EC 5142 Intro To DSP

Note Title

02-08-2011

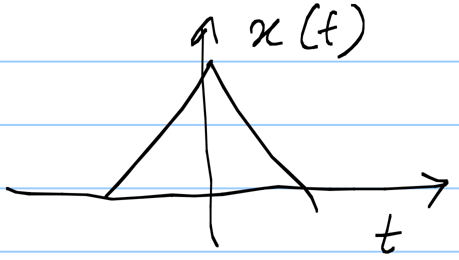
Basic operations on signals

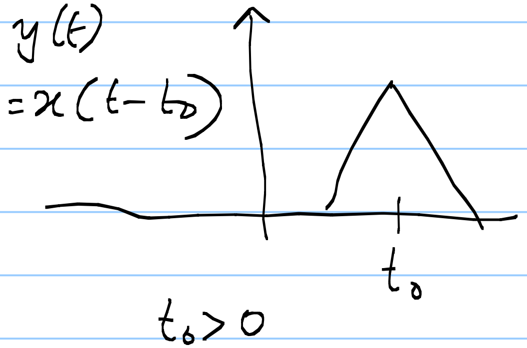
Operation on the independent variable.

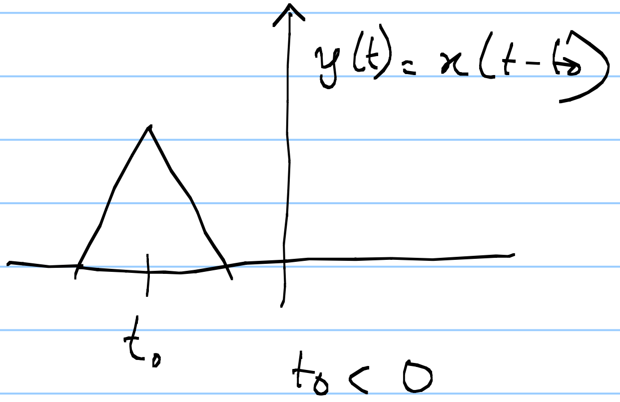
Shift:

$$y(t) = x(t-t_0)$$

$$y(t_0) = x(t_0 - t_0) = x(0)$$







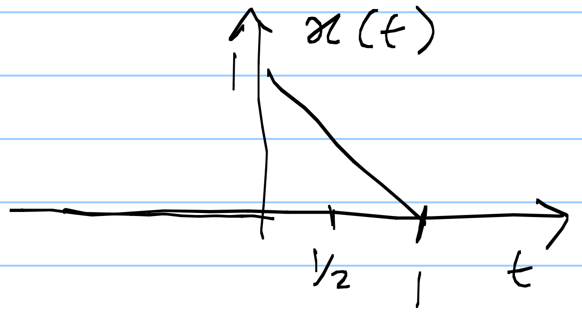
Scaling

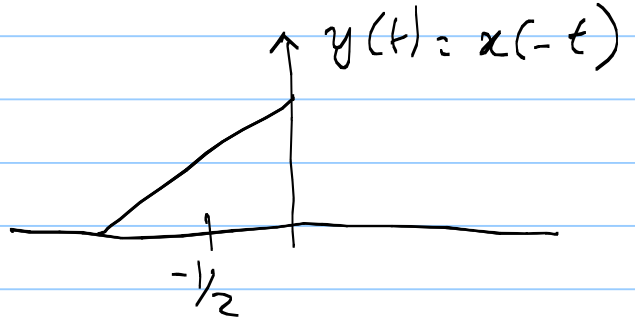
$y(t) = x(at)$ $\qquad a \in \mathbb{R}$

Eg:

$y(t) = x(-t)$ $\qquad a = -1$

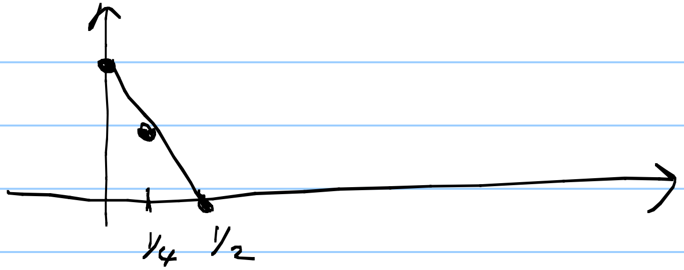
Reflection about the y-axis



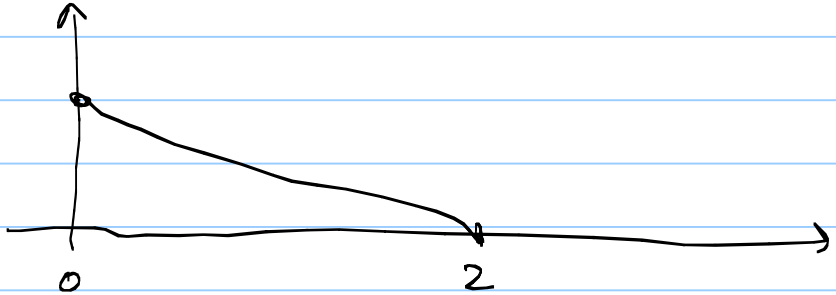


For $|a| \neq 1$:

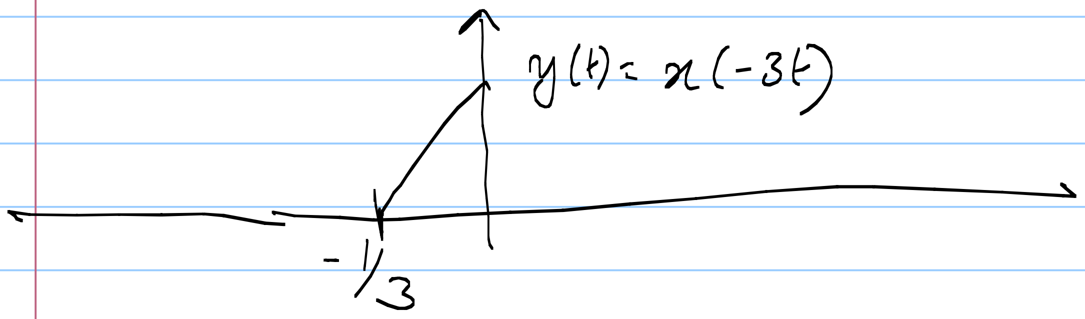
Eg: say $a = 2$, $y(t) = x(2t)$



Eg! $a = 1/2$ $y(t) = x(1/2\, t)$



Eg: $a = -3$ $y(t) = x(-3t)$

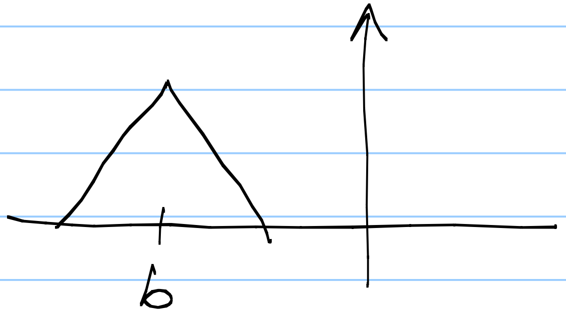


Affine transformation. $at + b$

$$y(t) = x(at+b)$$

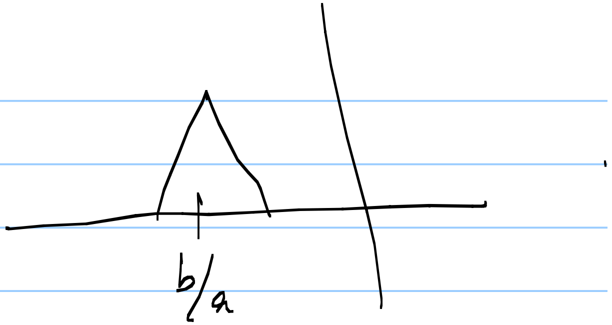
<u>Shift first, then scale</u>

$$w(t) = x(t+b)$$



$$w(at) = x(at + b)$$

$$\underline{\text{not}} \quad x(at + ab)$$



Scale first, then shift

$$w(t) = x(at)$$

$$w(t + b/a) = x[a(t + b/a)]$$

$$= x(at + b)$$

Non linear trans form:

$$X_M(s) = \int_0^\infty x(t)\, t^{s-1}\, dt$$ Mellin trans form

$$= \int_{-\infty}^{\infty} x(e^{-t})\, e^{-st}\, dt$$

= Laplace trans form of $x(e^{-t})$

Energy & Power of a Signal

Given $x(t)$, energy is defined as

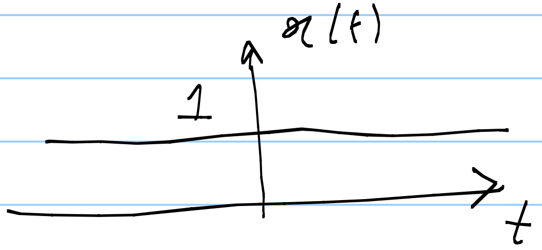
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \|x\|_2^2 \quad \text{(2-norm squared)}$$

Power is defined as:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

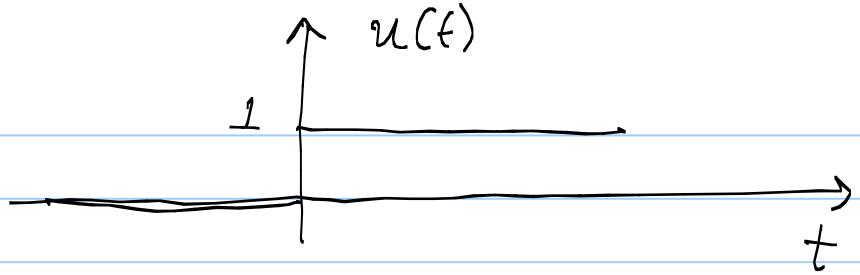
Elementary Signals:

$x(t) = 1 \quad \forall t$ "DC signal"



Unit Step Function

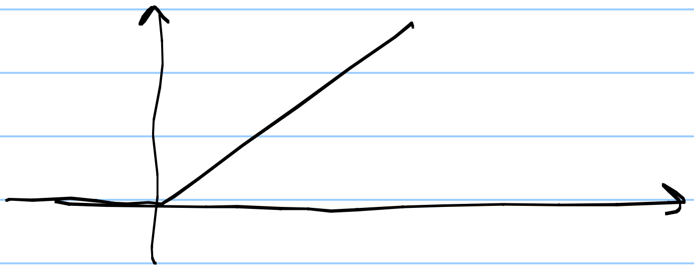
$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



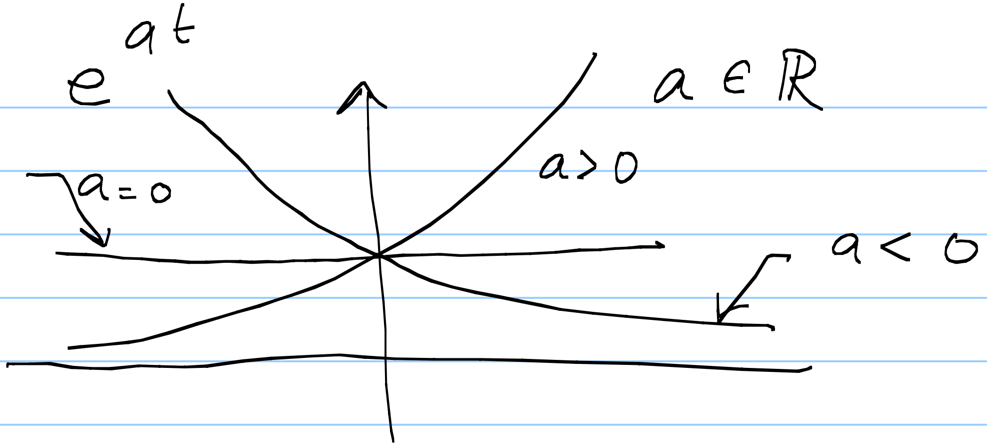
Variants:

For $t = 0$, $u(t)$ can be defined to be $0, 1, \frac{1}{2}$, any value, or undefined.

Unit ramp



$t\,u(t)$



$e^{j\Omega_0 t}$ $\Omega_0 \in \mathbb{R}$

$$= \cos \Omega_0 t + j \sin \Omega_0 t$$

$$e^{j\Omega_0 (t+T_0)} = e^{j\Omega_0 t + j\Omega_0 T_0} = e^{j\Omega_0 t} \cdot \underbrace{e^{j\Omega_0 T_0}}_{1}$$

$\Omega_0 T_0 = k\, 2\pi$

$T_0 = k \dfrac{2\pi}{\Omega_0}$

If T_0 is the period (fundamental period)

$T_0 = \dfrac{2\pi}{\Omega_0}$

For $e^{-j\Omega_0 t}$, $T_0 = \dfrac{2\pi}{\Omega_0}$

$e^{j\Omega_0 t}$, $T_0 = \dfrac{2\pi}{|\Omega_0|} \Rightarrow$ As $\Omega_0 \longrightarrow \infty$

$T_0 \longrightarrow 0$

Signal is periodic $\forall\, \Omega_0$ with period $\dfrac{2\pi}{|\Omega_0|} = T_0$

Most general exponential:

$$e^{st} = e^{(\sigma + j\Omega)t}$$

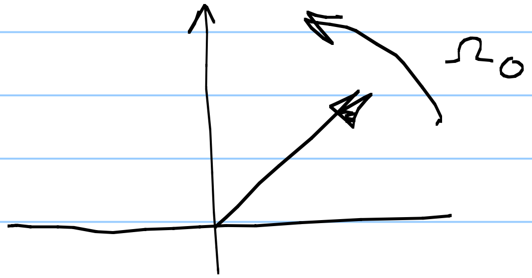
$\sigma + j\Omega$: complex frequency

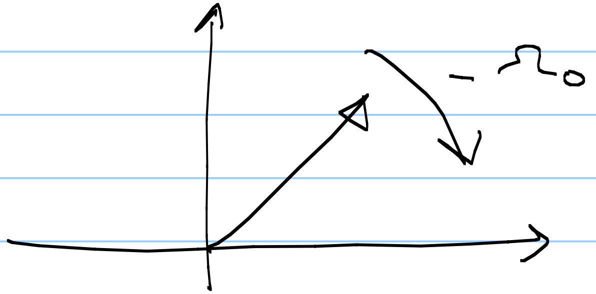
EC 5142 4th Aug. 2011

$e^{j\Omega_0 t}$ is periodic with period $\frac{2\pi}{|\Omega_0|}$

1) Always periodic for any Ω_0 w/ period $\frac{2\pi}{|\Omega_0|}$

2) Oscillates more & more rapidly as $\Omega_0 \uparrow$





$$e^{st} = e^{(\sigma+j\Omega)t} \quad e^{\sigma t} \cdot e^{j\Omega t}$$

$\sigma + j\Omega$: complex frequency

Continuous-Time Impulse

Dirac delta function

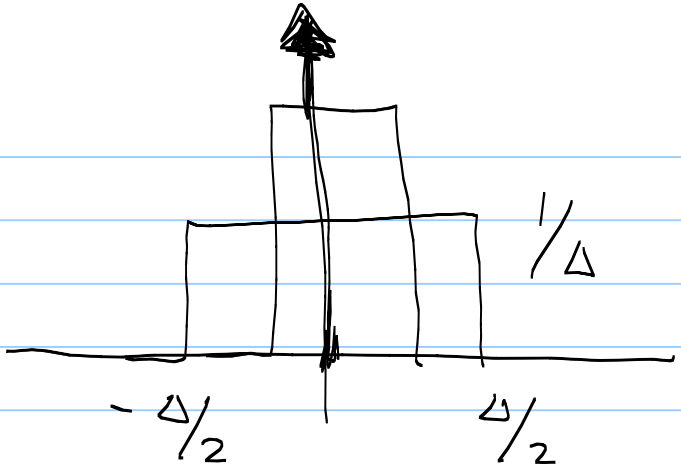
$$\delta(t) = 0 \qquad t \neq 0$$

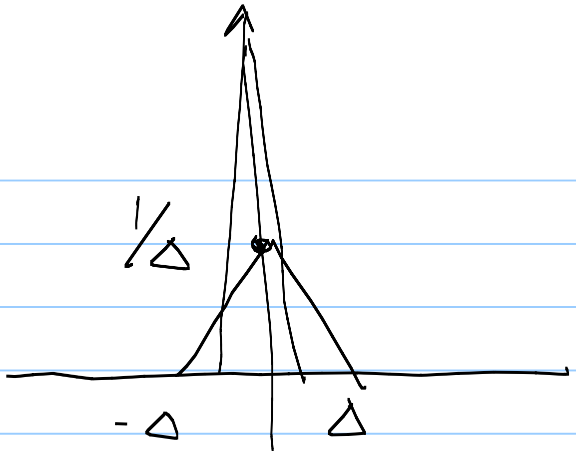
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Not a function in the usual sense.

