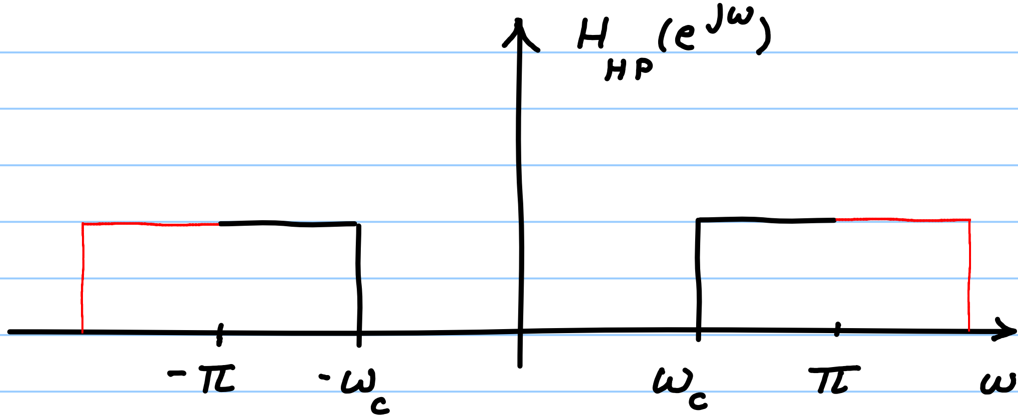
Note Title 29-08-2013

EE5330 Aug. 29, 2013

Ideal HPF

$$H_{HP}(e^{j\omega}) = 1 - H_{LP}(e^{j\omega})$$

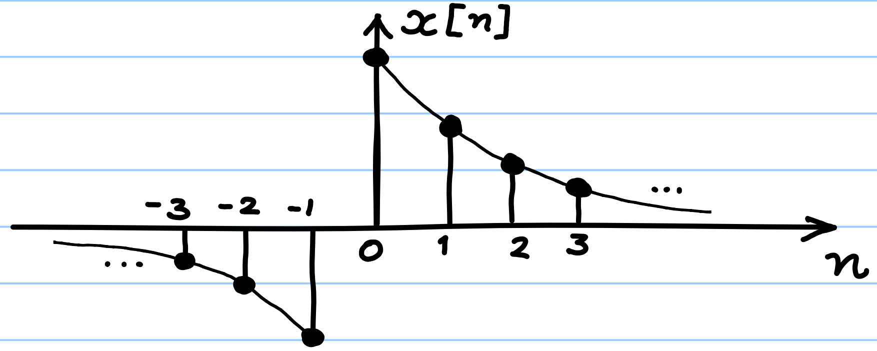


$$\longleftrightarrow \delta[n] - \frac{\sin \omega_c n}{\pi n}$$

Exercise

$$a^{|n|} \longleftrightarrow ?$$

Example



$$x[n] = a^{n} u[n] - a^{-n} u[-n-1]$$

$$\longleftrightarrow \frac{1}{1 - a e^{-j\omega}} + \frac{1}{1 - a^{-1} e^{-j\omega}} \qquad |a| < 1$$

$$\lim_{a \to 1} x[n] = \mathrm{sgn}[n] = \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

$$\lim_{a \to 1} X(e^{j\omega}) = \frac{2}{1 - e^{-j\omega}}$$

Thus,
$$\mathrm{sgn}[n] \overset{DTFT}{\longleftrightarrow} \frac{2}{1 - e^{-j\omega}}$$

sgn[n] and u[n] are related as follows: $u[n] = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}[n]$

Hence

$$u[n] \overset{DTFT}{\longleftrightarrow} \pi \tilde{\delta}(\omega) + \frac{1}{1 - e^{-j\omega}}$$

[Compare this with $u(t) \overset{CTFT}{\longleftrightarrow} \pi \delta(\Omega) + 1/j\Omega$]

Some properties of the DTFT:

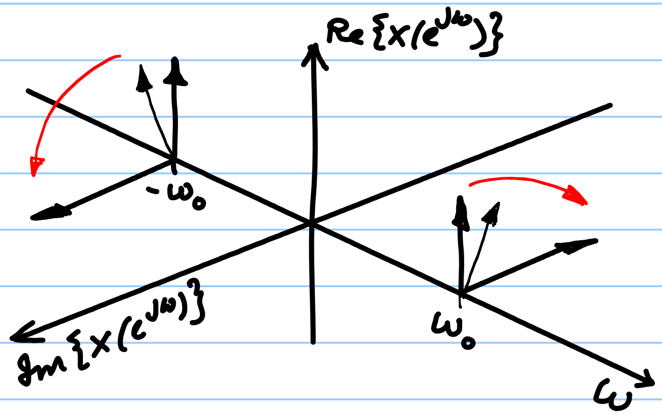
$$\sum_{n=-\infty}^{\infty} x[n]\, y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, Y^*(e^{j\omega})\, d\omega$$

$$x[n]\, y[n] \longleftrightarrow \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)\, Y(\omega-\theta)\, d\theta}_{\text{circular convolution in the frequency domain}}$$

