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Note Title 17-10-2011

$x[n] * h[n] \longleftrightarrow X(z)H(z) \quad RoC \supseteq RoC_x \cap RoC_h$

$x[n]\, h[n] \longleftrightarrow$ complex convolution

$a^n u[n] = x[n] \overset{z}{\longleftrightarrow} \dfrac{1}{1-az^{-1}} \quad |z| > |a|$

$h[n] = a^n u[n]$

$y[n] - a\, y[n-1] = x[n]$

$x[n] = b^n u[n]$

$$x(n) = a^n u(n)$$

$$\frac{1}{1-az^{-1}} \longrightarrow \boxed{\frac{1}{1-az^{-1}}} \longrightarrow \frac{1}{(1-az^{-1})^2}$$

Methods of Inversion

If $X(z)$ is rational, i.e. $X(z) = \dfrac{B(z)}{A(z)}$

$B(z)$: M^{th} order

$A(z)$: N^{th} order

$$= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 \; \cancel{a_0} + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{b_0 \prod_{l=1}^{M} \left(1 - z_l z^{-1}\right)}{\prod_{k=1}^{N} \left(1 - p_k z^{-1}\right)}$$

Suppose $b_0 = b_1 = b_2 = \cdots = b_{k-1} = 0$

$$X(z) = \frac{z^{-k} B_1(z)}{A(z)}$$

i.e. we can find the inverse transform of $\dfrac{B_1(z)}{A(z)}$

& shift the inverse transform appropriately.

Similarly for $A(z)$

Assume $M < N$, simple roots.

$$X(z) = \frac{B(z)}{\prod_{k=1}^{N} (1 - p_k z^{-1})} \longrightarrow \text{partial fractions}$$

$$= \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$ $A_k \rightarrow$ residues

Matlab:

residue(b, a)

$$A_k = \left. \frac{B(z)}{\prod_{\substack{l=1 \\ l \neq k}}^{N} (1 - p_l z^{-1})} \right|_{z = z_k}$$

$$\frac{A_k}{1 - p_k z^{-1}} \longrightarrow A_k (p_k)^n u(n)$$

or

$$-A_k (p_k)^n u[-n-1]$$

depending on RoC

$$X(z) = \frac{1}{1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}} = \frac{1}{(1 - \frac{1}{2} z^{-1})(1 - z^{-1})}$$

$$= \frac{2}{1 - z^{-1}} + \frac{-1}{1 - \frac{1}{2} z^{-1}}$$

(a) $|z| < \frac{1}{2}$ (b) $\frac{1}{2} < |z| < 1$ (c) $|z| > 1$

<u>$M \geq N$</u>

$$X(z) = \frac{B(z)}{A(z)} = \sum_{r=1}^{M-N} C_r z^{-r} + \left.\frac{B_1(z)}{A(z)}\right\} \text{proper from rational}$$

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 2 + \frac{B_1(z)}{A(z)}$$

proceed as before

<u>Repeated Roots:</u>

$$X(z) = \frac{G \prod_{i=1}^{M} (1 - a_i z^{-1})}{\prod_{q=1}^{Q} (1 - b_q z^{-1}) \prod_{\ell=1}^{R} (1 - r_\ell z^{-1})^{\sigma_\ell}} \qquad M < N$$

$$N = Q + \sum_{\ell=1}^{R} \sigma_\ell$$

$$X(z) = \sum_{q=1}^{Q} \frac{A_q}{1 - b_q z^{-1}} + \sum_{l=1}^{R} \sum_{k=1}^{\sigma_l} \frac{C_{lk}}{(1 - \gamma_l z^{-1})^k}$$

γ_l : root , σ_l = multiplicity

$$A_q = X(z)(1 - b_q z^{-1}) \Big|_{z = b_q}$$

fixed

$$C_{lk} = \frac{1}{(-\gamma_l)^{\sigma_l - k} (\sigma_l - k)!} \frac{d^{\sigma_l - k}}{d\xi^{\sigma_l - k}} \left[X(\xi^{-1})(1 - \sigma_l \xi)^{\sigma_l} \right] \Bigg|_{\xi = \frac{-1}{\gamma_l}}$$

Eg

$$X(z) = \frac{12 - 22z^{-1} + 16z^{-2}}{(1-2z^{-1})^3}$$

$R = 1$

$l = 1$

$\sigma_l = \sigma_1 = 3$

$$C_{1,3} = \frac{1}{(-2)^0\, 0!} \frac{d^0}{d\xi^0}\left[\frac{12 - 22\xi + 16\xi^2}{(1-2\xi)^3}(1-2\xi)^3\right]\Bigg|_{\xi = 1/2} = 5$$

$$C_{1,2} = \frac{1}{(-2)^1\, 1!} \frac{d^1}{d\xi^1}\left[\frac{12 - 22\xi + 16\xi^2}{(1-2\xi)^3}(1-2\xi)^3\right]\Bigg|_{\xi = 1/2} = 3$$

$$C_{11} = \frac{1}{(-2)^2\, 2!} \frac{d^2}{d\xi^2}\left[\frac{12-22\xi+16\xi^2}{(1-2\xi)^3}(1-2\xi)^3\right]\Bigg|_{\xi=1/2} = 4$$

$$X(z) = \frac{4}{1-2z^{-1}} + \frac{3}{(1-2z^{-1})^2} + \frac{5}{(1-2z^{-1})^3}$$

$$a^n u(n) \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

$$(n+1)a^n u(n) \longleftrightarrow \frac{1}{(1-az^{-1})^2} \qquad |z|>|a|$$

$$? \longleftrightarrow \frac{1}{(1-az^{-1})^3} \qquad |z|>|a|$$

Power Series Expansion

$$X(z) = \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

$$= 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots\cdots$$

$$\longleftrightarrow \{\ldots, 0, 0, \underset{\uparrow n=0}{1}, a, a^2, a^3, \cdots\} = a^n u[n]$$