Called functional or generalized function

Also called distribution

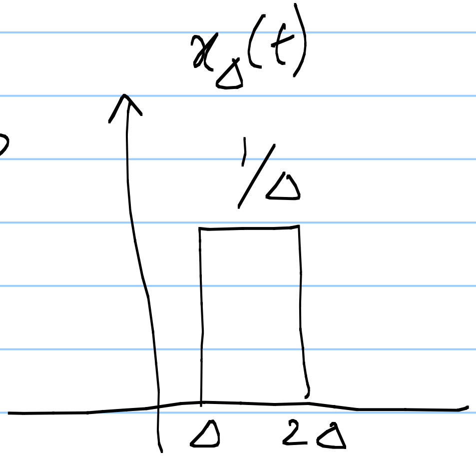




$$x_\sigma(t) = \frac{1}{\sqrt{2\pi}\ \sigma} e^{-t/2\sigma^2}$$

$x_\sigma(t) \rightarrow \delta(t)$ as $\sigma \rightarrow 0$

$\delta(0) = \infty$ wrong

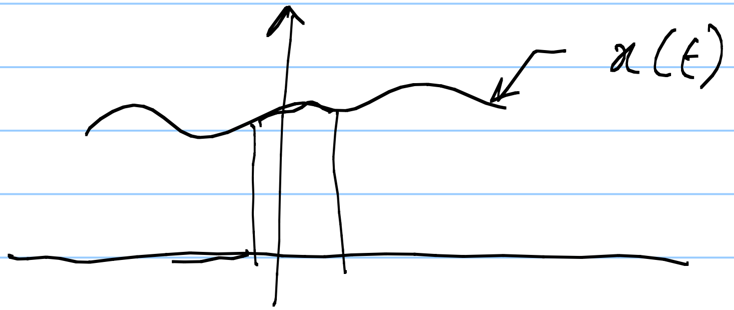


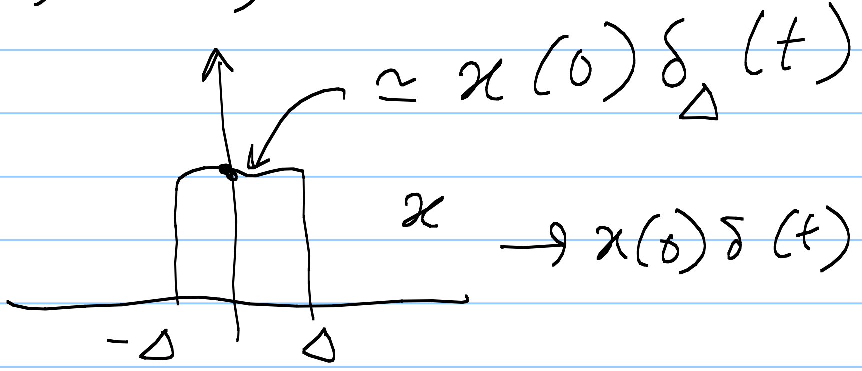
$$x_\Delta(t) \to \delta(t) \quad \text{as} \quad \Delta \to 0$$

Sifting property

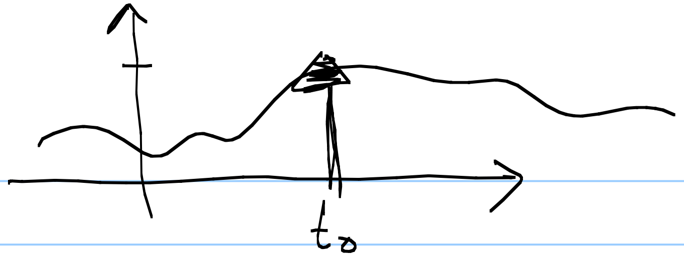
If $x(t)$ is continuous at $t = 0$, then

$$x(t)\,\delta(t) = x(0)\,\delta(t)$$





$\delta(t-t_0)$

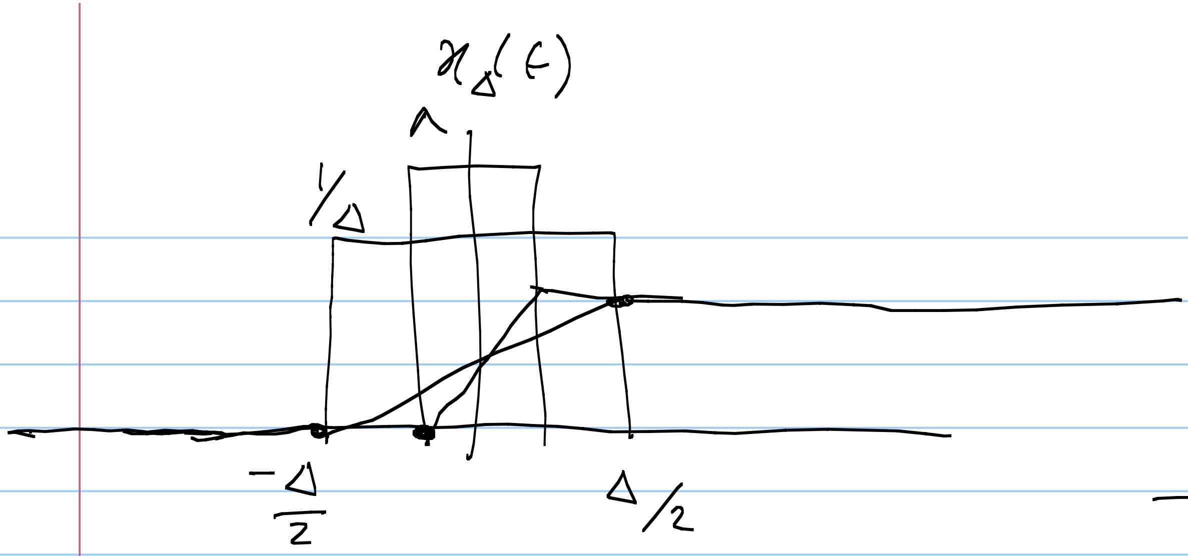


$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

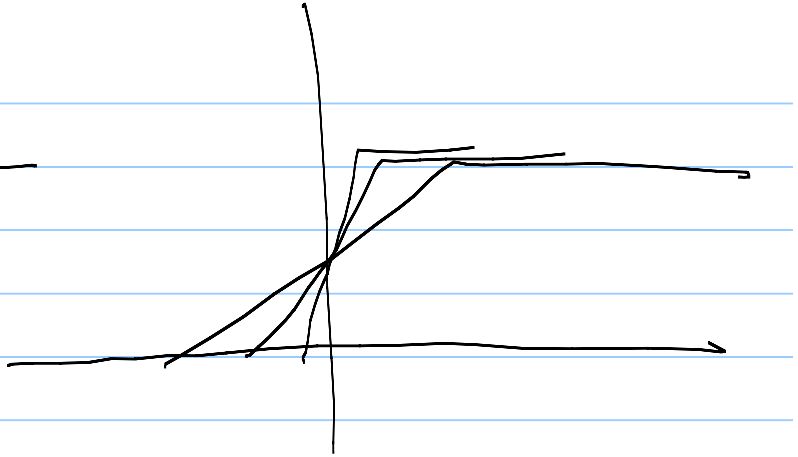
$$\delta(at) = \frac{1}{|a|}\,\delta(t) \quad \text{(verify!)}$$

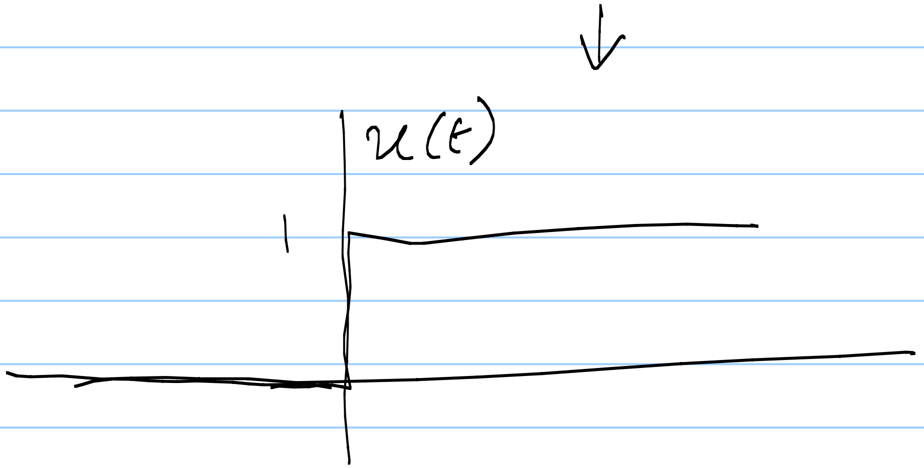
$$\delta(-t) = \delta(t) \quad \text{even function}$$

$$\delta(at+b) = \frac{1}{|a|}\,\delta(t+b/a) \quad \text{(verify)}$$



$$\int_{-\infty}^{t} x_\Delta(\tau)\,d\tau$$





$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

$$\frac{d}{dt} \int_{-\infty}^{t} \delta(\tau)\, d\tau = \delta(t) = \frac{d}{dt} u(t)$$

$$\delta(t) = \frac{d}{dt} u(t)$$

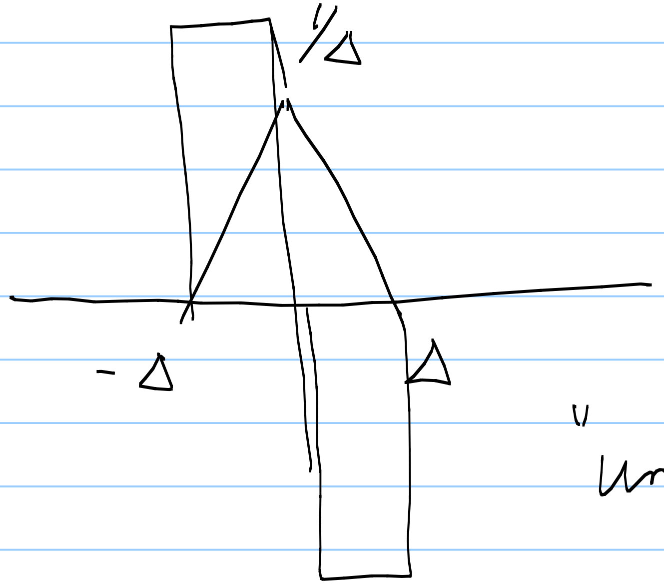
$$\frac{d}{dt}\left[f(t)\, u(t) \right] = u(t)\, f'(t) + f(t)\, u'(t)$$

$$= u(t)\, f'(t) + f(t)\, \delta(t)$$

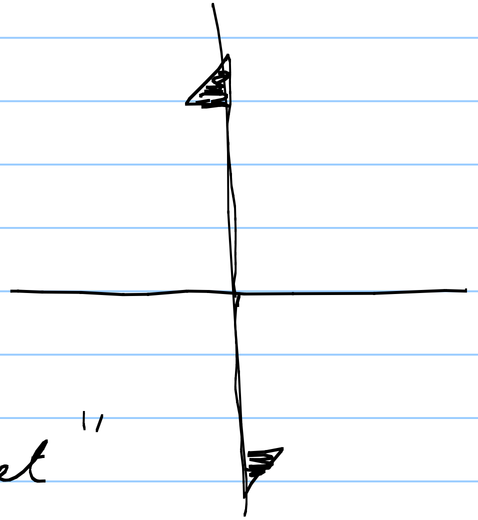
$$= u(t) f'(t) + f(0) \delta(t)$$

$f(t) u(t)$ = "right sided signal"

$f(t) u(-t)$ = "left sided signal"



$\longrightarrow$



"unit doublet"

$$\int_{-\infty}^{\infty} \delta'(t)\, dt = 0$$

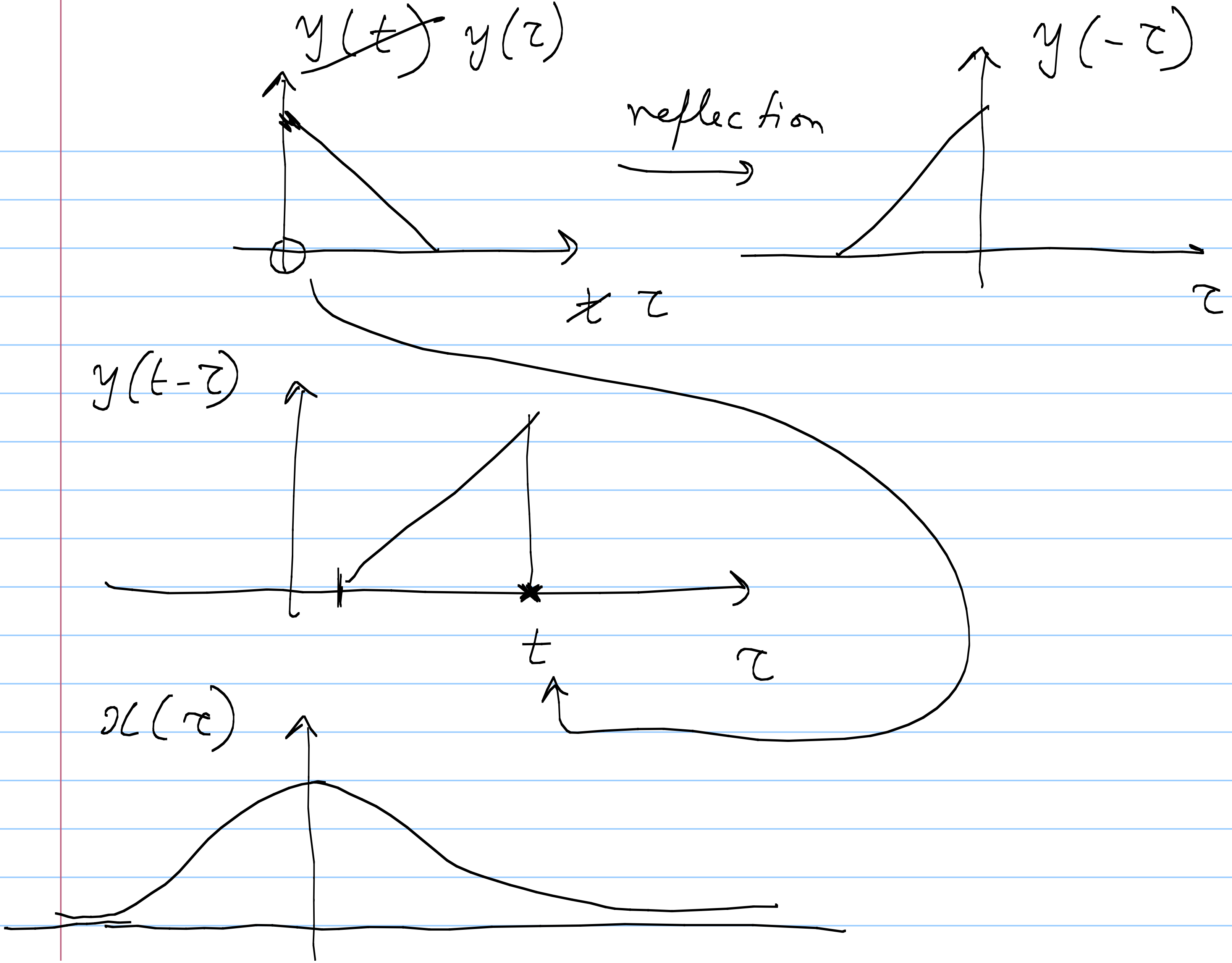
<u>Convolution</u>

$$x(t) * y(t) \overset{\text{def}}{=} \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau = z(t)$$

$$y(t) \longrightarrow y(\tau)$$

$$w(\tau) = y(-\tau) \qquad \text{reflection}$$

$$w(\tau - t) = y(-(\tau - t)) = y(t-\tau) \quad \text{translation}$$



~~y(t)~~ y(τ)
reflection
y(-τ)
~~t~~ τ
τ
y(t-τ)
t
τ
x(τ)

$$\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau = z(t)$$

http://www.jhu.edu/~signals/

$$\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$t - \tau = \lambda$$

$$\int_{-\infty}^{\infty} y(\lambda)\, x(t-\lambda)\, d\lambda$$ Commutative

Under certain conditions,

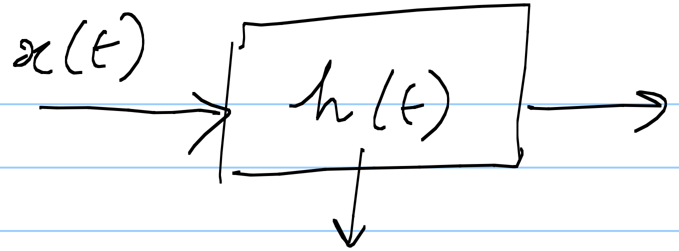
$$x * (y * z) = (x * y) * z \quad \text{associative}$$

