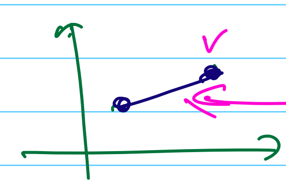
$$\boxed{A\bar{w} = \bar{b}}$$

Total Least Squares

$$\min_{\bar{w}} \left\| \bar{b} - \hat{\bar{b}} \right\|^2$$



line-fit



Moore Penrose Inverse

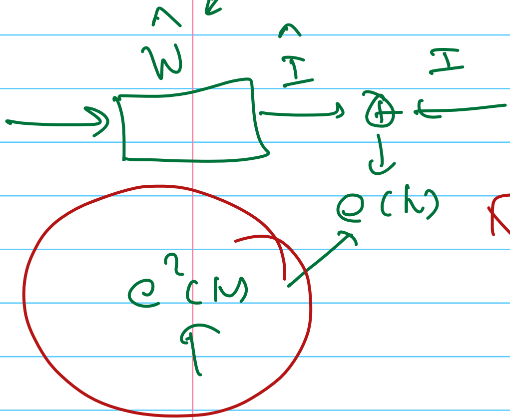
$$\bar{w}_{LS} = A^{\dagger}\bar{b}$$

$\dagger \rightarrow$ perp

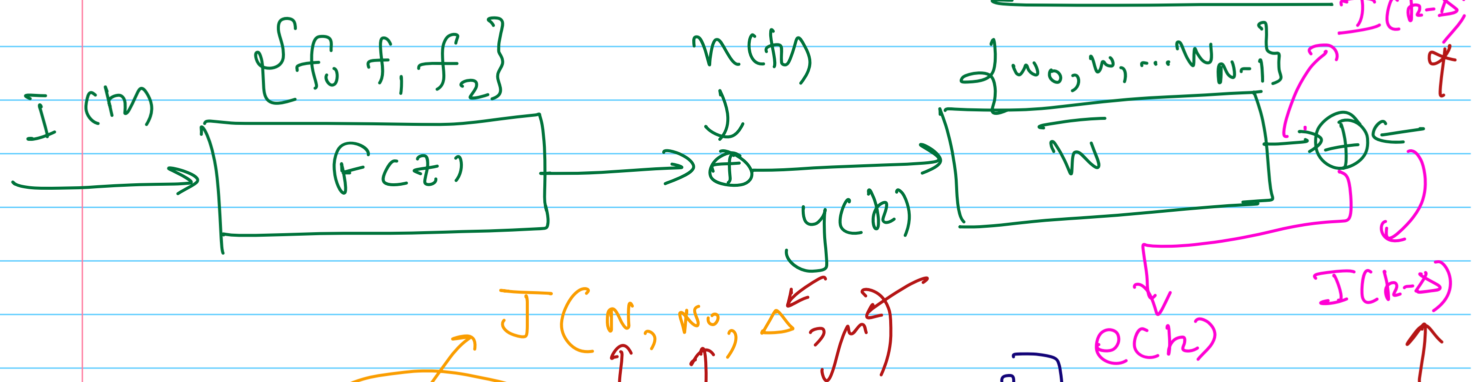
$$= (A^H A)^{-1} A\, \bar{b}$$

$$\underbrace{(A^H A)^{-1} A}_{\text{Pseudo-Inverse}}$$

$\bar{w}_{ZF}$



Linear - MMSE $\rightarrow$ finite order case



$J(N, w_0, \Delta, \mu)$

Performance Function $J(\bar{w}) = E\left[|e(k)|^2\right]$

LMMSE. $\min J(\bar{w})$ w.r.t. $\bar{w}_{M\times 1}$

Equaliser

No 2

Assume $\{f_0, f_1 \cdots f_{L-1}\}$ is known and N_0 is known

Estimate using Ergodic Assumption

Gradient Descent

μ

Learning Constant. Adaptively