Note: $X(\omega)$ is used insted of $X(e^{j\omega})$

Derive the above two properties from the corresponding z-transform counterparts by substituting $z = e^{j\omega}$

$$x[n-n_0] \longleftrightarrow \underbrace{e^{-j\omega n_0}}_{\text{rotates } X(e^{j\omega}) \text{ by an angle } \omega n_0} X(e^{j\omega})$$



$$\cos \omega_0 n \longleftrightarrow \pi\left[\tilde{\delta}(\omega-\omega_0) + \tilde{\delta}(\omega+\omega_0)\right]$$

$$\sin \omega_0 n \longleftrightarrow \frac{\pi}{j}\left[\tilde{\delta}(\omega-\omega_0) - \tilde{\delta}(\omega+\omega_0)\right]$$

DTFT Symmetry Properties

$$x[n] = x_R[n] + j\,x_I[n]$$

$$X(\omega) = X_R(\omega) + j\,X_I(\omega)$$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} \left[x_R[n] \cos \omega n + x_I[n] \sin \omega n \right]$$

$$X_I(\omega) = \sum_{n=-\infty}^{\infty} \left[x_I[n] \cos \omega n - x_R[n] \sin \omega n \right]$$

If $x[n] \in \mathbb{R}$, $\quad X_R(-\omega) = X_R(\omega)$

$$X_I(-\omega) = -X_I(\omega)$$

$$|X(\omega)|^2 = X_R^2(\omega) + X_I^2(\omega)$$

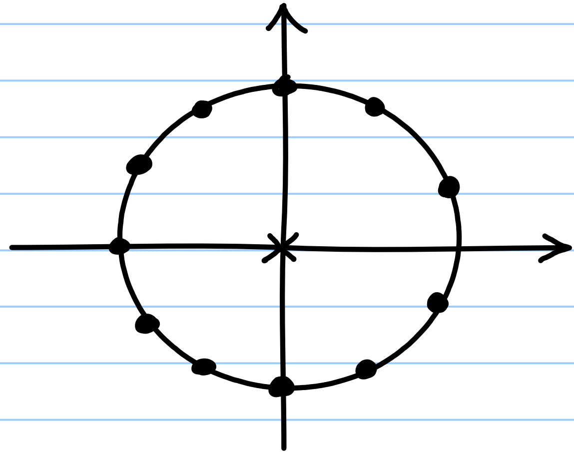
If $x[n]$ is real-valued, the magnitude of the DTFT is an even function of ω. The phase of the DTFT is an odd function of ω.