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$= x(t)$$

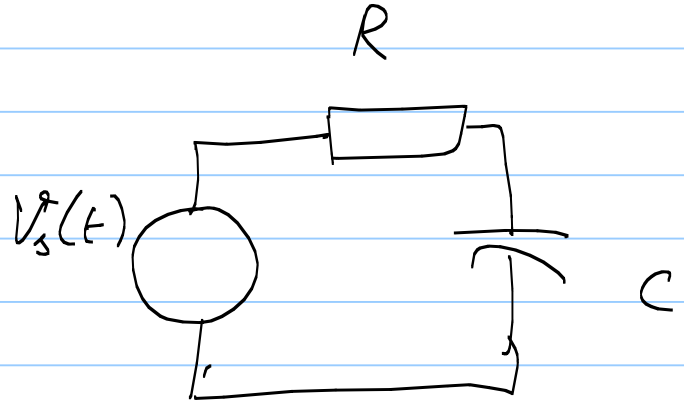
$$x(t) * \delta(t-t_0) = x(t-t_0) \quad \text{(verify)}$$

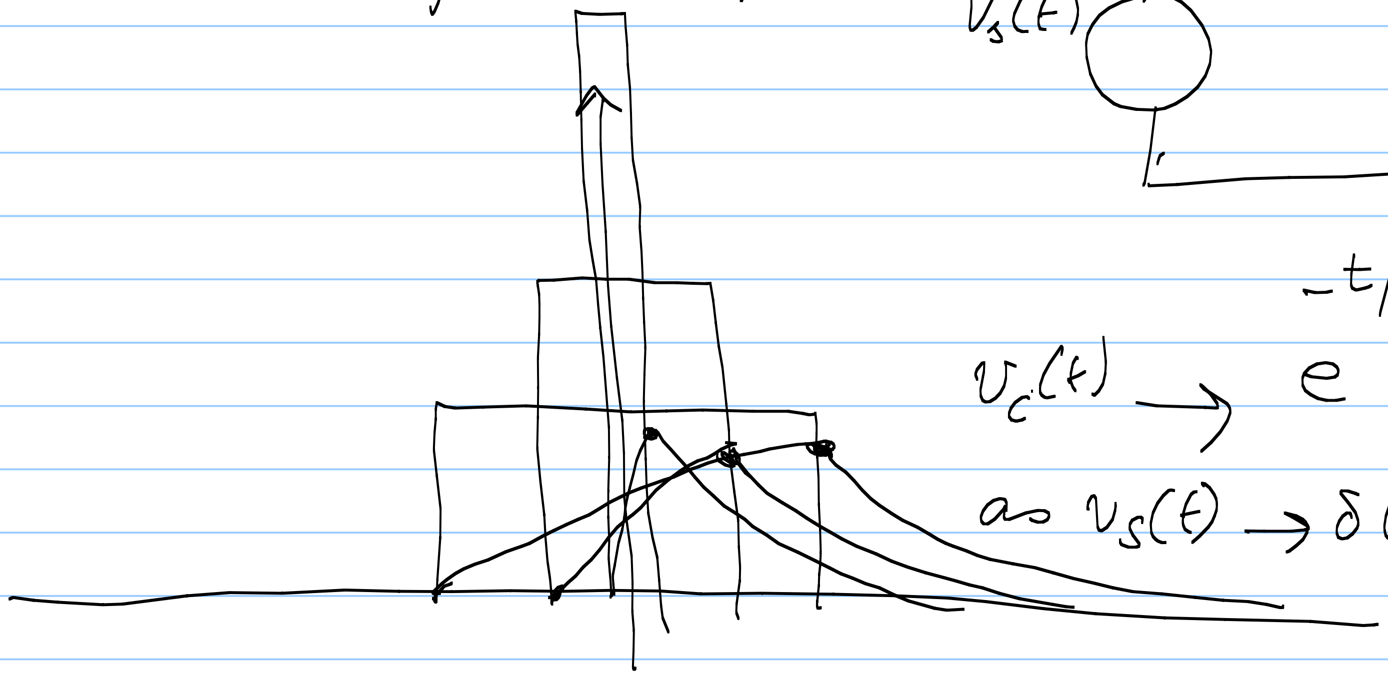
If $x(t) = \delta(t)$, $x(t) * \delta(t) = x(t) \Rightarrow \delta(t) * \delta(t) = \delta(t)$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

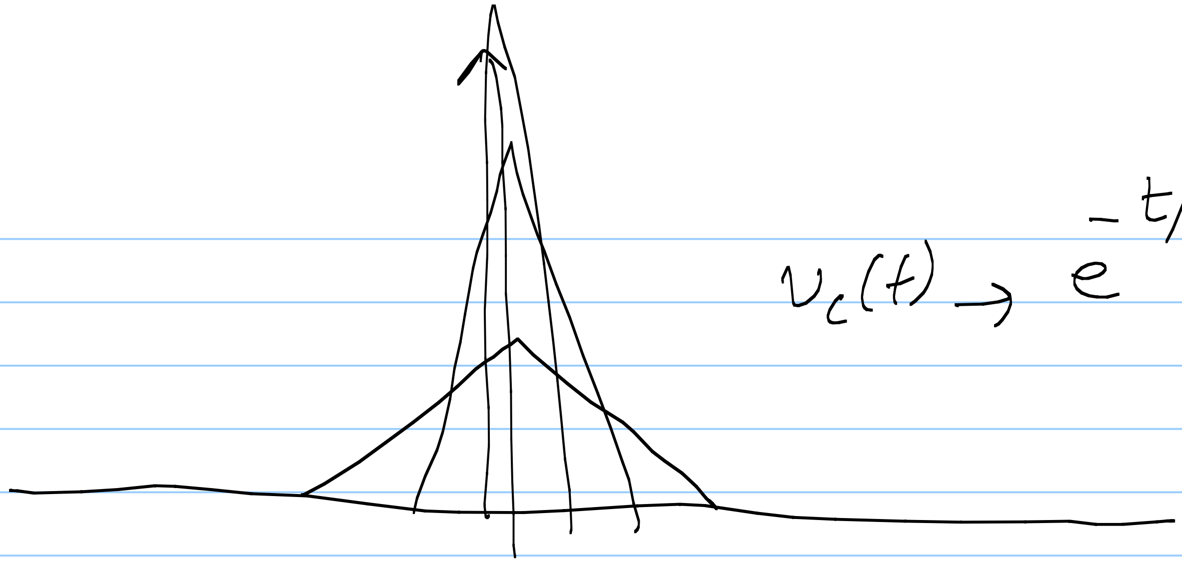
systems impulse response





$v_c(t) \rightarrow e^{-t/RC}\,u(t)$

as $v_s(t) \rightarrow \delta(t)$



$$v_c(t) \rightarrow e^{-t/RC} u(t)$$

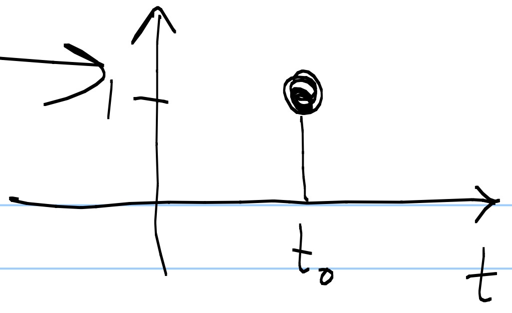
$$\delta^2(t) = \text{not defined}$$

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

WRONG!

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$



EC 5142 8^{th} Aug. 2011

Note Title 08-08-2011

Consider $x(t) * h(t)$

Suppose $h(t) = 0 \quad t < 0$

$h(t)$ can be written as $h(t)u(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, u(t-\tau)\, d\tau$$

For $\tau > t \quad u(t-\tau) = 0$

$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, \underbrace{u(t-\tau)}_{1}\, d\tau$$

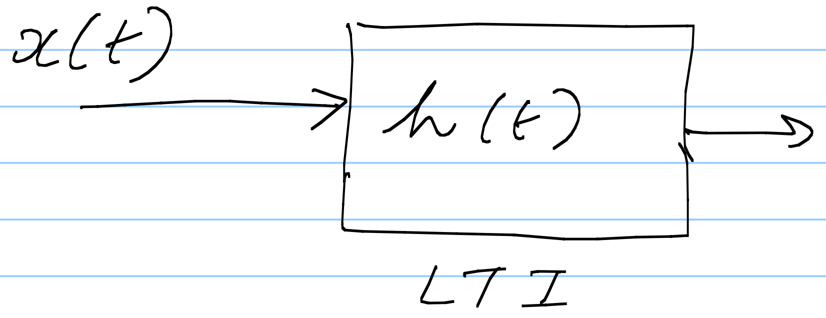
Now suppose $x(t) = 0 \quad t < 0$

i.e. $x(t)$ is written as $x(t)\,u(t)$

$$y(t) = \int_{-\infty}^{t} x(\tau)\,u(\tau)\,h(t-\tau)\,u(t-\tau)\,d\tau$$

$$= \int_{0}^{t} x(\tau)\,\underbrace{u(\tau)}_{1}\,h(t-\tau)\,\underbrace{u(t-\tau)}_{1}\,d\tau$$

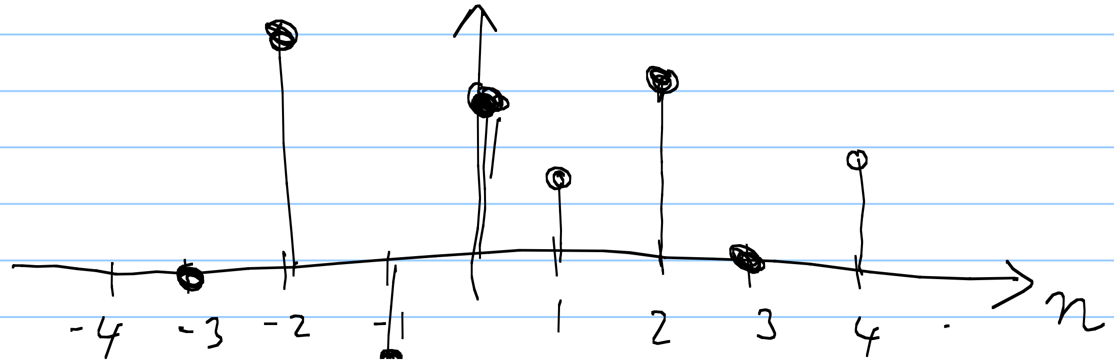
$$= \int_{0}^{t} x(\tau)\,h(t-\tau)\,d\tau$$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

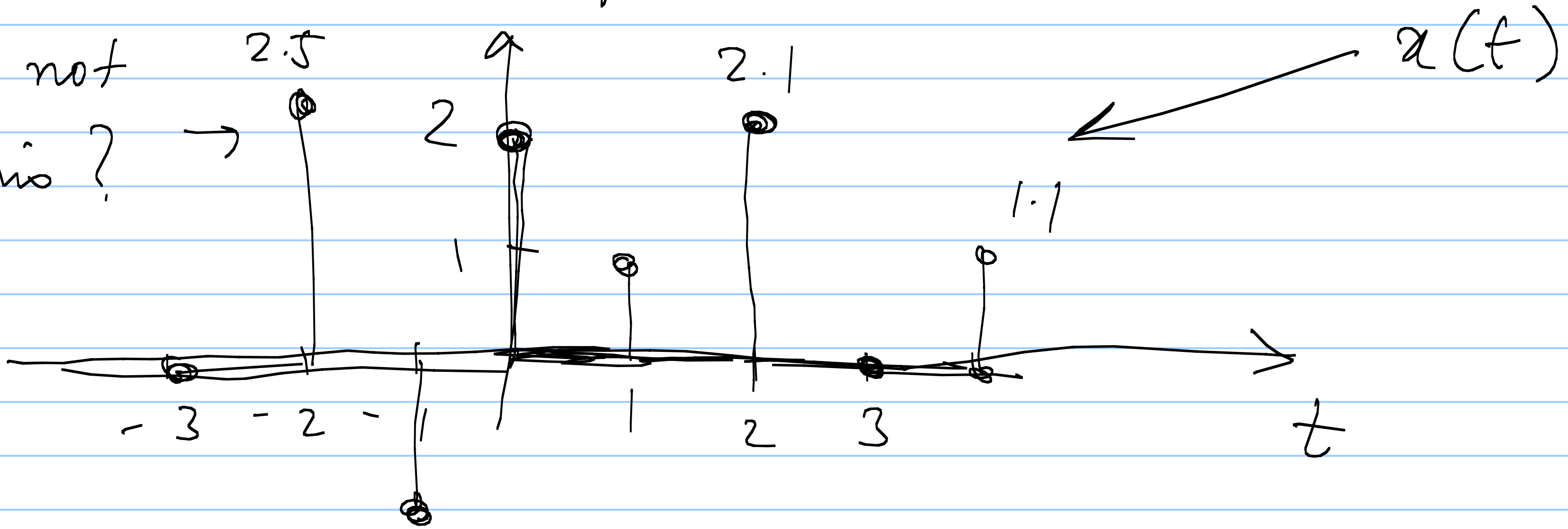
Discrete-time sequence

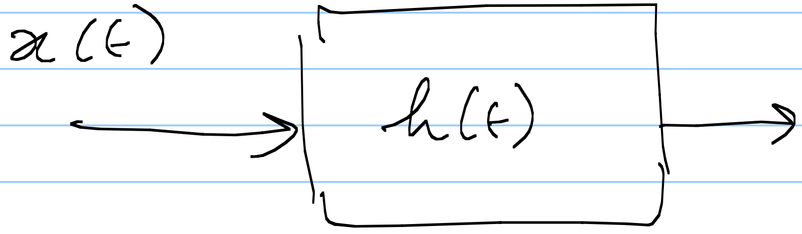
$x[n] \in \mathbb{C}$ $\qquad$ $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$



$x[1.5]$ = undefined, NOT zero " "

Why not this?





$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = 0$$

Unit impulse: $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

Unit step: $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \underbrace{n < 0} \end{cases}$

$= n \leq -1$

Analogous to complex exponential e^{st}, in discrete time we have

$$x[n] = z^n$$

$z = re^{jw}$ $\quad r > 0$

$x(n) = r^n e^{jwn}$

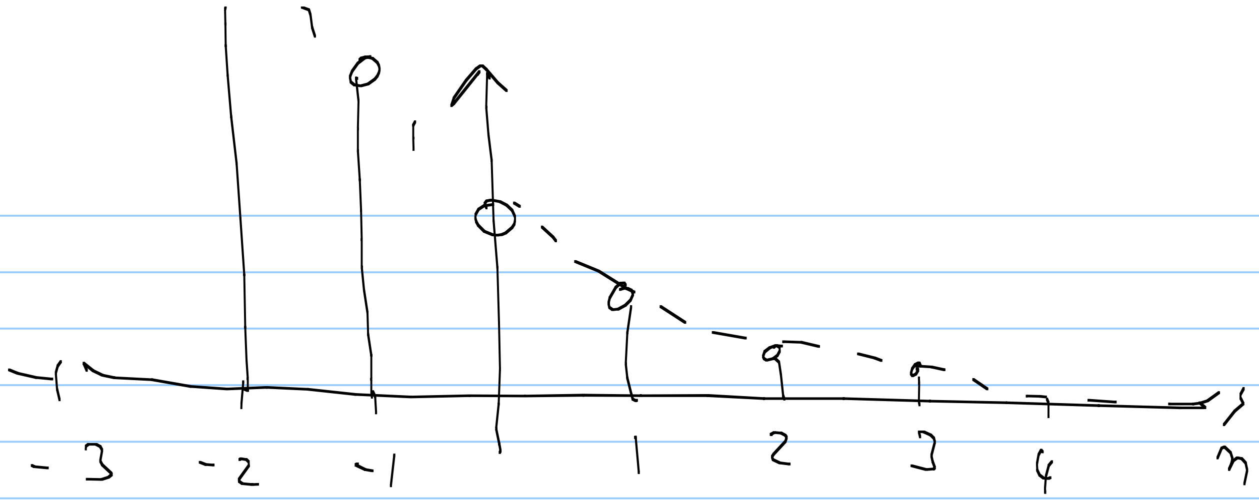
"DC": $x[n] = 1 \ \forall n$ $\quad r = 1, \ w = 0$

real exponential: r^n, $\quad w = 0$

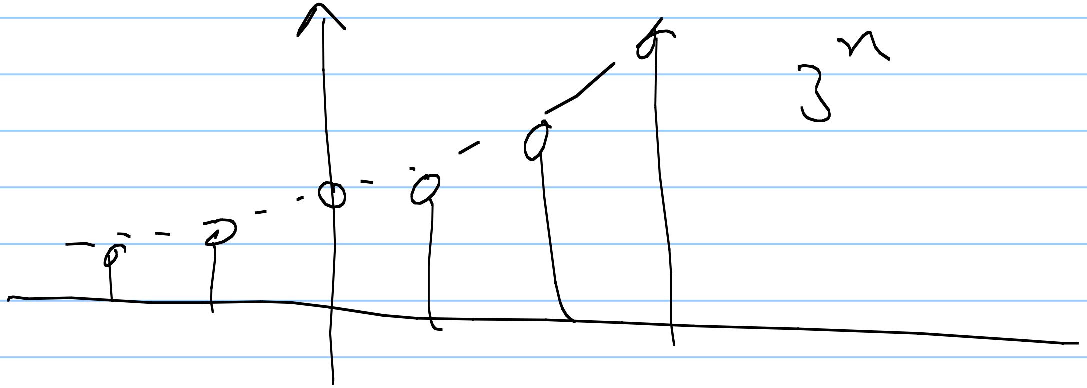
$|r| > 1$ growing exponential to the right

