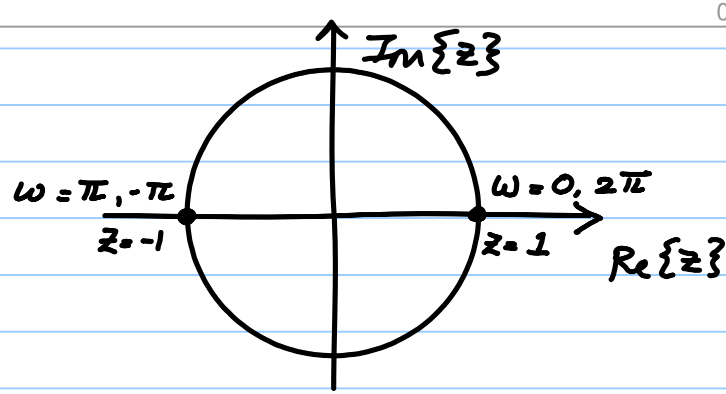
Note Title

06-08-2013

EE 5330 Aug. 6, 2013

$$X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}}$$



Some books use the notation $X(\omega)$ for the DTFT. In this case,

$$X(\omega)\Big|_{\omega=0} = X(0)$$

OTOH, Watch out for notation tripping you!

$$X(e^{j\omega})\Big|_{\omega=0} = X(1)$$

$X(e^{j\omega})$ makes explicit the 2π-periodicity

Example:

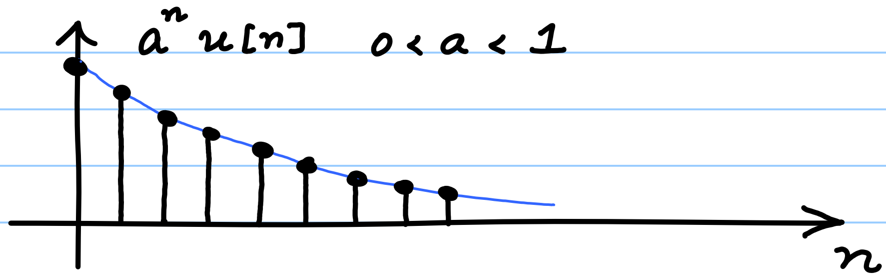
$$x[n] = a^n u[n]$$

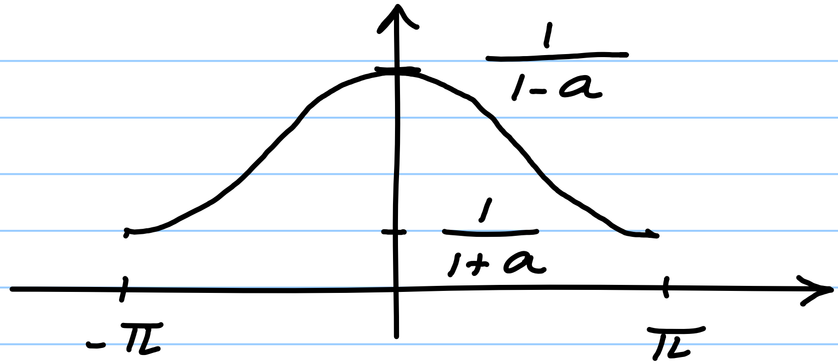
$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \frac{1}{1 - a e^{-j\omega}} \quad |a| < 1$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 + a^2 - 2a \cos \omega}}$$

