Note Title

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Properties of RoC for rational $X(z)$

1) RoC is, in general, an annular region of the form $r_1 < |z| < r_2$ [r_1 can be as small as 0, r_2 can be as large as ∞]

2) If $e^{j\omega} \in$ RoC, then the DTFT can be obtained by replacing z by $e^{j\omega}$

3) RoC cannot contain poles

4) If $x[n]$ is a finite duration signal, then the RoC is the entire z-plane, except possibly 0 and/or ∞

5) If $x[n]$ is a right-sided sequence, then the RoC is outside of a certain circle. ∞ may or may not belong to the RoC

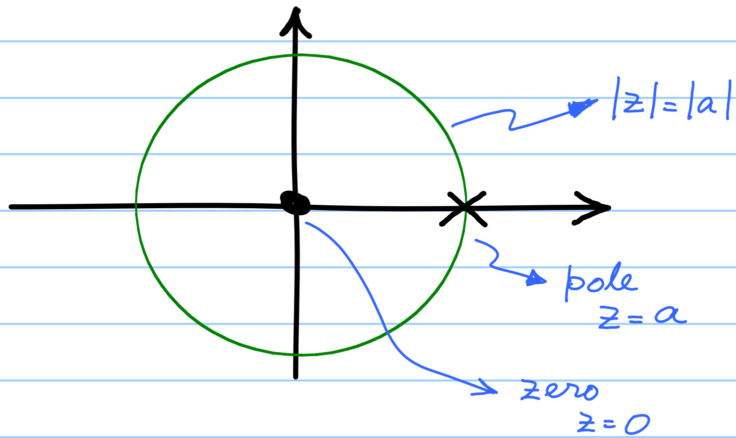
6) If $x[n]$ is a left-sided sequence, then the RoC is inside of a certain circle. 0 may or may not belong to the RoC

7) If $x[n]$ is a two-sided infinite sequence, then the RoC is in between two circles.

8) RoC must be a connected region. If the region is disconnected, the series expansion fails since it can be valid in only one region $\Rightarrow$ fails to be valid in the other regions.

Poles and Zeros Revisited

$$H(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

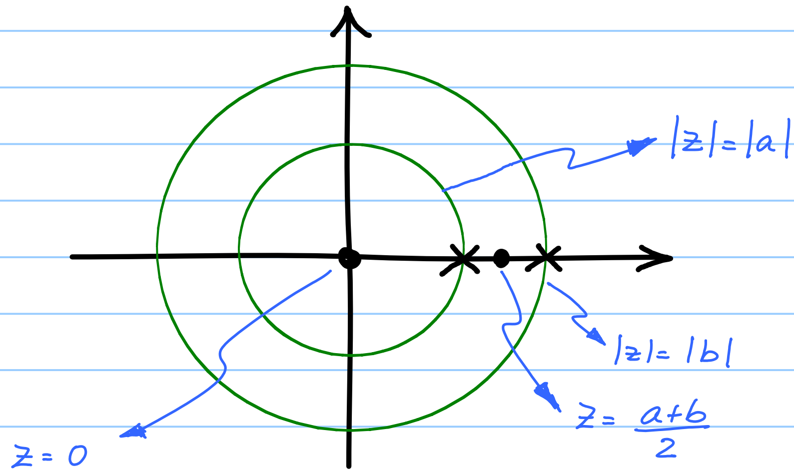


Pole : $z = a$

Zero : $z = 0$

$$a^n u[n] + b^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}}$$

$$= \frac{z\left(z - \frac{a+b}{2}\right)}{(z-a)(z-b)}$$



RoC : $|z| > |a| \cap |z| > |b|$

$$H(z) = \frac{1}{z-a}$$

Pole : $z = a$

Zero : ?

