Note Title 18-10-2011

EC 5142 Oct. 18 2011

$$X(z) = \frac{1}{1 - a z^{-1}}$$

$$|a z^{-1}| < 1 \Rightarrow |a| < |z|$$

$$= |z| > |a|$$

$$= 1 + a z^{-1} + a^2 z^{-2} + \dots$$

$$\longleftrightarrow \{0, 0, 0, \underset{\uparrow}{1}, a, a^2, \dots\} = a^n u[n]$$

$n = 0$

For anticausal resp., i.e. $|z| < |a|$, we need powers of z, rather than z^{-1}

Note Title 18-10-2011

$$X(z) = \frac{-a^{-1}z}{1 - a^{-1}z}$$

$$= -a^{-1}z\left[1 + a^{-1}z + a^{-2}z^{2} + \dots\right] \qquad |a^{-1}z| < 1$$

$$= |z| < |a|$$

$$= -a^{-1}z - a^{-2}z^{2} - a^{-3}z^{3} - \dots$$

$n = -1$ $\qquad$ $n = -2$ $\qquad$ $n = 0$

$$\longleftrightarrow \left\{ \dots, -a^{-3}, -a^{-2}, -a^{-1}, 0, 0, 0 \right\}$$

$$-a^{n}u[-n-1]$$

Long Division

$$
\begin{array}{r|l}
 & 1 + az^{-1} + \cdots \\
\hline
1 - az^{-1} & 1 \\
 & \underline{1 - az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1} - a^2 z^{-2}} \\
 & \qquad a^2 z^{-2} \\
 & \qquad \vdots
\end{array}
$$

causal response

Anticausal response:

$$-a^{-1}z - a^{-2}z^{2} - \dots$$

$$-a^{-1}z + 1 \;\big)\; 1$$

$$\underline{1 - a^{-1}z}$$

$$a^{-1}z$$

$$\underline{a^{-1}z - a^{-2}z^{2}}$$

$$a^{-2}z^{2}$$

$$\vdots$$

$$a^{|n|} = x[n] = a^{n}u[n] + a^{-n}u[-n-1]$$

$$a^{n} u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

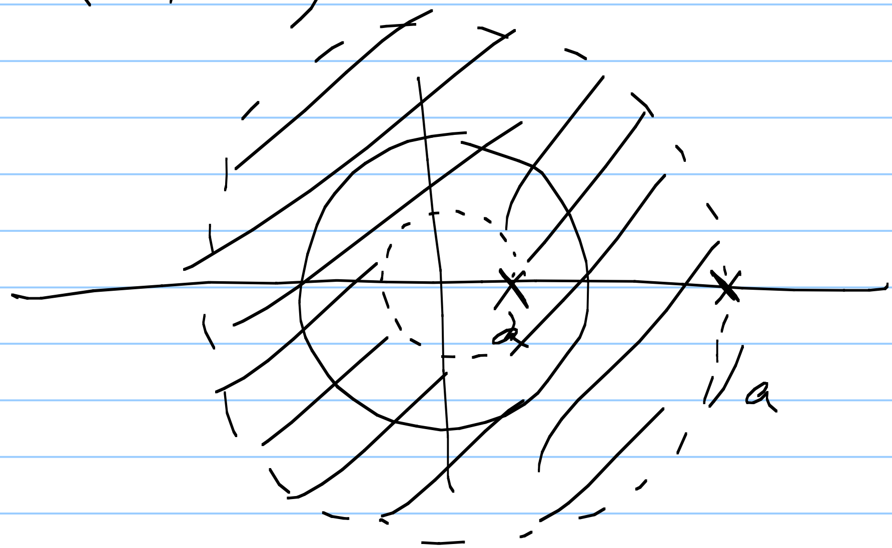
$$a^{-n} u[-n] \longleftrightarrow \frac{1}{1-az} \qquad |z| < |1/a|$$

$$a^{-n-1} u[-n-1] \longleftrightarrow \frac{z}{1-az} \qquad |z| < |1/a|$$

$$-a^{-n} u[-n-1] \longleftrightarrow \frac{az}{1-az} \qquad |z| < |1/a|$$

$$a^{|n|} \longleftrightarrow \frac{1}{1-az^{-1}} + \frac{az}{1-az} \qquad |a| < |z| < \frac{1}{|a|}$$

$$= \frac{1-a^2}{1+a^2-a(z+z^{-1})}$$



$$X(z) = b_{-q} z^{q} + b_{-(q-1)} z^{q-1} + \cdots + b_{-1} z + b_0 + b_1 z^{-1}$$

$$+ b_2 z^{-2} + \cdots + b_p z^{-p}$$

$$\longleftrightarrow \{ 0, 0, \underset{\substack{\uparrow \\ n=-q}}{b_{-q}}, b_{-(q-1)}, \ldots, b_{-1}, \underset{\substack{\uparrow \\ n=0}}{b_0}, b_1, \ldots, b_p, 0, \ldots \}$$

$$X(z) = z^{2} \left(1 - \tfrac{1}{2} z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

