Note Title

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Paley-Wiener Theorem

Let $h[n] = 0$ for $n < 0$ and let $h[n] \in \ell_2$. Let $h[n]$ possess DTFT $H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$. Then

$$\int_{-\pi}^{\pi} |\ln |H(e^{j\omega})|| \, d\omega < \infty$$

Conversely, if $|H(e^{j\omega})| \in \mathcal{L}_2[-\pi, \pi)$ and $\int_{-\pi}^{\pi} |\ln |H(e^{j\omega})|| \, d\omega < \infty$, then there exists $\theta(\omega)$ s.t. the filter with transfer function $H(e^{j\omega}) = |H(e^{j\omega})| \cdot e^{j\theta(\omega)}$ has an impulse response that is causal.

Observations

(i) $H(e^{j\omega})$ cannot be zero over an interval.

(ii) $H(e^{j\omega})$ cannot be constant over an interval.

(iii) The transition from passband to stopband cannot be abrupt.

(iv) The real and imaginary parts of $H(e^{j\omega})$ cannot be independent.

To see how the real and imaginary parts of $H(e^{j\omega})$ are related, we proceed as follows.

Any $h[n]$ can be written as

$$h[n] = h_e[n] + h_o[n]$$

where

$$h_e[n] = \frac{h[n] + h[-n]}{2} \qquad h_o[n] = \frac{h[n] - h[-n]}{2}$$

If $h[n] = 0$ for $n < 0$, $h[n]$ and $h[-n]$ do not overlap except at $n = 0$. Hence, $h[n]$ can be recovered from $h_e[n]$ as follows:

$$h[n] = 2h_e[n]u[n] - h_e[n]\delta[n]$$

$$= 2h_e[n]u[n] - h[0]\delta[n] \quad (\because h_e[0] = h[0])$$

OTOH, since $h_o[0] = 0$ always, we can recover $h[n]$ from $h_o[n]$ for $n>0$ only. $h[0]$ information is needed for full recovery.

Recall $\quad h[n] \longleftrightarrow H(e^{j\omega}) = H_R(e^{j\omega}) + j\, H_I(e^{j\omega})$

$$h_e[n] = \frac{h[n]+h[-n]}{2} \longleftrightarrow \frac{H(e^{j\omega}) + H(e^{-j\omega})}{2}$$

If $h[n] \in \mathbb{R}$, then $H(e^{-j\omega}) = H^*(e^{j\omega})$. Hence,

$$h_e[n] \longleftrightarrow H_R(e^{j\omega})$$

Also recall

$$u[n] \longleftrightarrow \pi\, \tilde{\delta}(\omega) + \frac{1}{1-e^{-j\omega}}$$

Hence,

$$2\,h_e[n]\,u[n] \longleftrightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta})\left[\pi\tilde{\delta}(\omega-\theta) + \frac{1}{1-e^{-j\overline{\omega-\theta}}}\right]d\theta$$

$$= H_R(e^{j\omega}) + \frac{1}{\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta})\frac{1}{1-e^{j\,\overline{\omega-\theta}}}\,d\theta$$

But

$$\frac{1}{1-e^{-j\omega}} = \frac{1-\cos\omega - j\sin\omega}{2-2\cos\omega} = \frac{1}{2} - \frac{j}{2}\cot\left(\frac{\omega}{2}\right)$$

Hence,

$$2h_e[n]\,u[n] \longleftrightarrow H_R(e^{j\omega}) + \frac{1}{2\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta})d\theta - \frac{j}{2\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta})\cot\left(\frac{\omega-\theta}{2}\right)d\theta$$

$$= H_R(e^{j\omega}) + \underbrace{h_e[0]}_{h[0]} - \frac{j}{2\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta$$

Hence

$$2h_e[n]u[n] - h[0] = h[n] \longleftrightarrow H(e^{j\omega}) = H_R(e^{j\omega}) + j\,H_I(e^{j\omega})$$

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j\,H_I(e^{j\omega}) = H_R(e^{j\omega}) - \frac{j}{2\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta$$

Hence, equating the real and imaginary parts, we get,

$$\boxed{H_I(e^{j\omega}) = \frac{-1}{2\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta}$$

Similarly, one can show

$$\boxed{H_R(e^{j\omega}) = h[0] + \frac{1}{2\pi}\int_{-\pi}^{\pi} H_I(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta}$$

The above are called the Discrete Hilbert Transform relationships.

Thus, for a causal sequence, the real and imaginary parts of $H(e^{j\omega})$ are not independent. If the real and imaginary parts are related, does it imply that the magnitude and phase are also related?