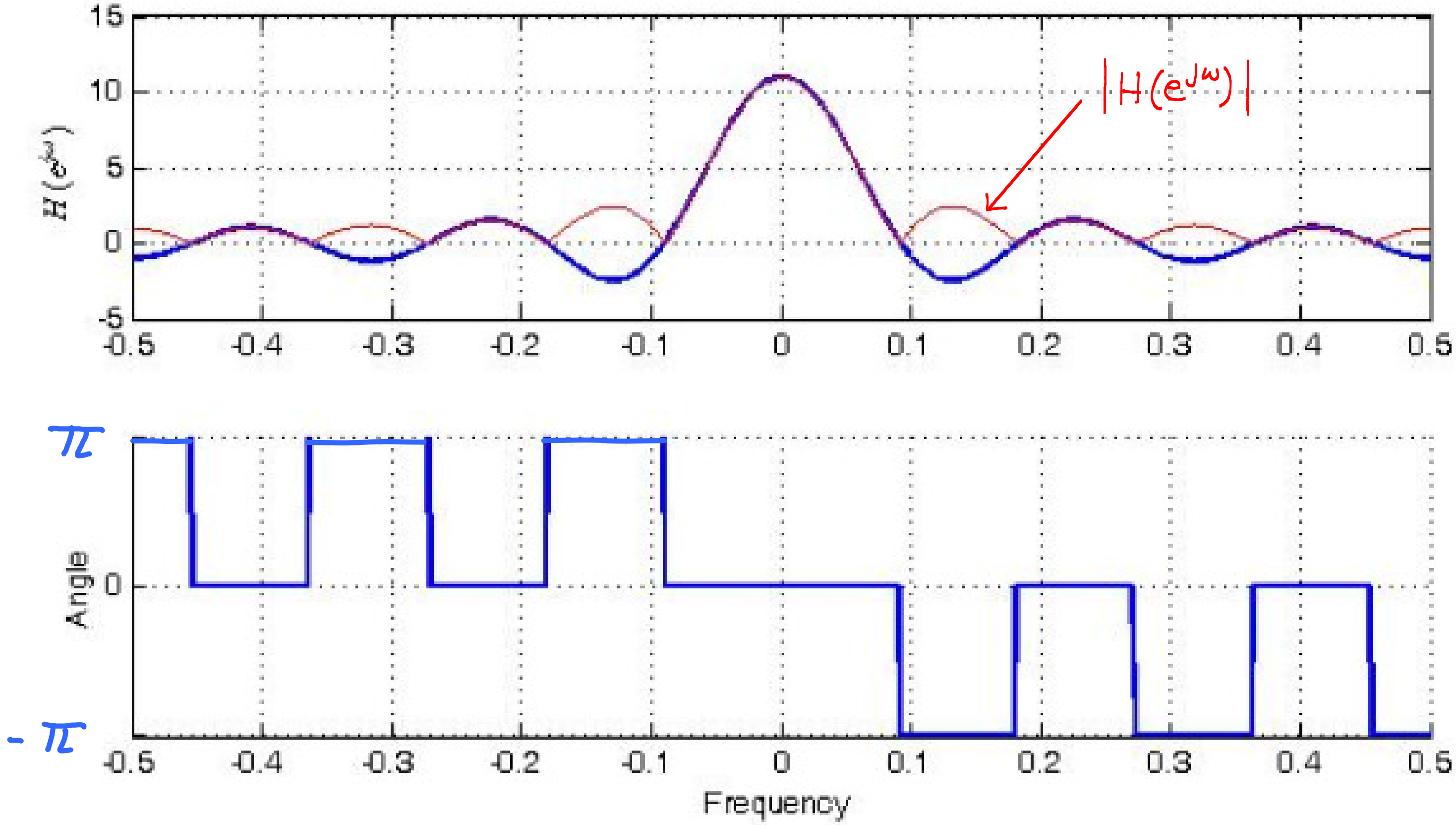
Recall that if $x[n] = x^*[n]$, then $X(\omega) = X^*(-\omega)$

<u>Example</u>

$$h[n] = 1 \qquad -M \le n \le M$$

$$H(e^{j\omega}) = \frac{\sin (2M+1)\omega/2}{\sin \omega/2}$$





15
10
5
0
-5
$H(e^{j\omega})$
$|H(e^{j\omega})|$
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5
π
0
$-\pi$
Angle
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5
Frequency

Exercise If $x[n] = j x_I[n]$, i.e., purely imaginary, then what symmetry, if any, does the DTFT possess?

Exercise

$$A \cos(\omega_0 n + \phi) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y[n]$$

Let $h[n]$ be real-valued. Then, $H(e^{j\omega}) = H^*(e^{-j\omega})$

$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$ where $\theta(\omega) = \measuredangle H(e^{j\omega})$

Show that $y[n] = A\,|H(e^{j\omega_0})| \cos(\omega_0 n + \phi + \theta(\omega_0))$

Is the above $x[n]$ an eigensignal?

Stability

An LTI system is BIBO stable if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$$\Rightarrow \sum_{n=-\infty}^{\infty} |h[n] \cdot e^{j\omega n}| < \infty$$

i.e., unit circle is part of RoC.

Causality

For a causal system with transfer function $H(z)$, the RoC is of the form $|z| > r_{max}$, where r_{max} is the radius of the furthermost pole (we have assumed $H(z)$ is rational).

When is a causal system stable?

Consider the furthermost pole. In the partial fraction expansion, it will give rise to (assuming simple pole)

$$\frac{A_k}{1 - p_k z^{-1}} \longleftrightarrow A_k (p_k)^n u[n]$$

$$\Rightarrow \sum_{n=0}^{\infty} |h[n]| < \infty \quad \boxed{\text{iff}} \quad |p_k| < 1$$

$\Rightarrow$ **all poles must lie inside the unit circle**

Since $r_{max} < 1$, the unit circle is now part of the RoC, which condition must be satisfied for BIBO stability.

If p_k is not a simple pole,

$$\frac{(n+1)(n+2)\cdots(n+M-1)}{(M-1)!} a^n u[n] \longleftrightarrow \frac{1}{(1-p_k z^{-1})^M} \qquad |z| > |p_k|$$

$$\sum_{n=0}^{\infty} n^{\ell} |p_k|^n < \infty \quad \boxed{\text{iff}} \quad |p_k| < 1 \quad \text{for ANY } \ell$$

Hence all poles must lie strictly inside the unit circle for a causal system.

For an anticausal system, the RoC is of the form $|z| < r_{min}$, where r_{min} is the radius of the innermost pole. In this case, for stability, all poles must lie strictly outside the unit circle. Once again the unit circle is part of the RoC, which is essential for BIBO stability.