$x[n] =$

$$X(z) = e^{z} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \qquad |z| < \infty$$

$$\longleftrightarrow \left\{ \dots, \underset{n=-3}{\frac{1}{3!}}, \underset{n=-2}{\frac{1}{2!}}, \underset{n=-1}{\frac{1}{1!}}, \underset{n=0}{\frac{1}{1}}, 0, 0, \dots \right\} = x[n]$$

$$X(z) = e^{z^{-1}} \longleftrightarrow x[-n]$$

$$X(z) = \ln(1 + az^{-1}) \qquad |z| > |a|$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n z^{-n}}{n}$$

$$\longleftrightarrow$$

Can also get same answer using differentiation in the z-domain property

Contour Integral Method

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$\frac{1}{2\pi j} \oint X(z)\, z^{k-1}\, dz = \frac{1}{2\pi j} \oint \left[\sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \right] z^{k-1}\, dz$$

$$\stackrel{?}{=} \sum_{n=-\infty}^{\infty} x[n]\, \frac{1}{2\pi j} \oint z^{k-n-1}\, dz$$

$$\frac{1}{2\pi j}\oint z^{n}\,dz = \begin{cases} 1 & n=-1 \\ 0 & n\neq -1 \end{cases}$$

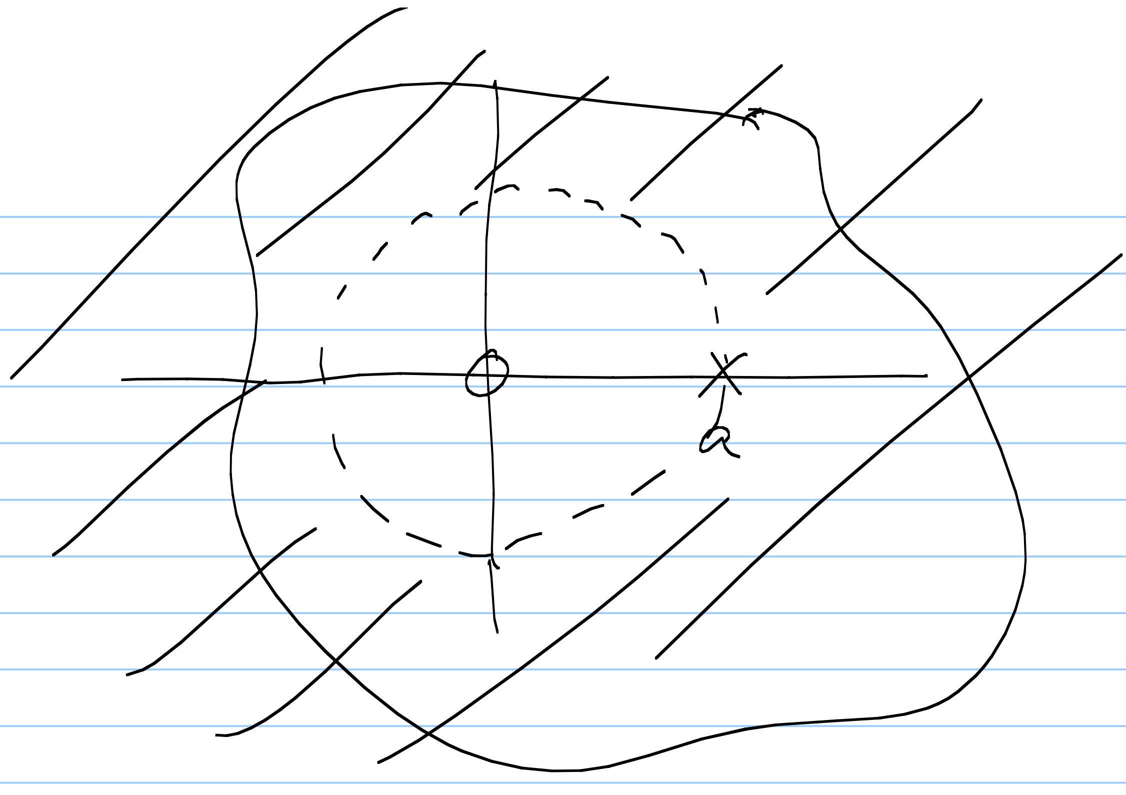
$$\frac{1}{2\pi j}\oint z^{k-n-1}\,dz = \begin{cases} 1 & k=n \\ 0 & k\neq n \end{cases}$$

$$\frac{1}{2\pi j}\oint X(z)\,z^{k-1}\,dz = x[k]$$

$$X(z) = \frac{1}{1-az^{-1}}$$

$$= \frac{z}{z-a}$$

$$|z|>|a|$$



$$\frac{1}{2\pi j} \oint \frac{z}{z-a} z^{n-1} \, dz$$

$$= \frac{1}{2\pi j} \oint \frac{z^{n}}{z-a} \, dz$$

Using Residue theorem:

$$\mathrm{Res}\left[\frac{X(z)}{z-a}\right]_{z=a} = X(z)\Big|_{z=a} = a^{n} \qquad n \geq 0$$

$$x[n] = a^{n} \qquad n \geq 0$$

[Can also see Markushavich]

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]$$

$$a_0 = 1$$

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{l=0}^{M} b_l x[n-l]$$

Applying z-transform,

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{l=0}^{M} b_l z^{-l} X(z)$$

$$Y(z)\underbrace{\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right]}_{\to A(z)} = X(z) \underbrace{\sum_{l=0}^{M} b_l z^{-l}}_{\to B(z)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

$H(z)$: system transfer function

rational transfer function

$$H(z) = \frac{B(z)}{A(z)} = \frac{Y(z)}{X(z)}$$

$$A(z)\,Y(z) = B(z)\,X(z)$$