EE 5330 Aug. 22, 2013

We can now see why $x[n] = 1$ has no z-transform.

Recall

$$u[n] = \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

Hence,

$$u[-n] = \frac{1}{1 - z} \qquad |z| < 1$$

$$u[-n-1] = \frac{z}{1 - z}$$

$$= \frac{-1}{1 - z^{-1}} \qquad |z| < 1$$

$$x[n] = 1 = u[n] + u[-n-1]$$

$$u[n] \xrightarrow{\text{RoC}} |z| > 1 \qquad u[-n-1] \xrightarrow{\text{RoC}} |z| < 1$$

Since $|z| > 1 \cap |z| < 1 = \phi \Rightarrow x[n] = 1$ has no z-transform!

Similarly, a^n has no z-transform

<u>Exercise</u>

Find the z-transform of $a^{|n|}$. Plot the pole-zero plot.
For what values of 'a' does the transform exist?

Hint: $a^{|n|} = a^n u[n] + a^{-n} u[-n-1]$

Observe the differences between modulation and time-reversal.

$$(-1)^n x[n] \longleftrightarrow X(-z)$$

$$x[-n] \longleftrightarrow X(z^{-1})$$

$$(-1)^n x[n] \xleftrightarrow{DTFT} X(e^{j\overline{\omega\pm\pi}})$$

$$x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$$

If the $X(\omega)$ notation is used, this will be written as $X(-\omega)$. Do not confuse this with $X(-z)$. Be careful when comparing books that use different notation

7) <u>Time-Domain Convolution</u>

Let $p[n] = \sum_{k=-\infty}^{\infty} x[k]\, y[n-k]$

Then $\quad P(z) = X(z)Y(z) \quad RoC \supseteq RoC_x \cap RoC_y$

<u>Proof:</u>

$$P(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x[k]\, y[n-k] \right] z^{-n}$$

$$\stackrel{?}{=} \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} y[n-k]\, z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, z^{-k}\, Y(z)$$

$$= X(z)\, Y(z)$$

Hence $\quad P(z) = X(z)Y(z)$

For the DTFT,

$$x[n] * y[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega})Y(e^{j\omega})$$

This property forms the basis for

$\boxed{\text{FREQUENCY SELECTIVE FILTERING}}$

Example

$$x[n] = a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$y[n] = -b^n u[-n-1] \longleftrightarrow \frac{1}{1-bz^{-1}} \qquad |z| < |b|$$

$$x[n] * y[n] \longleftrightarrow \frac{1}{(1-az^{-1})(1-bz^{-1})} \qquad |a| < |z| < |b|$$

Exercise

Evaluate the convolution in the time domain — make sure you get the limits fixed correctly for $n<0$ and $n \geq 0$.

What happens when $a \rightarrow b$?

Repeat for $a^n u[n] * b^n u[n]$

8) Product Theorem

$$x[n]\, y[n] \longleftrightarrow \frac{1}{2\pi j} \oint_C X(\xi)\, Y(z/\xi) \frac{d\xi}{\xi}$$

Knowledge of inversion integral is needed to prove this.

9) Initial Value Theorem :

Let $x[n] = 0$ for $n < 0$

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$\lim_{z \to \infty} X(z) = x[0]$$

If $x[n] = 0$ for $n < 1$, then

$$X(z) = x[1]z^{-1} + x[2]z^{-2} + \cdots$$

In this case,

$$\lim_{z \to \infty} z\,X(z) = x[1]$$ and so on.

10) <u>Final Value Theorem</u>

Let $x[n] = 0$ for $n < M$

Define $v[n] = x[n] - x[n-1]$

Hence $V(z) = (1 - z^{-1}) X(z)$

$$V(z) = \sum_{n=M}^{\infty} \left(x[n] - x[n-1] \right) z^{-n}$$

$$\lim_{z \to 1} V(z) = \lim_{z \to 1} \sum_{n=M}^{\infty} \left(x[n] - x[n-1] \right) z^{-n}$$

$$\stackrel{?}{=} \sum_{n=M}^{\infty} \left(x[n] - x[n-1] \right)$$

$$= \operatorname{Lt}_{N \to \infty} \sum_{n=M}^{N} \left(x[n] - x[n-1] \right)$$

$$= \operatorname{Lt}_{N \to \infty} \left[\begin{array}{l} \cancel{x[M]} - \cancelto{0}{x[M-1]} \\ + \cancel{x[M+1]} - \cancel{x[M]} \\ + x[M+2] - \cancel{x[M+1]} \\ + \quad \cdots \\ + \cancel{x[N-1]} + x[N-2] \\ + x[N] - \cancel{x[N-1]} \end{array} \right]$$

$$= \operatorname{Lt}_{N \to \infty} x[N]$$

$$= x[\infty]$$

Hence, $\operatorname{Lt}_{z \to 1} V(z) = \boxed{\operatorname{Lt}_{z \to 1} (1 - z^{-1}) X(z) = x[\infty]}$