$|r| < 1$ " " to the left

$x[n] = (1/2)^n$



$x[n] = 3^n$



If $-1 < r < 0$, then signs alternate

$$x[n] = \left(-1/2\right)^n = (-1)^n \left(1/2\right)^n$$

Let's look at $e^{j\omega_0 n}$ & compare it with $\underbrace{e^{j\Omega_0 t}}$

↳ 1. always periodic w/ period $T = \frac{2\pi}{|\Omega_0|}$

2. Oscillates more & more rapidly as $\Omega_0 \uparrow$

$$x[n] = e^{j\omega_0 n}$$

If $x[n+N] = x[n]$, then $e^{j\omega_0 (n+N)} = e^{j\omega_0 n}$

$$\Rightarrow e^{j\omega_0 N} = 1$$

$$\omega_0 N = 2\pi k$$

$$2\pi f_0 N = 2\pi k$$

$$\boxed{f_0 = k/N}$$

Hence $e^{j\omega_0 n}$ is periodic with period N if and only if $f_0 = \frac{k}{N}$ i.e. a rational number

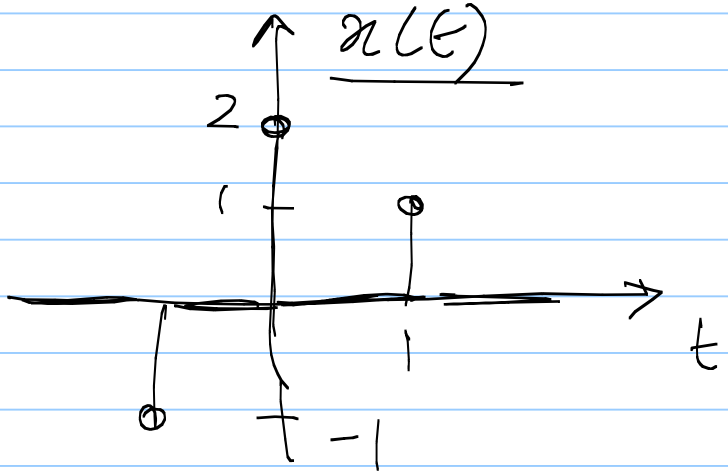
Now if $\omega_1 = \omega_0 + 2\pi$

$$e^{j\omega_1 n} = e^{j(\omega_0 + 2\pi)n} = e^{j\omega_0 n} \cdot e^{j2\pi n}$$

$$= e^{j\omega_0 n}$$

EC 5142 9th Aug. 2011

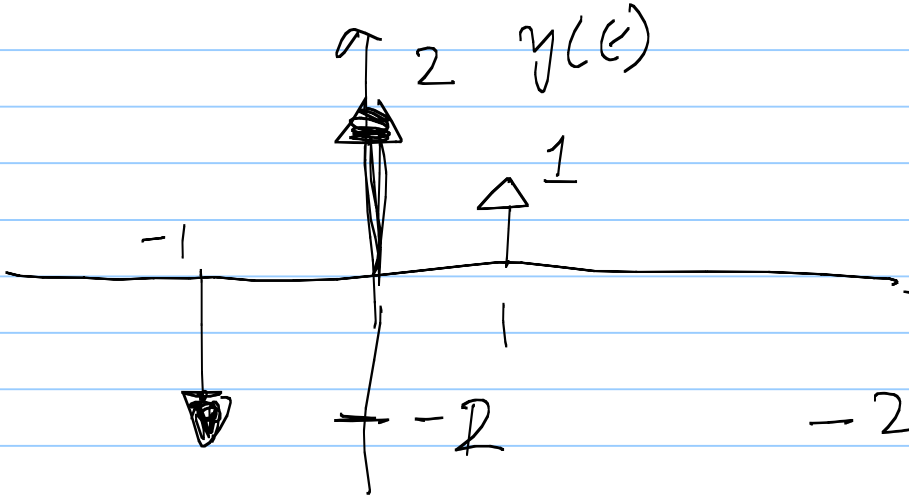
Note Title 09-08-2011



$x(1) = 1$

$x(0) = 2 \neq 2\delta(t)$

$\int_{-\infty}^{\infty} x(t)\,dt = 0$



$\int_{-\infty}^{\infty} y(t)\,dt$.

$-2+2+1 = \underline{1}$

$f(x) = \quad 1 \quad x$ is irrational

$\qquad\quad 0 \quad x$ is rational

$$e^{jw_0 n} = x[n]$$

$$x[n+N] = x[n] \Rightarrow \underbrace{\frac{w_0}{2\pi}}_{f_0} = \frac{k}{N}$$

$e^{j\Omega_0 t}$ is always periodic w/ periodic

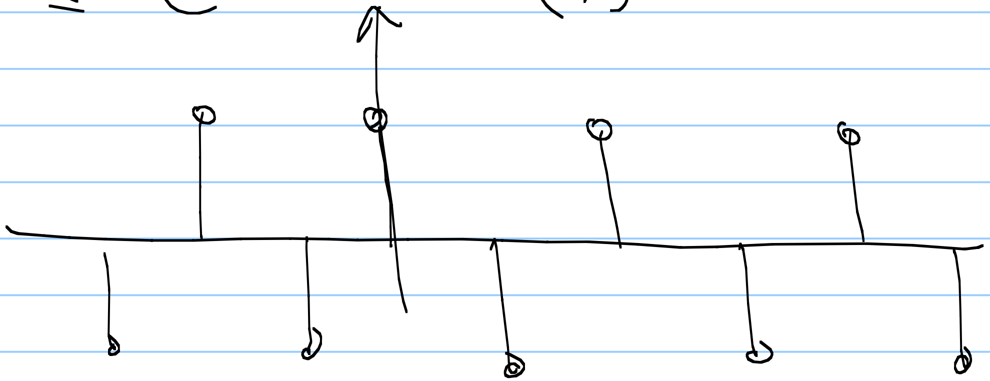
$$T = \frac{2\pi}{|\Omega_0|}$$

$$e^{j(\omega_0+2\pi)n} = e^{j\omega_0 n}$$

If $\omega_0 = 0$ $\quad e^{j\omega_0 n} = \underline{1} \quad \forall n$

If $\omega_0 = 2\pi$, $e^{j2\pi n} = 1 \quad \forall n$

$$e^{j\pi n} = e^{-j\pi n} = (-1)^n$$



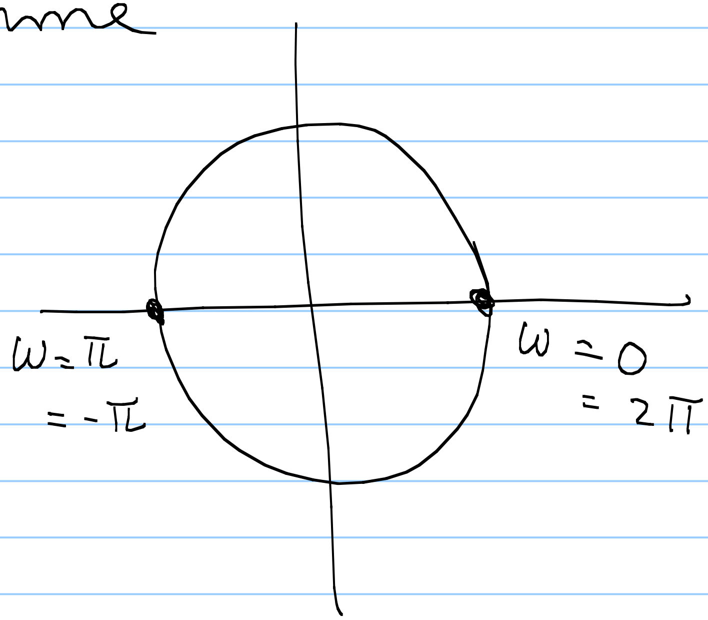
$$e^{j\,1.1\pi n} = \cos(1.1\pi n) + j\sin(1.1\pi n)$$

e^{jn} = not periodic in time

$= e^{j(1+2\pi)n}$

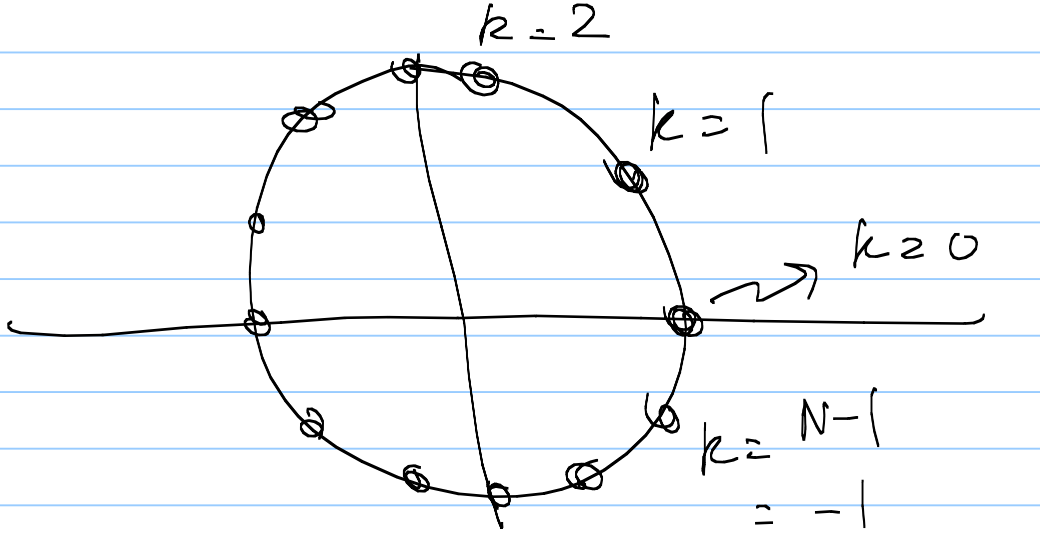
$e^{j\omega_0 n}$, $\frac{\omega_0}{2\pi} = \frac{k}{N}$

$f_0 \rightarrow (0,1)$



$$f = \overbrace{0, \frac{1}{N}, \frac{2}{N}, \frac{3}{N}, \ldots, \frac{N-1}{N}}, N$$

N distinct complex exponential



e^{jw_0n} & e^{-jw_0n} are two linearly independent signals.

$$x_1[n] = e^{jw_0n} = \cos w_0n + j\sin w_0n$$

$$x_2[n] = e^{-jw_0n} = \cos w_0n - j\sin w_0n$$

$$a_1x_1[n] + a_2x_2[n] = 0$$

$$\Rightarrow a_1 = a_2 = 0 \qquad a_1, a_2 \in \mathbb{C}$$

$$\cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$\cos 2\pi \frac{1}{N} n \xrightarrow{} k=1 \qquad \cos 2\pi \frac{-1}{N} n \xrightarrow{} k=-1 = N-1$$

$$\underbrace{2\pi f_0}_{\omega_0} n$$

$$\sin 2\pi \frac{1}{N} n \qquad \sin 2\pi \frac{-1}{N} n$$

EC 5142 10^{th} Aug. 2011

$x[n] = e^{j\omega_0 n}$: We want $x[n+N] = x[n]$

$$\Rightarrow f_0 = \frac{W_0}{2\pi} = \frac{k}{N} \Rightarrow \quad k = 0, 1, \ldots, N-1$$

Discrete-Time Convolution:

$$y[n] = \sum_{k=-\infty}^{\infty}{}' \, x[k] \, h[n-k]$$

$$= x[n] * h[n]$$

$$y(t) = x(t) \cdot h(t)$$

$$y(ct) = x(ct) \cdot h(ct)$$

$$y(t) = x(t) * h(t)$$

$$y(ct) = x(ct) * h(ct)$$ WRONG

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

find the correct expression

$$y(ct) = \int_{-\infty}^{\infty} x(\tau)\, h(ct-\tau)\, d\tau$$

Let $\left.\begin{aligned} x[n+N] &= x[n] \\ h[n+N] &= h[n] \end{aligned}\right\}$ same period N

$$\sum_{k=-\infty}^{\infty} \underbrace{x[k]\, h[n-k]}_{\text{periodic w/ period } N}$$

$= \quad 0, \quad \infty, \quad \text{or} \quad -\infty$

Suppose,

$$\sum_{k=0}^{N-1} x[k]\, h[n-k] \overset{\text{def}}{=} x[n] \underset{\underset{\text{optional suffix}}{\uparrow\ \underbrace{N}}}{\circledast} h[n]$$

circular convolution

periodic convolution

$$x(t+T) = x(t) \quad ; \quad h(t+T) = h(t)$$

$$x(t) \circledast h(t) = \int_0^T x(\tau)\, h(t-\tau)\, d\tau$$

For linear convolution, if $x[n]$ is of length M, & $h[n]$ is of length N, then

$x[n] * h[n] = y(n) \longrightarrow$ is of length $M+N-1$

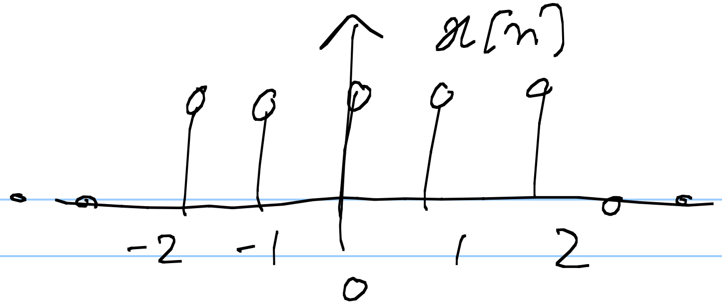
(Verify)

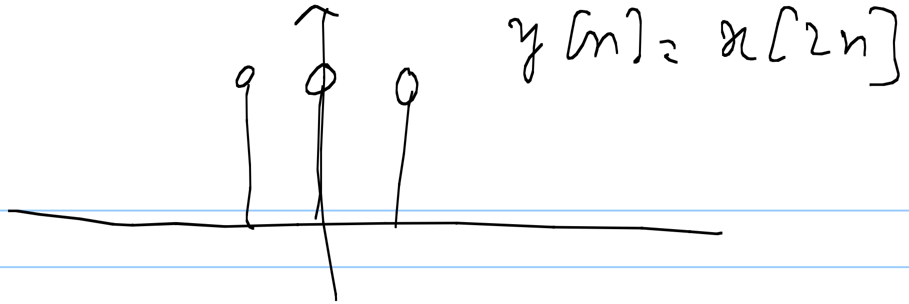
Scaling in Discrete Time

$x[an]$ $\qquad an \in \mathbb{Z}$

Eg: $y(n) = x[2n]$

$y[0) = x[0]$, $y[1] = x[2]$, $y[-1] = x[-2]$, ...





$y[n] = x[n/2]$

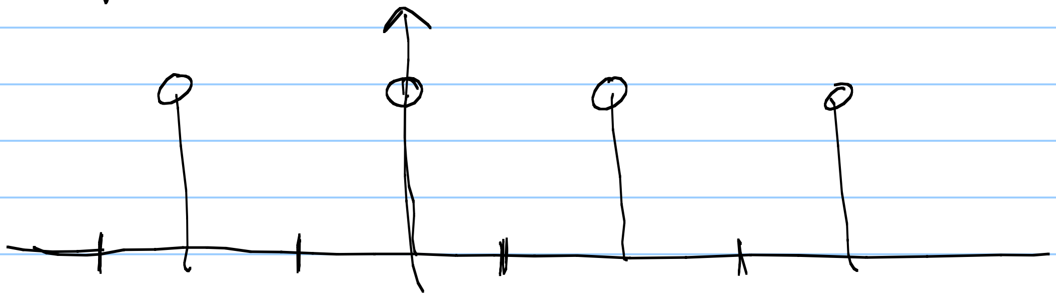
$y[0] = x[0]$

$y[2] = x[1]$

$y[-2] = x[-1]$

$y[1] = x[1/2] = \underline{\text{undefined}}$

$y[-1] = x[-1/2] = \underline{\text{undefined}}$



It is common to set $y[n] = 0$ when $n = 2k+1$

$$y[n] = x[mn + p]$$

$$w[n] = x[n+p]$$

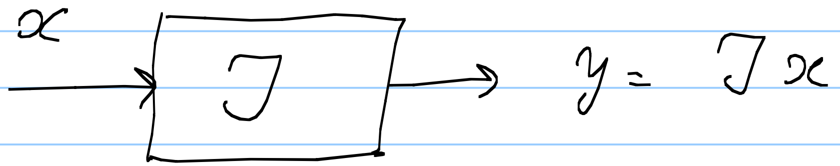
$$w[mn] = x[mn+p]$$

Is the other way possible? i.e., scaling first & then shifting?