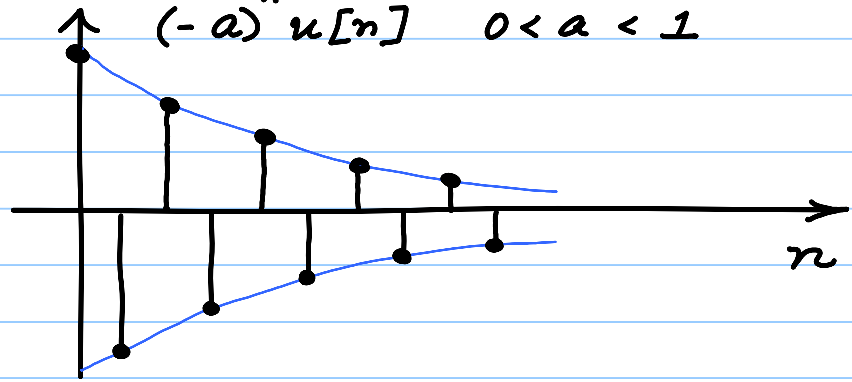
If $0 < a < 1$,

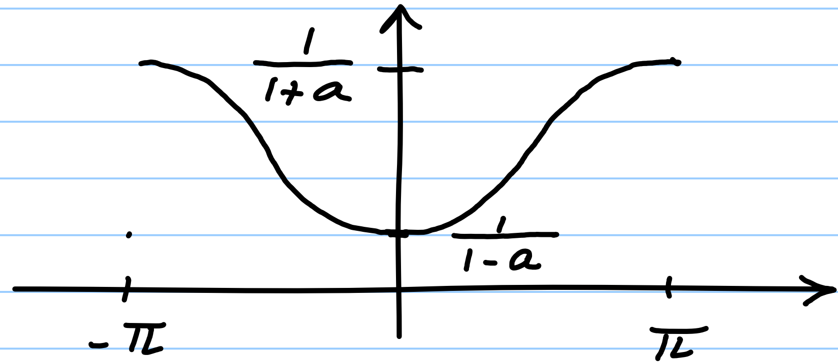




" lowpass signal "

$(-a)^n u[n] \quad 0 < a < 1$





" highpass signal "

Example

$$x[n] = 2^n u[n]$$

DTFT does not exist !

OTOH,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Let $z = r e^{j\omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, r^{-n} e^{-j\omega n}$$

Can think of $X(z)$ as the DTFT of $x[n]\, r^{-n}$

Hence, for $2^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} 2^n r^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{r}\right)^n e^{-j\omega n}$$

$$= \frac{1}{1 - \frac{2}{r} e^{-j\omega}} \qquad \boxed{\text{provided } r > 2}$$

$$= \frac{1}{1 - 2z^{-1}} \qquad |z| > 2$$

since $z = re^{j\omega}$

Note that if $x[n] = a^n$ (not $a^n u[n]$)

then DTFT does not exist even if $|a| < 1$.

This is because the summation does not converge

over the range $-\infty < n < 0$.

We will see later that a^n does not possess

Z-transform either.

If $x[n] = 1$ (i.e. a^n with $a = 1$), DTFT exists

$$1 \longleftrightarrow \pi\,\delta(\omega) + \frac{1}{1 - e^{-j\omega}} \qquad -\pi \le \omega < \pi$$

impulse shows up!

(we will derive this later)

Convergence of the Z-transform

(1) Uniform Convergence.

Consider the unilateral Z-transform defined as

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

$X(z)$ is said to converge uniformly at $z = z_0$ to $X(z_0)$ if

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \quad \text{s.t. } \forall m > N \quad \left| \sum_{n=0}^{m} x[n]\, z_0^{-n} - X(z_0) \right| < \varepsilon$$

In this course, we will focus only on absolute convergence, discussed next.

(2) Absolute Convergence:

Let $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$X(z)$ converges absolutely at $z = z_0$ to $X(z_0)$ if $|X(z_0)|$ is finite

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n} \right| \quad \text{since } z = re^{j\omega}$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]| r^{-n}$$

