Note Title 27-08-2013

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If $M \geq N$, we must first divide to get quotient and remainder.

Thus, $X(z)$ is transformed to the form

$$X(z) = \sum_{r=0}^{M-N} c_r z^{-r} + \underbrace{\sum_{k=1}^{N} \frac{A_k}{1 - q_k z^{-1}}}_{\text{num. degree } M' < N}$$

<u>Example</u>

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}} = 2 + \frac{-1 + 5z^{-1}}{1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}}$$

$$= 2 + \frac{-9}{1 - \frac{1}{2} z^{-1}} + \frac{8}{1 - z^{-1}}$$

Based on RoC, three different time-domain sequences are possible

(i) $|z| > 1$: $2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$

(ii) $\frac{1}{2} < |z| < 1$: $2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] - 8u[-n-1]$

(iii) $|z| < 1/2$: $2\delta[n] + 9\left(\frac{1}{2}\right)^n u[-n-1] - 8u[-n-1]$

Repeated Roots:

$$X(z) = \frac{G \prod_{i=1}^{M} (1 - a_i z^{-1})}{\prod_{q=1}^{Q} (1 - b_q z^{-1}) \prod_{\ell=1}^{R} (1 - \gamma_\ell z^{-1})^{\sigma_\ell}}$$

where $M < N = Q + \sum_{\ell=1}^{R} \sigma_\ell$

$$X(z) = \sum_{q=1}^{Q} \frac{A_q}{1 - b_q z^{-1}} + \sum_{\ell=1}^{R} \sum_{k=1}^{\sigma_\ell} \frac{C_{\ell,k}}{(1 - \nu_\ell z^{-1})^k}$$

ν_ℓ = root

σ_ℓ = multiplicity

$$A_q = X(z)(1 - b_q z^{-1}) \Big|_{z = b_q}$$

$$C_{\ell,k} = \frac{1}{(-\nu_\ell)^{\sigma_\ell - k} (\sigma_\ell - k)!} \frac{d^{\sigma_\ell - k}}{d\xi^{\sigma_\ell - k}} \left[X(\xi^{-1})(1 - \sigma_\ell \xi)^{\sigma_\ell} \right] \Bigg|_{\xi = \frac{1}{\nu_\ell}} \qquad k = 1, 2, \ldots, \sigma_\ell$$

Example

$$X(z) = \frac{12 - 22z^{-1} + 16z^{-2}}{(1 - 2z^{-1})^3} \qquad R = 1 \qquad \sigma_l = 3$$

$$C_{1,3} = \frac{1}{(-2)^0\, 0!} \frac{d^0}{d\xi^0}\left[\frac{12 - 22\xi + 16\xi^2}{(1 - 2\xi)^3} (1-2\xi)^3 \right]\Bigg|_{\xi = \frac{1}{2}} = 5$$

$$C_{1,2} = \frac{1}{(-2)^1\, 1!} \frac{d}{d\xi}\left[\frac{12 - 22\xi + 16\xi^2}{(1 - 2\xi)^3} (1-2\xi)^3 \right]\Bigg|_{\xi = \frac{1}{2}} = 3$$

$$C_{1,1} = \frac{1}{(-2)^2\, 2!} \frac{d^2}{d\xi^2}\left[\frac{12 - 22\xi + 16\xi^2}{(1 - 2\xi)^2} (1-2\xi)^3 \right]\Bigg|_{\xi = \frac{1}{2}} = 4$$

$$X(z) = \frac{4}{1 - 2z^{-1}} + \frac{3}{(1 - 2z^{-1})^2} + \frac{5}{(1 + 2z^{-1})^3}$$

To proceed further we need ROC information.

You will need results similar to the following:

$$\frac{(n+1)(n+2)\cdots(n+M-1)}{(M-1)!}a^n u[n] \longleftrightarrow \frac{1}{(1-az^{-1})^M} \qquad |z|>|a| \qquad M \geq 2$$

Contour Integral Method

$$x(z) = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$$

$$= \sum_i \left[\text{residues of } X(z) z^{n-1} \text{ evaluated at the poles encircled by } C \right]$$

For multiple poles, say an m^{th} order pole at $z = z_0$, $X(z)z^{n-1}$ can be written as $\frac{\Gamma(z)}{(z-z_0)^m}$. The residue at z_0 is

$$\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \Gamma(z) \Big|_{z=z_0}$$

To verify that the inversion integral does indeed give back $x[n]$, we proceed as follows:

$$\frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz = \frac{1}{2\pi j} \oint_C \left[\sum_{k=-\infty}^{\infty} x[n] z^{-k} \right] z^{n-1} dz$$

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{1}{2\pi j} \oint_C z^{n-k-1} dz$$

Recall $$\frac{1}{2\pi j} \oint_C z^{n} dz = \begin{cases} 1 & n = -1 \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\sum_{k=-\infty}^{\infty} x[k] \frac{1}{2\pi j} \oint_C z^{n-k-1} dz = x[n]$$

Alternately,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Let $z = re^{j\omega}$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n}$$

$X(re^{j\omega})$ is a 2π-periodic function in ω and hence

$x[n]r^{-n}$ can be thought of as the Fourier Series coefficients!

Thus,

$$x[n] r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega}) e^{j\omega n} d\omega$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega}) r^{n} e^{j\omega n} d\omega$$

Let $z = re^{j\omega}$

$$dz = j\underbrace{re^{j\omega}}_{z}\,d\omega$$

$$d\omega = \frac{dz}{jz}$$

We can thus convert the real-integral into a contour integral by invoking the principle of analytic continuation. Hence,

$$x[n] = \frac{1}{2\pi j}\oint X(z)\,z^{n-1}\,dz$$

as before.