Think about this

Systems

Takes an input, processes it, & produces an output.



(1) Linear

(a) additivity: if $x_1 \longrightarrow y_1$

& $x_2 \longrightarrow y_2$

and $x_1 + x_2 \longrightarrow y_1 + y_2$

then additivity holds

(b) Homogeneity:

$a \in \mathbb{C}$

If $x_1 \rightarrow y_1$

$a . x_1 \rightarrow a . y_1$

then homogeneity holds

If both additivity & homogeneity hold, the system is <u>linear</u>

$$a_1 x_1 + a_2 x_2 \rightarrow a_1 y_1 + a_2 y_2$$ (superposition)

$y(t) = \frac{dx(t)}{dt}$ verify that this is linear

$y(t) = x^2(t)$ verify that this is nonlinear

$y[n] = b_0 x[n] + b_1 x[n-1] + a_1 y[n-1]$

(linear)

One important consequence of linearity is

If $x(t) = 0$ $\forall t$, $y(t) = 0$ for all t

$$x_1(t) \longrightarrow y(t)$$

$$0 \cdot x(t) \longrightarrow 0 \cdot y(t)$$

$$0 \longrightarrow 0$$

EC5142 11th Aug. 2011

Classification of Systems.

1) Linearity

Additivity + homogeneity = linearity

$v(t) = i(t) \cdot R$ linear system

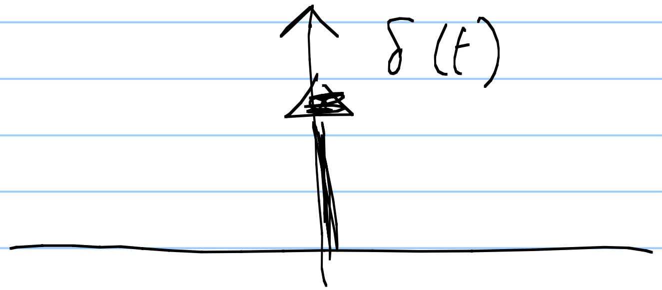
$\underbrace{a} \cdot x \longrightarrow a \cdot y$

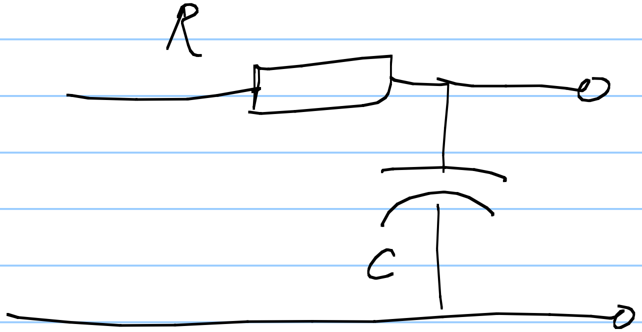
$a \in \mathbb{C}$

$0 \cdot x \longrightarrow 0 \cdot y$

If $x(t) = 0 \ \forall t$, then $y(t) = 0 \ \forall t$

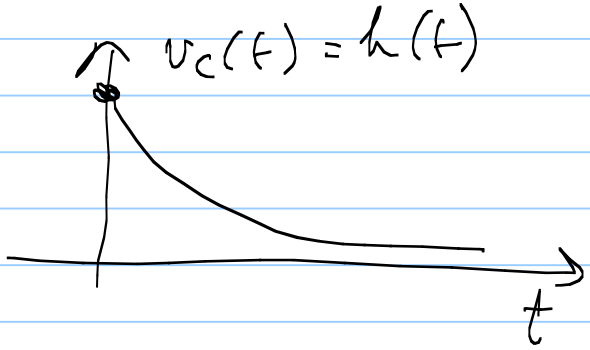
If $x(t) = 0$ for $t < t_0$, then $y(t) = 0$ for $t < t_0$





$$h(t) = e^{-t/RC} u(t) = v_c(t)$$

$$v_c(0^-) = 0$$



2) Causality

If the Output $y(t)\Big|_{t=t_0}$ depends only on

i/p $x(t)$ for $t \leq t_0$ & $y(t)$ for $t < t_0$

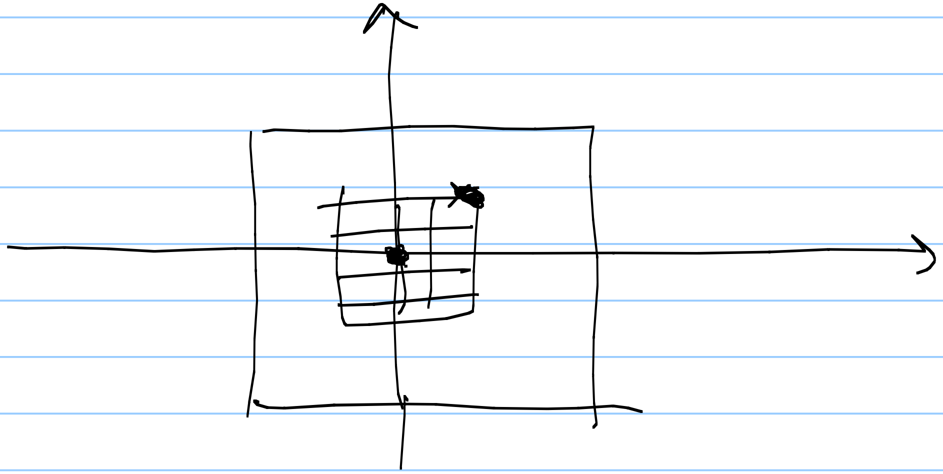
then the system is causal

Eg: $y(t) = c\,x(t)$: causal

Eg: $y(t) = x(-t)$ non causal

since for $t<0$, the o/p is a fn. of $x(t)$ for $t>0$.

Eg: $y(t) = x(t) \cos(t+1)$ causal



3) Memory or Memoryless

If o/p at $t = t_0$ depends only i/p only at $t = t_0$ then the system is memoryless.

$v(t) = i(t) \cdot R$ memoryless

$v_c(t) = \int_{-\infty}^{t} i_c(\tau)\, d\tau$ has memory

$y[n] = x[n] + a\, y[n-1]$ has memory
causal, linear

$y[n] = x[n] + b\, x[n+1] + a\, y[n-1]$ has memory
linear
non causal

4) Time Invariance

If $x(t) \longrightarrow y(t)$

& $x(t-t_0) \longrightarrow y(t-t_0)$

then the system is time invariant

$$y(t) = t\,x(t)$$

$$y(t-t_0) = (t-t_0)\,x(t-t_0) \qquad \text{———} \quad (1)$$

$$x(t) \longrightarrow x(t-t_0) = w(t) \longrightarrow t\,w(t)$$

$$t\,x(t-t_0) \quad - (2)$$

$\Rightarrow$ Time variant

$y(t) = x^2(t)$ nonlinear

time invariant

Linearity + time invariance : important

Sub class of systems

LTI systems

(5) <u>Stability</u> Bounded i/p Bounded o/p (BIBO) Stability

If $|x(t)| \leq B_x$ produces $y(t)$ s.t.

$|y(t)| \leq B_y$, then the system is BIBO stable

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

If $x(t) = u(t)$, $y(t) = t\, u(t)$

$e^{jw_0 n}$: $x[n+N] = x[n]$

$f_0 = \frac{w_0}{2\pi} = \frac{k}{N}$ $k = 0, 1, \ldots, N-1$

For some k & N, N need not be the fundamental period.

Even & Odd

If $x(t) = x(-t)$, then $x(t)$ is even

($x(t)$ is real valued)

$\cos(t)$: even

$\sin(t)$: odd

$x(t) = -x(-t)$ then $x(t)$ is odd

For complex signals

If $x(t) = x^*(-t)$, then it is conjugate even
conjugate symmetric

$x(t) = -x^*(-t)$, conjugate antisymmetric

$e^{j\Omega_0 t} \longrightarrow \left(e^{-j\Omega_0 t}\right)^* = e^{j\Omega_0 t}$: conjugate symmetric

Real Valued :

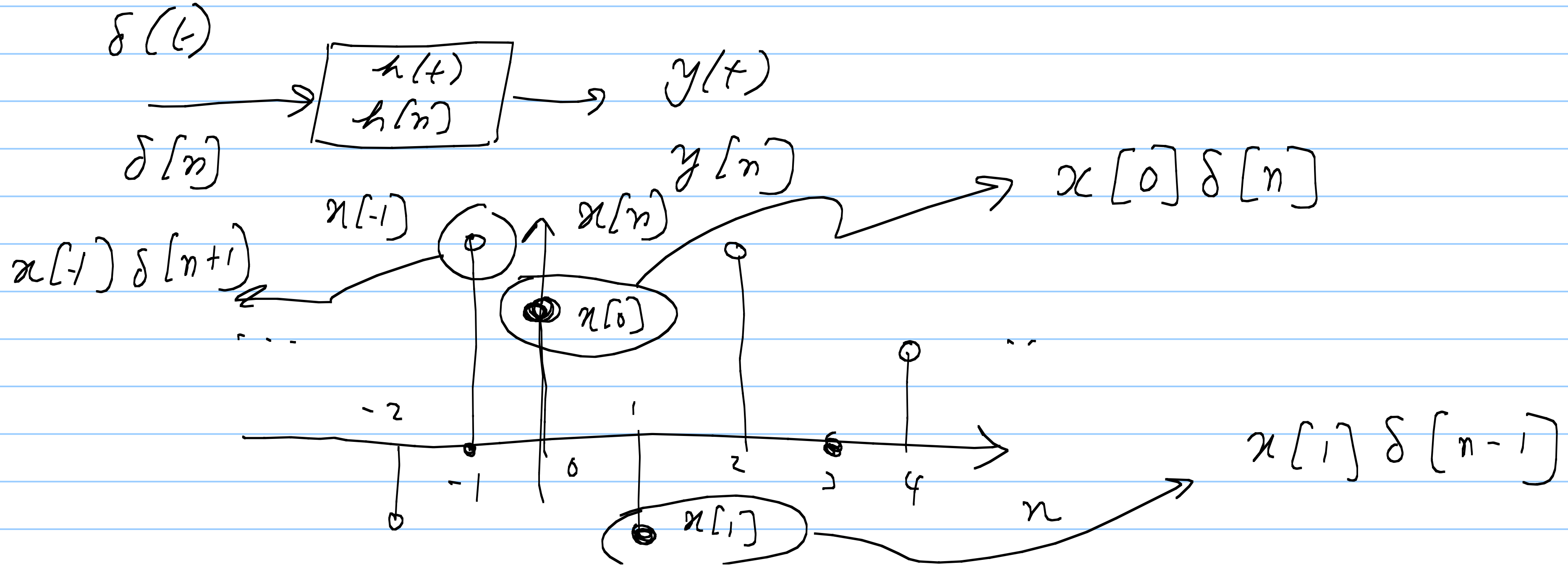
Any signal can be expressed as a sum of even & odd parts.

$$x(t) = \underbrace{x_o(t)}_{\text{odd part}} + \underbrace{x_e(t)}_{\text{even part}}$$

$$x_e(t) = \frac{x(t) + x(-t)}{2} \qquad x_o(t) = \frac{x(t) - x(-t)}{2}$$

EC 5142 17th Aug 2011

Systems that are both linear & time invariant form an important class LTI



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$\delta[n] \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow h[n-k]$$ time invariance

$$x[k]\,\delta[n-k] \longrightarrow x[k]\,h[n-k]$$ homogeneity

$$\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$ additivity

$$\underbrace{\sum x[k]\,\delta[n-k]}$$

$$x[n] \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = y[n]$$

$$x(t) \longrightarrow \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = y(t)$$

For causal systems, $h(t) = 0 \quad t < 0$

necessary & sufficient

Necessity: $\delta(t)$ is applied at $t = 0$

& hence $h(t)$ cannot be non zero for $t < 0$

for a causal system

Sufficiency! $h(t) = 0 \quad t < 0$

$$y(t_0) = \int_{-\infty}^{\infty} x(\tau)\, h(t_0 - \tau)\, d\tau$$

$$= \int_{-\infty}^{t_0} x(\tau)\, h(t_0 - \tau)\, d\tau$$

$\Rightarrow$ $y(t_0)$ depends only on i/p values $t \leq t_0$

Stability

$|x(t)| \leq B_x$ bounded i/p

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau \right|$$

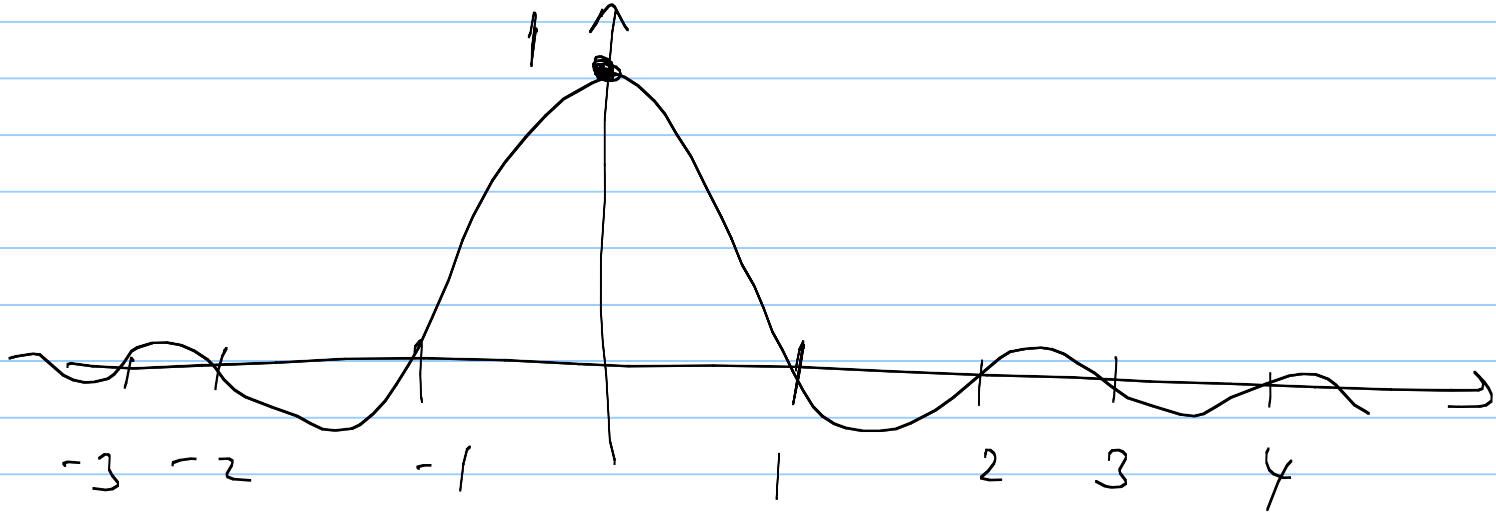
$$\le \int_{-\infty}^{\infty} \underbrace{|x(t-\tau)|}_{\le B_x} \, |h(\tau)|\, d\tau$$

$$|y(t)| \le B_x \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

For bounded o/p we require $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \quad \text{for DT systems}$$

$$\frac{\sin \pi t}{\pi t}$$



"Sinc" function

sometimes also called "analog sinc"