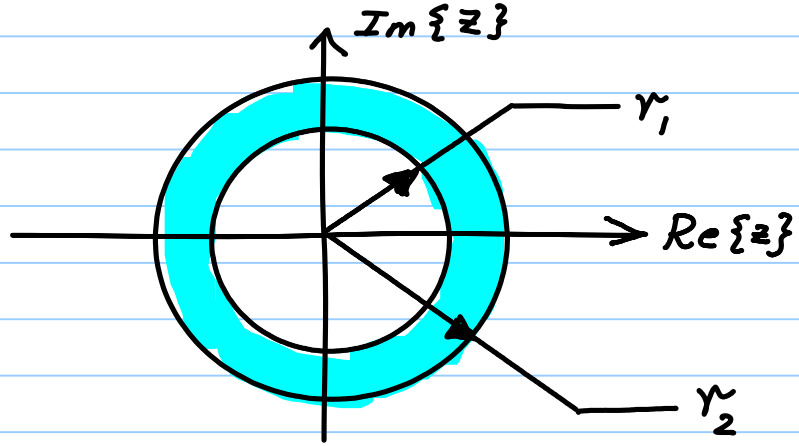
$$\sum_{n=-\infty}^{\infty} |x[n]|\, r^{-n} = \underbrace{\sum_{n=0}^{\infty} \frac{|x[n]|}{r^{n}}}_{\text{causal part}} + \underbrace{\sum_{n=1}^{\infty} |x[-n]|\, r^{n}}_{\text{anti-causal part}}$$

If $\exists\, r_1$ s.t. the above converges, then the sum converges for all $r > r_1$. This is because

$$\frac{1}{r^n} < \frac{1}{r_1^{\,n}} \quad \text{for } r > r_1$$

If $\exists\, r_2$ s.t. the above converges, then the sum converges for all $r < r_2$. This is because

$$r^n < r_1^{\,n} \quad \text{for } r < r_2$$



Thus, the convergence region is, in general, an ANNULAR region of the form $r_1 < |z| < r_2$

$\boxed{\begin{array}{l} r_1 \text{ can be as small as zero} \\ r_2 \text{ can be as large as infinity} \end{array}}$

If $z_0 = r_0 e^{j\omega_0} \in$ RoC, then $|z_0| \in$ RoC, i.e. if it converges at $\omega = \omega_0$, it converges for all $\omega \in [0, 2\pi) \Rightarrow r_0 e^{j\omega} \in$ RoC

<u>Example</u>

$$a^n u[n] \longleftrightarrow \frac{1}{1 - a z^{-1}} \qquad |z| > |a| \Rightarrow r_1 = |a|$$

$$r_2 = \infty$$

$$-a^n u[-n-1] \longleftrightarrow \frac{1}{1 - a z^{-1}} \qquad |z| < |a| \Rightarrow r_1 = 0$$

$$r_2 = |a|$$

Example

Let $x[n] = \frac{1}{n} \quad n = 1, 2, \ldots.$

What can you say about the ROC ?

Consider $\sum_{n=1}^{\infty} \frac{1}{n}$. Does this series converge ?

Compare the following two series :

$$1, \underbrace{\frac{1}{2}}_{1 \text{ term}}, \underbrace{\frac{1}{3}, \frac{1}{4}}_{2 \text{ terms}}, \underbrace{\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}}_{4 \text{ terms}}, \underbrace{\frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \ldots, \frac{1}{16}}_{8 \text{ terms}}, \frac{1}{17}, \ldots$$

$$1, \quad \underbrace{\frac{1}{2},}_{\substack{\text{sum}\\ \frac{1}{2}}} \quad \underbrace{\frac{1}{4}, \frac{1}{4},}_{\substack{\text{sum}\\ \frac{1}{2}}} \quad \underbrace{\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8},}_{\substack{\text{sum}\\ \frac{1}{2}}} \quad \underbrace{\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \ldots, \frac{1}{16}}_{\substack{\text{sum}\\ \frac{1}{2}}}, \frac{1}{32}, \ldots$$

Term-by-term, the 1^{st} series is greater than the 2^{nd} series, and the second series diverges. By the comparison test,

$$\sum_{n=1}^{\infty} \frac{1}{n} \longrightarrow \infty$$

Note: If the series converges, $a_n \to 0$
If $a_n \to 0$, the series need not converge.

Now, what can you say about ROC of $X(z)$? Specifically, is the unit circle part of the ROC?