To investigate behaviour at $z = \infty$, make the transformation $z = \frac{1}{s}$

$$Y(s) = X(z)\Big|_{z=\frac{1}{s}}$$

$$= \frac{1}{\frac{1}{s} - a}$$

$$= \frac{s}{1-as}$$

$\Rightarrow$ $s = 0$ is a zero of $Y(s)$

$\Rightarrow$ $z = \infty$ is a zero of $X(z)$

$$\{b_0, b_1, b_2, \ldots, b_M\} \longleftrightarrow \overbrace{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}^{B(z)}$$

RoC: $|z| > 0$

$$H(z) = \frac{b_0 z^M + b_1 z^{M-1} + \ldots + b_{M-1} z + b_M}{z^M} = \frac{B_1(z)}{z^M}$$

Zeros: roots of $B_1(z)$ There are M zeros in the finite z-plane

Poles: M^{th} order pole at $z = 0$

$\boxed{\text{A pole or a zero at } z=0 \text{ is called a TRIVIAL pole or zero}}$

Neglecting the trivial pole at $z=0$, the above is called an "All-Zero Filter". This Filter is FIR.

Similarly,

$$H(z) = \frac{1}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

is called an "All-Pole Filter". It has a trivial zero of order N. This Filter is IIR.

In general, $H(z) = \dfrac{B(z)}{A(z)}$ is called as a "Pole-Zero Filter"

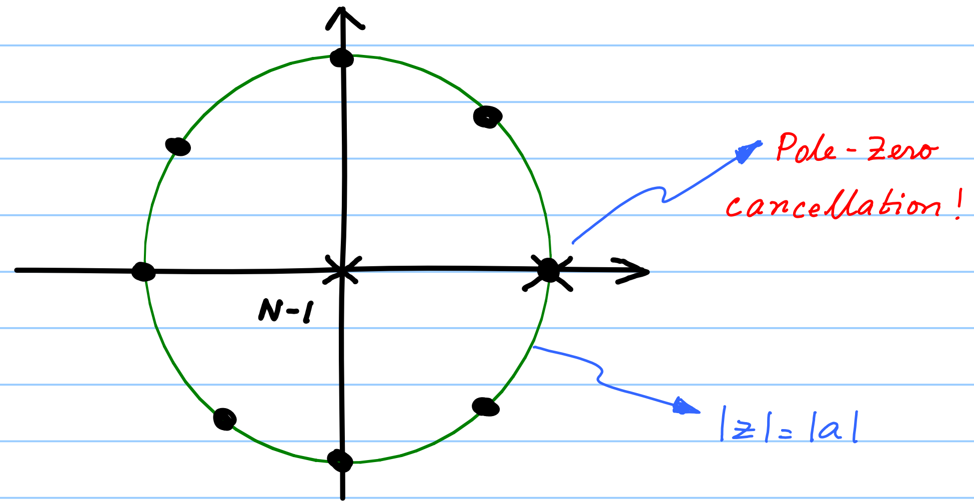
If there are uncancelled non-trivial poles, this filter will be IIR

Let $h[n] = a^n \quad 0 \le n \le N-1$

$$H(z) = 1 + a z^{-1} + a^2 z^{-2} + \cdots + a^{N-1} z^{-(N-1)} \qquad |z| > 0$$

$$= \frac{1 - a^N z^{-N}}{1 - a z^{-1}}$$

$$= \frac{z^N - a^N}{z^{N-1}(z-a)}$$



N zeros lie on the circle $|z| = |a|$. The pole at $z = a$ cancels with the zero at $z = a$. There is an $(N-1)^{th}$ order trivial pole.

In the time-domain, the input-output relationship can be shown to take up either of the following forms:

$$y[n] = x[n] + a\,x[n-1] + \cdots + a^{N-1}x[n-N+1]$$

or

$$y[n] = a\,y[n-1] + x[n] - a^{N}x[n-N]$$ ← Corresponds to a recursive implementation of the above non-recursive difference eqn.

Any pole introduced in the recursive implementation must necessarily get cancelled, since the given filter is FIR.
FIR filters cannot have uncancelled non-trivial poles!