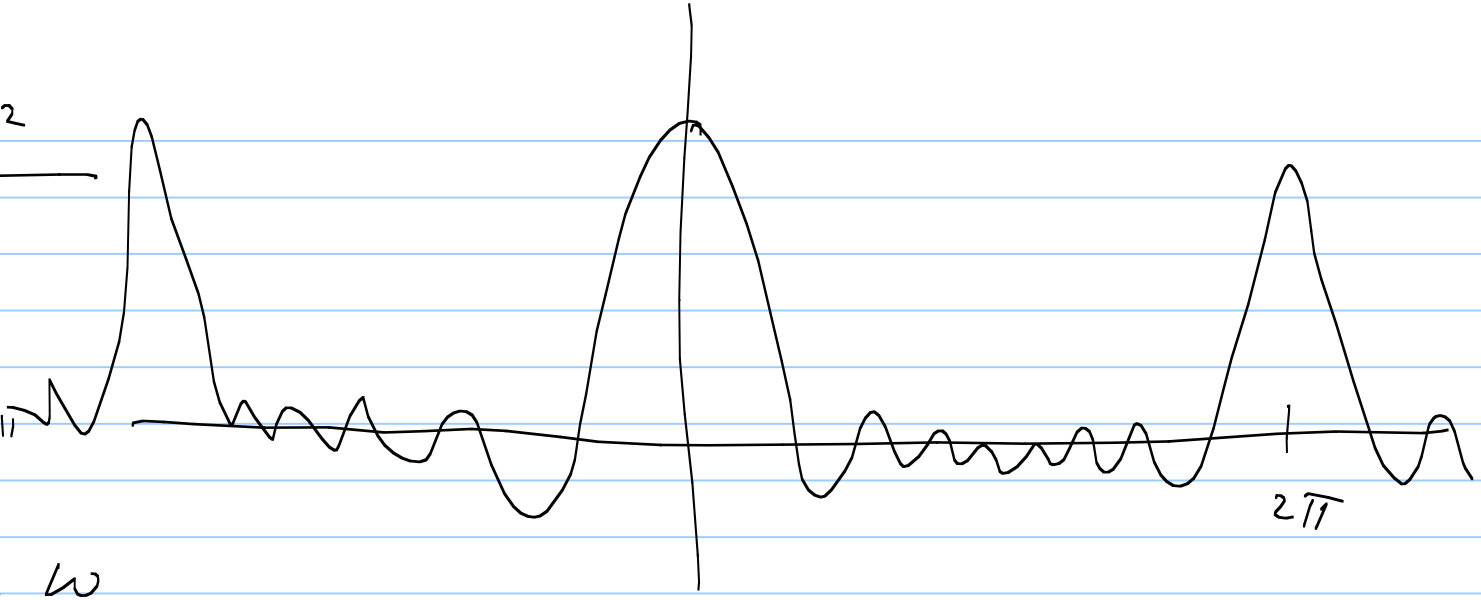
non periodic

$$\frac{\sin(2N+1)\,\omega/2}{\sin \omega/2}$$

"digital sinc"

periodic in ω

w/ period 2π



Dirichlet kernel

Matlab: diric

Systems represented by Linear Const. Coeff.
Differential Equations (LCCDE) are a very
Differce
important class

EC 5142 22nd Aug. 2011

$$\frac{d^k}{dt^k} a_k e^{st} = a_k s^k e^{st}$$

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{l=0}^{M} b_l \frac{d^l}{dt^l} x(t)$$

Homogeneous soln is obtained by solving

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = 0$$

If $y(t) = e^{st}$

$$e^{st}\left[\sum_{k=0}^{N} a_k s^k\right] = 0$$

$$\Rightarrow \sum_{k=0}^{N} a_k s^k = 0$$

If the roots are distinct,

$$y_h(t) = \sum_{k=1}^{N} C_k e^{S_k t}$$

c_k : are determined by the auxiliary conditions

$e^{s_k t}$: natural modes of the system

Recall.

$$Y_I e^{-at} + \frac{1}{a-b}\left[-e^{-at} + e^{-bt}\right] u(t)$$

For $Y_I = 0$, we get $\underbrace{-\frac{1}{a-b} e^{-at}}_{\text{natural mode}} + \underbrace{\frac{1}{a-b} e^{-bt}}_{\text{forced response}}$ $t \geq 0$

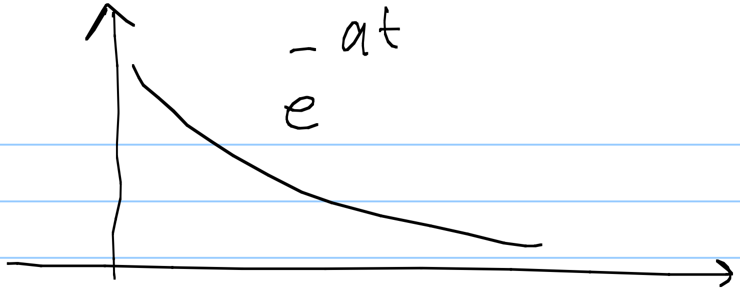
Repeated roots:

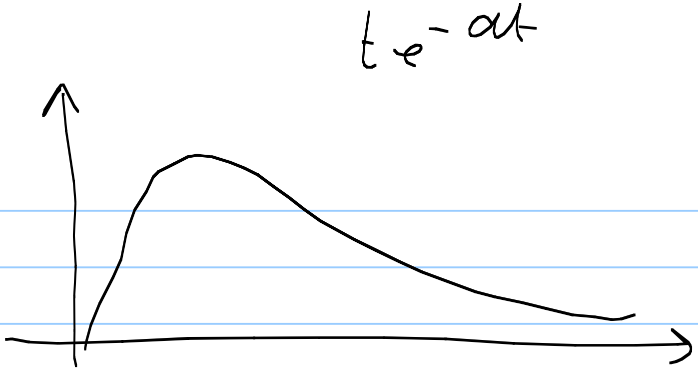
Eg1 $e^{st}(s+a)^2 = 0$

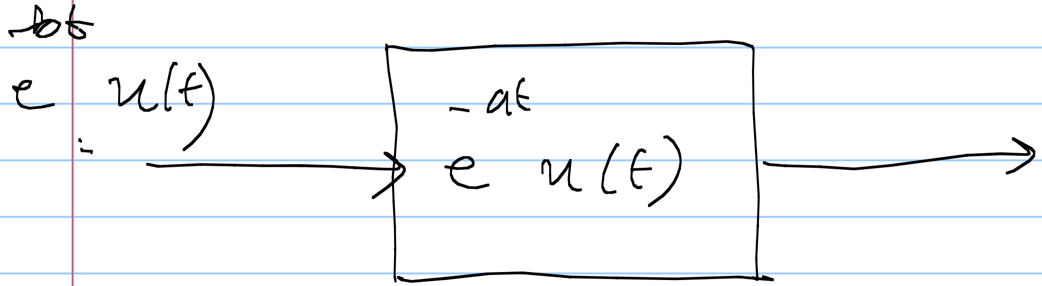
Modes are: $c_1 e^{-at}$, $c_2 t e^{-at}$

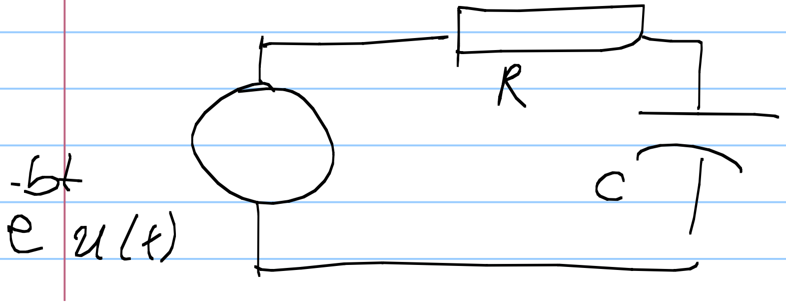
For multiplicity 'r'

$$c_1 e^{-at},\ c_2 t e^{-at},\ c_3 t^2 e^{-at}, \ldots, C_r t^{r-1} e^{-at}$$









$$\frac{1}{a-b}\left[e^{-bt} - e^{-at}\right] u(t)$$

If $a = b$, soln is of the form $\underbrace{K e^{-at}}_{\text{natural mode}} + C \cdot t e^{-at}$

$$e^{-at} u(t) * e^{-at} u(t) = \int_0^t e^{-a\tau} e^{-a(t-\tau)} d\tau$$

$$= e^{-at} \int_0^t d\tau$$

$$= t e^{-at} u(t)$$

Look up Lathi for zero-input & zero state responses.

$$x(t) = e^{st} \longrightarrow \boxed{T} \longrightarrow T e^{st}$$

$$\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

For LTI : $y(t) = e^{st} \, H(s)$

$$= \lambda \cdot x(t)$$

Reminiscent of $\underline{A}\,\underline{x} = \lambda\,\underline{x}$

$$x(t) = e^{j\Omega t} \longrightarrow \boxed{} \longrightarrow y(t) = e^{j\Omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\Omega\tau} d\tau$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{H(j\Omega)}$$

$H(j\Omega)$ is also denoted by $H(\Omega)$

↳ just notation

$$\sum_{k=1}^{N} a_k e^{s_k t} \longrightarrow \boxed{LTI} \longrightarrow \sum_{k=1}^{N} H(s_k) a_k e^{s_k t}$$

$$\sum_{k=1}^{N} a_k e^{j\Omega_k t} \longrightarrow \boxed{LTI} \longrightarrow \sum_{k=1}^{N} a_k H(j\Omega_k) e^{j\Omega_k t}$$

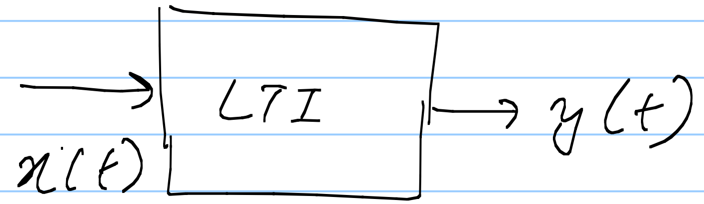
No new frequencies !!

We will assume

$$\underbrace{\sum_{k=-\infty}^{\infty} a_k e^{j\Omega_k t}} \longrightarrow \boxed{LTI} \longrightarrow \sum_{k=-\infty}^{\infty} a_k H(j\Omega_k) e^{j\Omega_k t}$$

very powerful way of representing certain types of $x(t)$

EC 5142 23rd Aug. 2011



$$\hookrightarrow \sum_{k=-\infty}^{\infty} a_k e^{j\Omega_k t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} \overbrace{H(j\Omega_k)}^{H(\Omega_k)} a_k e^{j\Omega_k t}$$

No new frequencies are introduced by an LTI system

If $x(t+T) = x(t)$, then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \qquad \Omega_0 = \frac{2\pi}{T}$$

Fourier Series representation

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}}$$ synthesis eqn

Inner product of $x(t)$ & $y(t)$: $\int_{R} x(t)\, y^*(t)\, dt$

$$\int_0^T x(t) \cdot e^{-jl\Omega_0 t} \, dt = \int_0^T \left[\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \right] e^{-jl\Omega_0 t} \, dt$$

$$\overset{?}{=} \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_0^T e^{j(k-l)\Omega_0 t} \, dt}$$

$$= a_k \, \delta_{kl} \, T$$

$\hookrightarrow \delta_{kl} = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$

$$= a_\ell T$$

$$\boxed{a_\ell = \frac{1}{T}\int_0^T x(t)\, e^{-j\ell\Omega_0 t}\, dt}$$

analysis equation

$$\cos\Omega_0 t = \frac{1}{2}e^{j\Omega_0 t} + \frac{1}{2}e^{-j\Omega_0 t}$$

$$a_1 = \frac{1}{2} = a_{-1}$$

$\sin \Omega_0 t$: $a_1 = \frac{1}{2j}$ $a_{-1} = -\frac{1}{2j}$

$$\cos \Omega_0 t + \sin^2 \Omega_0 t = \cos \Omega_0 t + \frac{1}{2}(1 - \cos 2\Omega_0 t)$$

$$= \cos \Omega_0 t + \underbrace{\frac{1}{2}}_{a_0} - \frac{1}{2}\cos 2\Omega_0 t$$

$a_0 = \frac{1}{2}$ $a_2 = a_{-2} = -\frac{1}{4}$

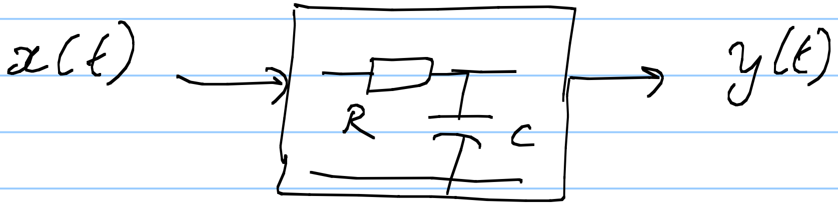
$a_1 = \frac{1}{2} = a_{-1}$

EC 5142 Aug. 24 2011

$$x(t) = \cos \Omega_0 t + \sin^2 \Omega_0 t$$

$$= \cos \Omega_0 t + \frac{1}{2}\left(1 - \cos 2\Omega_0 t\right)$$

$$a_0 = \frac{1}{2} \qquad a_1 = a_{-1} = \frac{1}{2} \qquad a_2 = a_{-2} = -\frac{1}{4}$$



$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$H(j\Omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\Omega t}\, dt = \frac{1}{1+j\Omega RC}$$

$$y(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\Omega_0 t}$$

$$c_0 = a_0 \, H(j\Omega)\Big|_{\Omega=0} = \frac{1}{2}$$

$$c_1 = \frac{1}{2} \frac{1}{1+j\Omega_0 RC} \qquad c_{-1} = \frac{1}{2} \frac{1}{1-j\Omega_0 RC}$$

$$C_2 = -\frac{1}{4} \frac{1}{1+j2\Omega_0 RC} \qquad C_{-2} = -\frac{1}{4} \frac{1}{1-2j\Omega_0 RC}$$

For real-valued $x(t)$, $x(t) = x^*(t)$

$$x^*(t) = \left[\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \right]^*$$

$$= \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\Omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_{-k}^{*} e^{jk\Omega_0 t}$$

$$= x(t)$$

$$\Rightarrow \quad a_k = a_{-k}^{*} \quad \Rightarrow \quad |a_k| = |a_{-k}|$$

Again, for real-valued $x(t)$

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k e^{jk\Omega_0 t} + \sum_{k=1}^{\infty} a_{-k} e^{-jk\Omega_0 t}$$

$$= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\Omega_0 t} + a_k^* e^{-jk\Omega_0 t} \right]$$

$$a_k = A_k e^{j\theta_k}$$

$$= a_0 + \sum_{k=1}^{\infty} A_k e^{j(k\Omega_0 t + \theta_k)} + A_k e^{-j(k\Omega_0 t + \theta_k)} \Big]$$

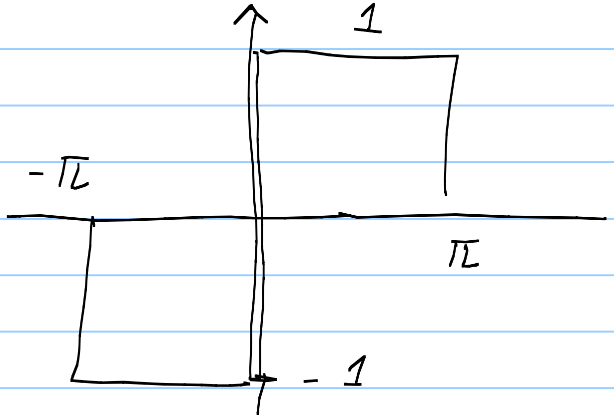
$$= a_0 + \sum_{k=1}^{\infty} 2A_k \cos(k\Omega_0 t + \theta_k)$$

$$A_k \cos(k\Omega_0 t + \theta_k) = \underbrace{A_k \cos\theta_k}_{B_k} \cos k\Omega_0 t - \underbrace{A_k \sin\theta_k}_{C_k} \sin k\Omega_0 t$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} B_k \cos k\Omega_0 t - C_k \sin k\Omega_0 t$$

Eg:

$x(t)$



$$\Omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

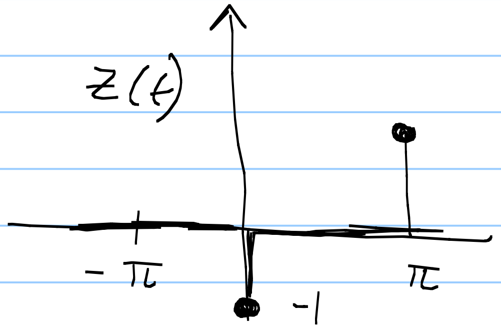
$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\Omega_0 t}\,dt = \begin{cases} \frac{2}{j\pi k} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

$$x(t) = \frac{4}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\cdots\right]$$

$$x(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$

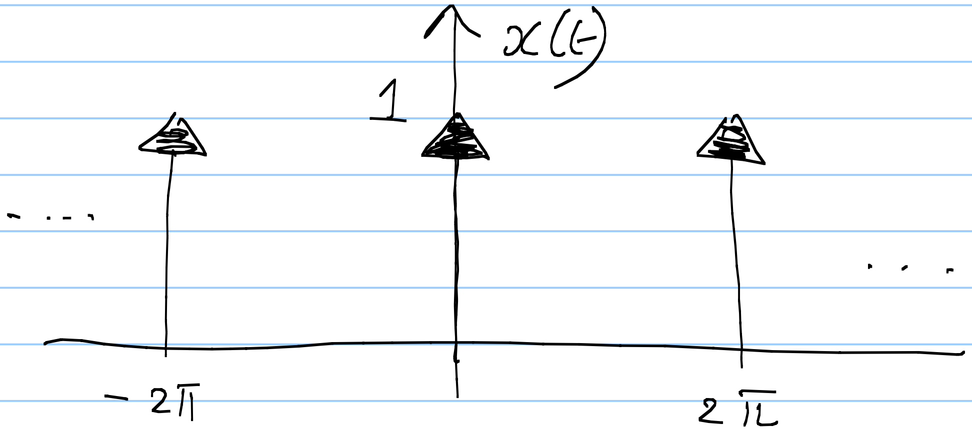
$$y(t) = \begin{cases} -1 & -\pi < t \leq 0 \\ 1 & 0 < t \leq \pi \end{cases}$$

$$y(t) = x(t) +$$



$y(t)$ has the same set of FS coeffs. as $x(t)$ since $z(t)$'s FS coeffs. are ZERO.

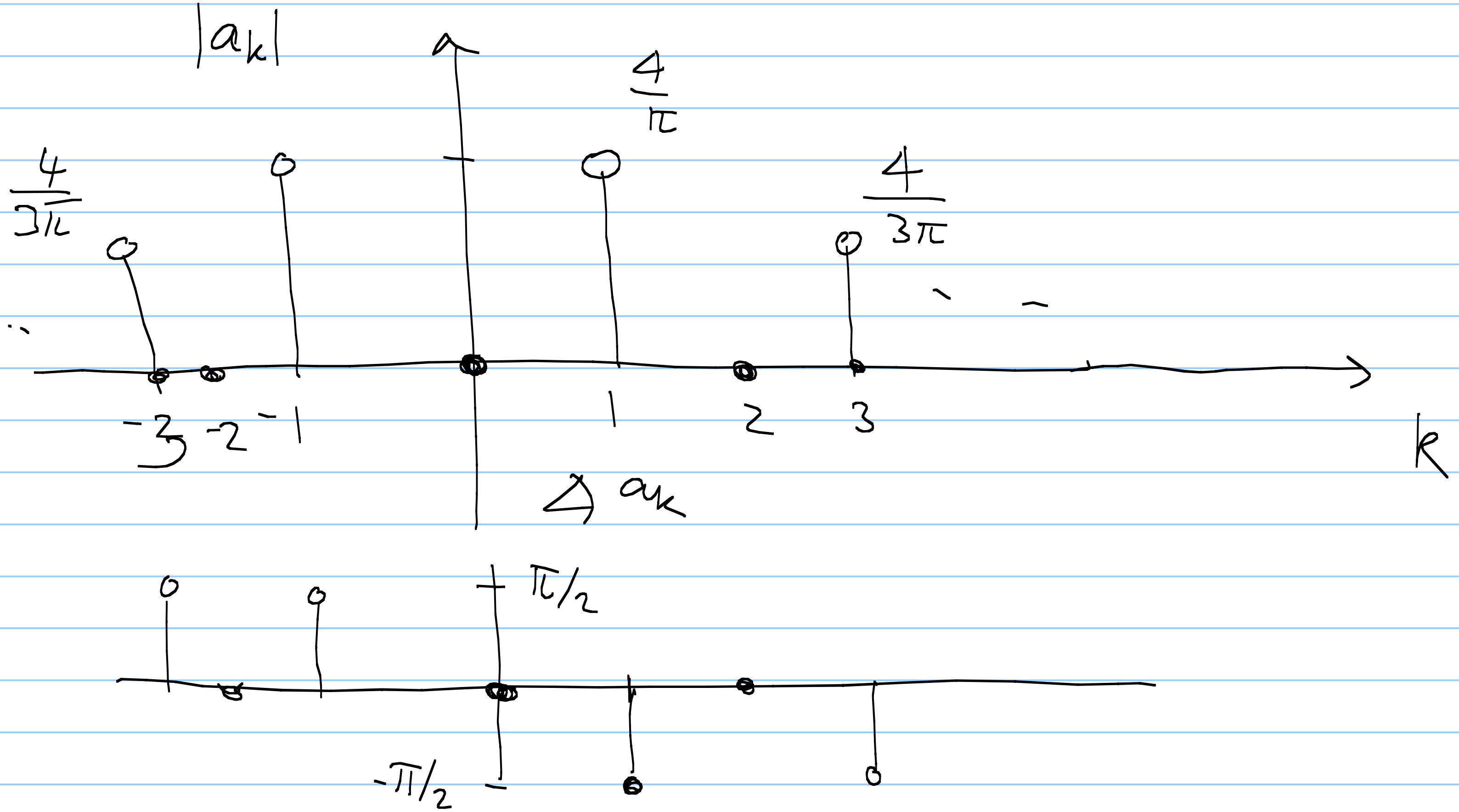
Eg!

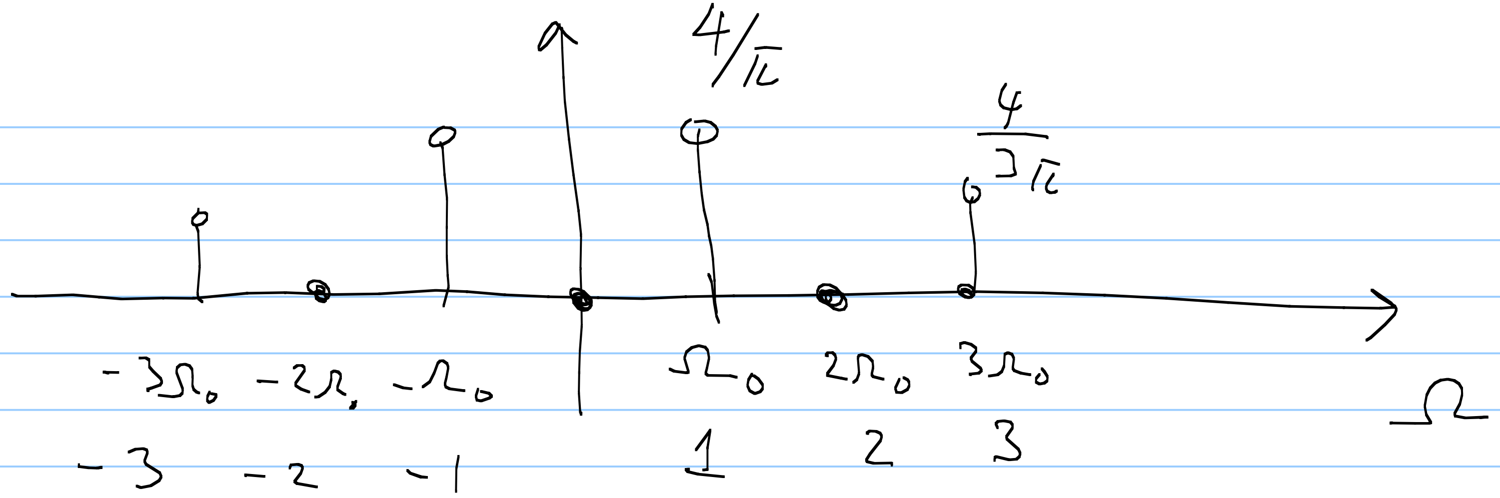


$$a_k = \frac{1}{2\pi} \quad \forall k$$

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t + 2k\pi)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$




4/π
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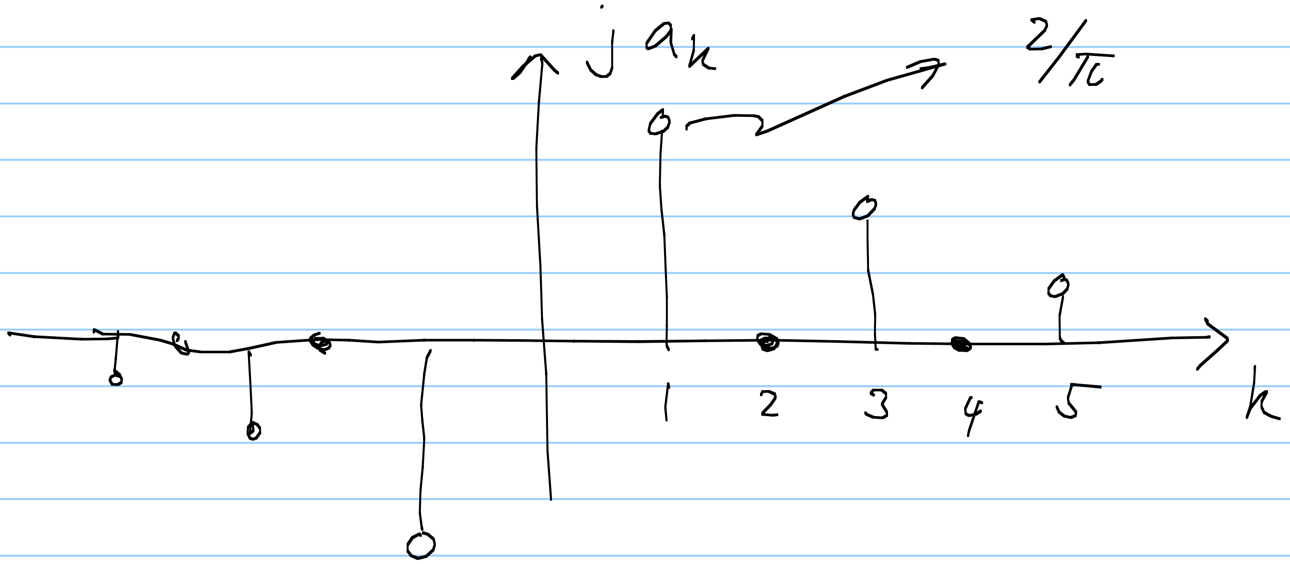
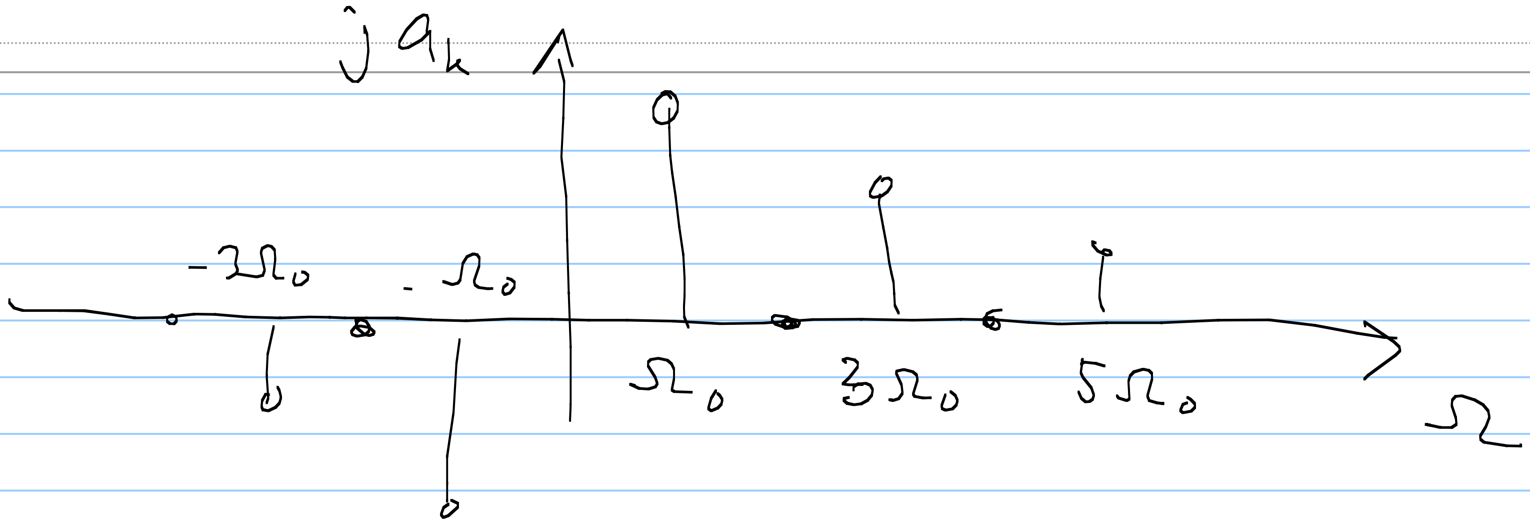
EC 5142 29th Aug. 2011

$$a_k = \begin{cases} \dfrac{2}{j\pi k} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$





Symmetry

$x(t)$ is real-valued

$$x(t) = \sum_{k=0}^{\infty} A_k \cos k\Omega_0 t - \sum_{k=1}^{\infty} B_k \sin k\Omega_0 t$$

Suppose $x(t)$ is even, i.e. $x(t) = x(-t)$

$$x(-t) = \sum_{k=0}^{\infty} A_k \cos k\Omega_0 t + \sum_{k=1}^{\infty} B_k \sin k\Omega_0 t$$

$$= x(t)$$

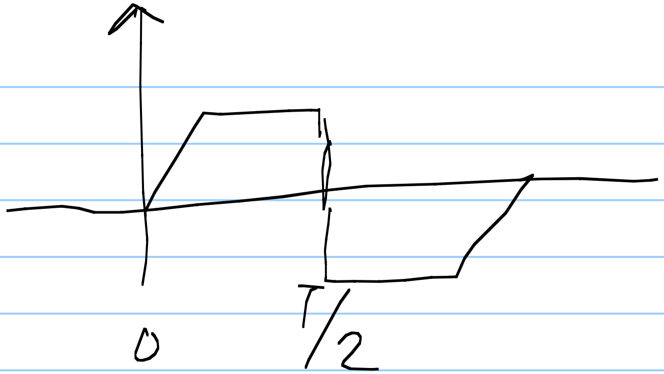
$$\Rightarrow \quad B_k = 0$$

Suppose $x(t) = -x(-t)$ i.e. odd

$$\Rightarrow \quad A_k = 0$$

Halfwave (odd) Symmetry)

$$x(t + T/2) = -x(t)$$



$$\frac{1}{T}\int_0^{T/2} x(t)\, e^{-jk\Omega_0 t}\, dt + \underbrace{\frac{1}{T}\int_{T/2}^{T} x(t)\, e^{-jk\Omega_0 t}\, dt}_{t + T/2 = \lambda}$$

$a_k = 0 \qquad k = \text{even}$

What about halfwave even symmetry?

$$x(t+T/2) = x(t)$$

Properties

(1) Linearity $\quad x_1(t+T) = x_1(t) \longleftrightarrow a_k$

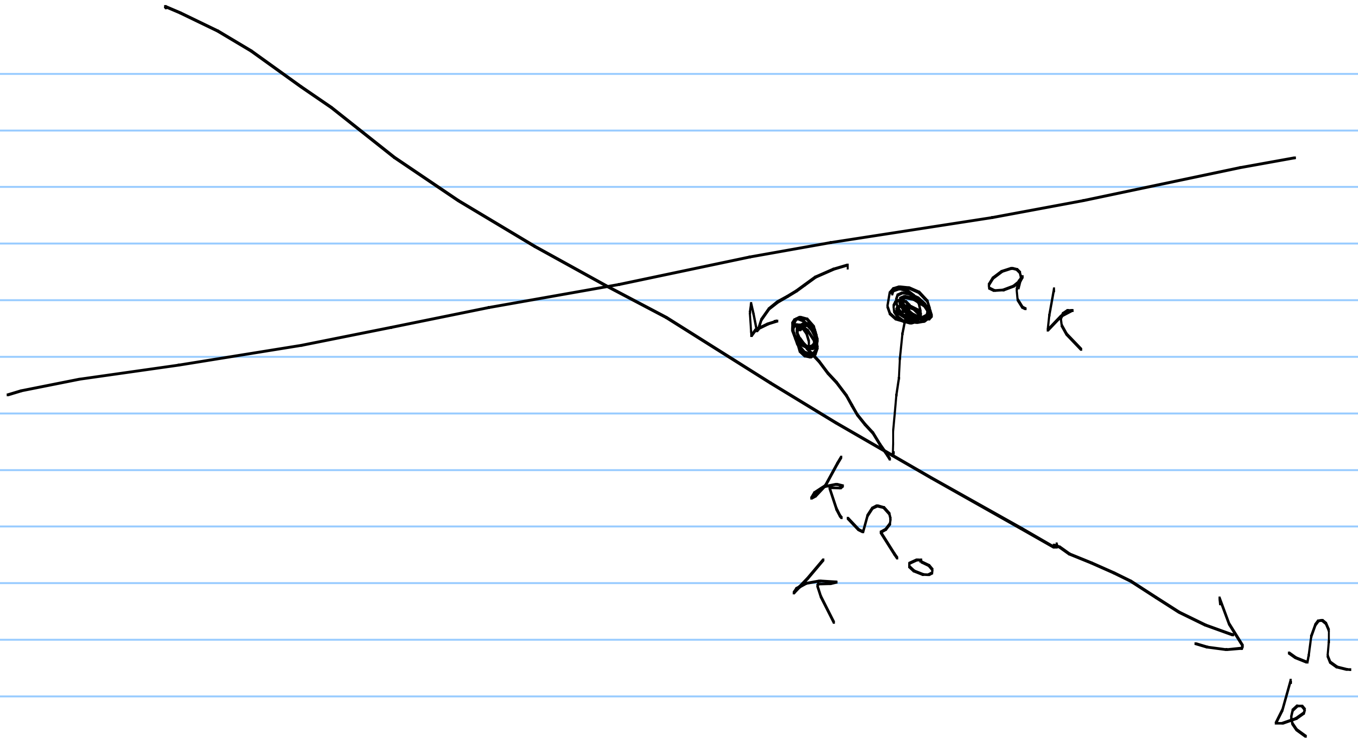
$$x_2(t+T) = x_2(t) \longleftrightarrow b_k$$

$$c_1 x_1(t) + c_2 x_2(t) \xleftrightarrow{FS} c_1 a_k + c_2 b_k$$

(2) Time Shift

$$y(t) = x(t - t_0) \longleftrightarrow e^{-jk\Omega_0 t_0} a_k = b_k$$

$$|b_k| = |a_k|$$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k

$$y(t) \longleftrightarrow b_k$$

(3) $= x(-t) \longleftrightarrow b_k = a_{-k}$

$x(t)$ is real valued $\Rightarrow a_{-k} = a_k^*$

$x(t)$ is even $\Rightarrow a_{-k} = a_k$

$a_k = a^*_k = a_{-k}$ real & even

(4) $y(t) = x(ct)$

$$x(t) = x(t + T_x)$$

$$y(t) = x(ct) = x(ct + T_x)$$

$$y\left(t - \frac{T_x}{c}\right) = x\left(c\left(t - \frac{T_x}{c}\right) + T_x\right)$$

$$= x(ct)$$

$$\Rightarrow \quad T_y = \frac{T_x}{c}$$

$$b_k = \frac{1}{T_y} \int_0^{T_y} y(t) e^{-jk\frac{2\pi}{T_y}t} dt$$

$$= \frac{1}{T_y} \int_0^{T_y} x(ct) e^{-jk\frac{2\pi}{T_y}t} dt$$

Letting $ct = \lambda$, we get

$$b_k = \frac{1}{T_y} \int_0^{T_y} y(t) e^{-jk\frac{2\pi}{T_y}t} dt = a_k$$

Time Scaling:

$y(t) = x(ct)$

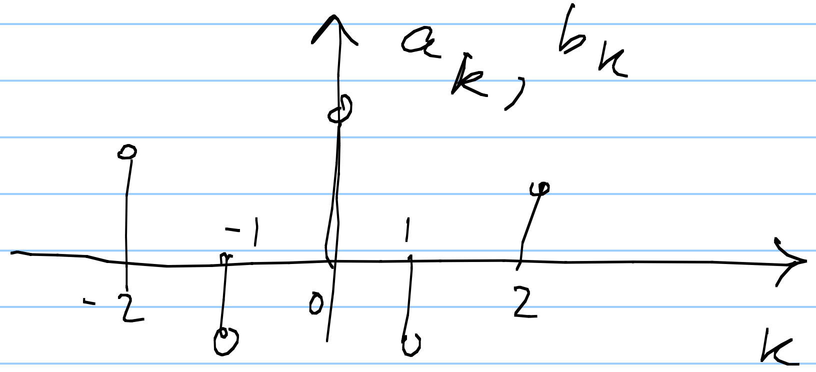
$y(t) \longleftrightarrow b_k$ period T_y

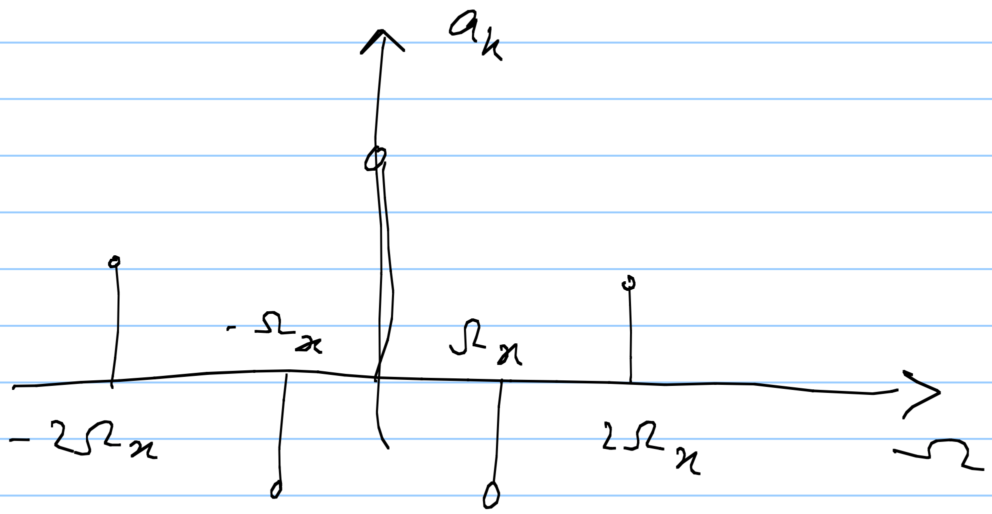
$x(t) \longleftrightarrow a_k$ period T_x

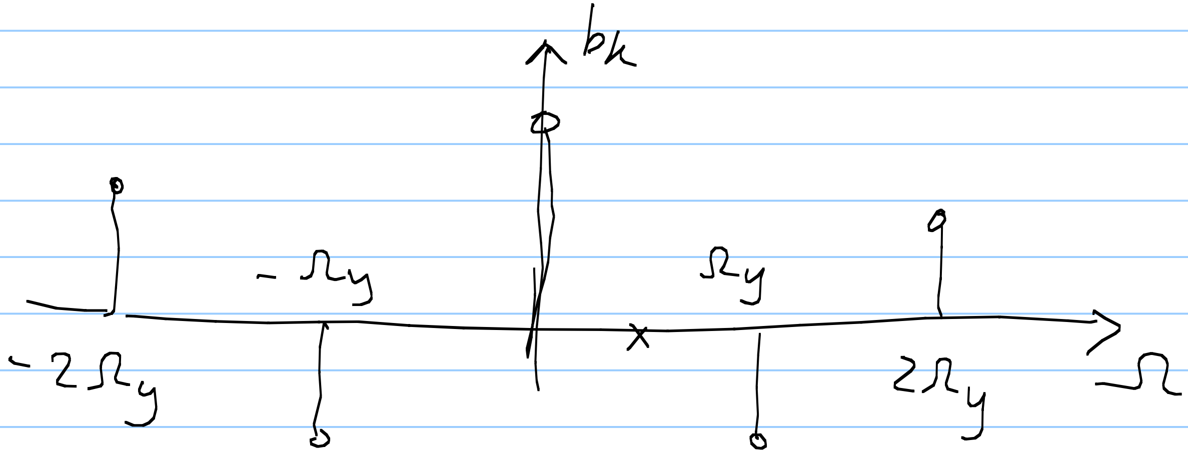
$$T_y = \frac{T_x}{c}$$

$$\Omega_y = c\,\Omega_x$$

$b_k = a_k$


a_k, b_k
-1
1
-2
0
2
k


a_k
$-\Omega_x$
Ω_x
$-2\Omega_x$
$2\Omega_x$
Ω


b_k
$-\Omega_y$
Ω_y
x
$-2\Omega_y$
$2\Omega_y$
Ω

Multiplication in the time domain

$$x(t)\,y(t) \longleftrightarrow ?$$

$$\frac{1}{T}\int_T \left[\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}\right]\left[\sum_{l=-\infty}^{\infty} b_l e^{jl\Omega_0 t}\right] e^{-jm\Omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T \left[\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} a_k b_l e^{j(k+l)\Omega_0 t}\right] e^{-jm\Omega_0 t}\, dt$$

Let $k + l = r$ & simplify

$$x(t)\, y(t) \longleftrightarrow a_k * b_k$$

Convolution in the time domain

$$z(t) = x(t) \circledast y(t) \longleftrightarrow a_k \cdot b_k$$

$$z(t) = \int_T x(\tau)\, y(t-\tau)\, d\tau$$

Differentiation :

$$x(t) \longleftrightarrow a_k$$

$$x'(t) \longleftrightarrow$$

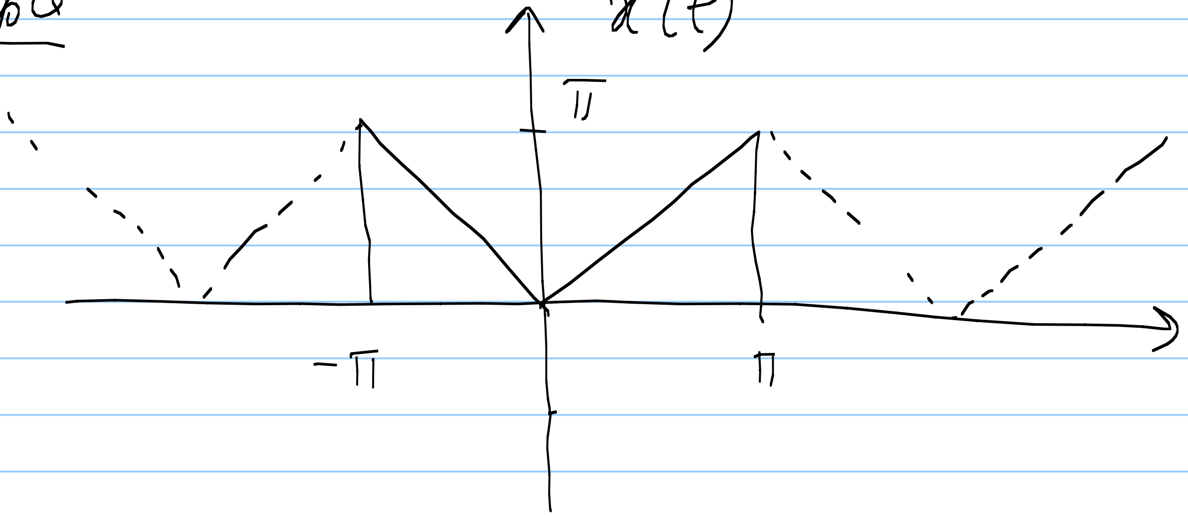
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

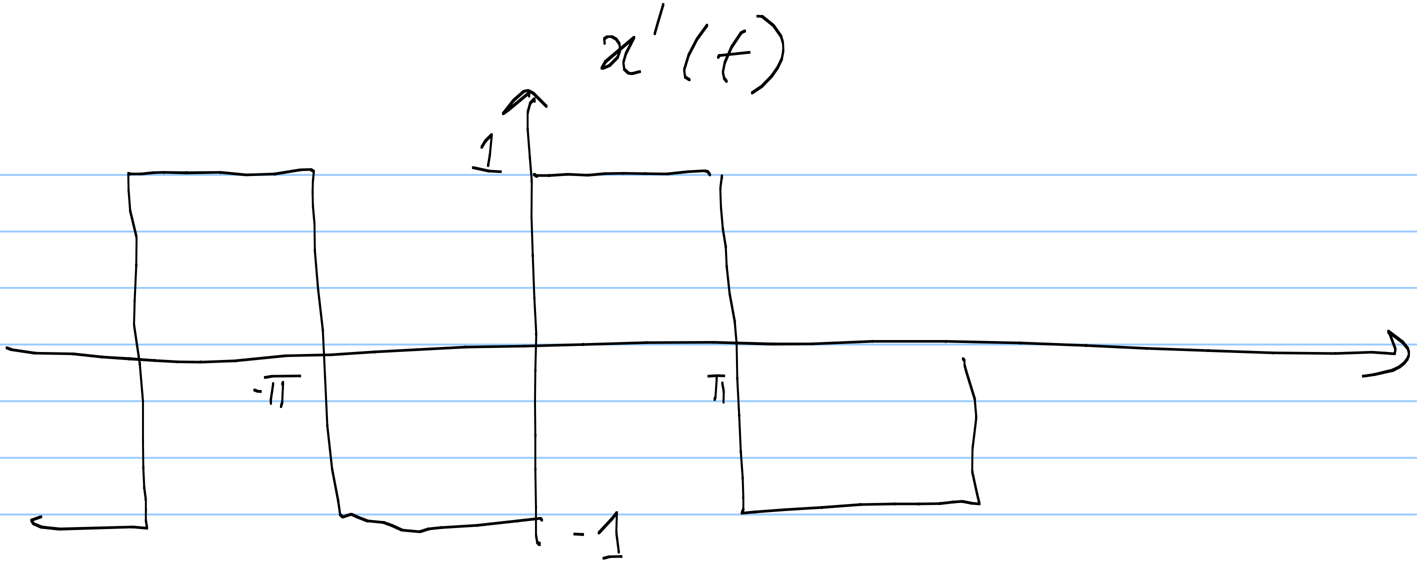
$$\frac{dx}{dt} = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}\right]$$

$$\overset{?}{=} \sum_{k=-\infty}^{\infty} jk\Omega_0 a_k \, e^{jk\Omega_0 t}$$

$$x'(t) \longleftrightarrow jk\Omega_0 a_k$$

Example





$$x'(t) \longleftrightarrow \begin{cases} \frac{2}{jk\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

$$x(t) \longleftrightarrow a_k$$

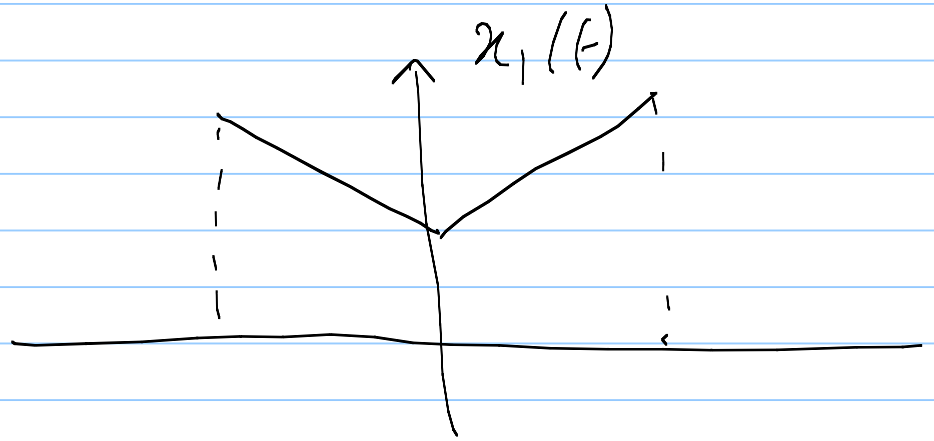
$$x'(t) \longleftrightarrow jk\Omega_0 \, a_k = b_k$$

$$b_k = \begin{cases} \dfrac{2}{jk\pi} & k = \text{odd} \\ 0 & k \text{ even} \end{cases}$$

$$a_k = \frac{2}{jk\pi \cdot jk\Omega_0} \qquad k \text{ odd}$$

$$= \frac{-2}{\pi \Omega_0 k^2} \quad k \neq 0 \quad \Omega_0 = 1$$

Now consider



$$x_1'(t) = x'(t)$$

Hence a_0 has to be computed separately

If the N^{th} derivative of $x(t)$ becomes impulsive, then the FS coefficients decay as $\frac{1}{K^N}$

Integration

$$\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{a_k}{jk\Omega_0} \quad k \neq 0$$

$$a_0 = 0$$

Parseval's Relation

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\frac{1}{T}\int_T x(t)\,x^*(t)\,dt$$

$$= \frac{1}{T}\int_T \left[\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}\right]\left[\sum_{l=-\infty}^{\infty} a_l^* e^{-jl\Omega_0 t}\right] dt$$

$$\overset{?}{=} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_k a_l^* \frac{1}{T} \int_T e^{j(k-l)\Omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2$$

EC 5142 Sep. 5 2011

For Fourier series, the basis functions were

$e^{jk\Omega_0 t}$ $k = 0, \pm 1, \pm 2, \ldots$

Generalized FS expansion :

$$x(t) = \sum_k{}' a_k \phi_k(t)$$

Note Title 05-09-2011

$$\langle x(t), \phi_\ell \rangle = \sum_k a_k \langle \phi_k(t), \phi_\ell(t) \rangle$$

If $\langle \phi_k(t), \phi_l(t) \rangle = \delta_{k\ell}$

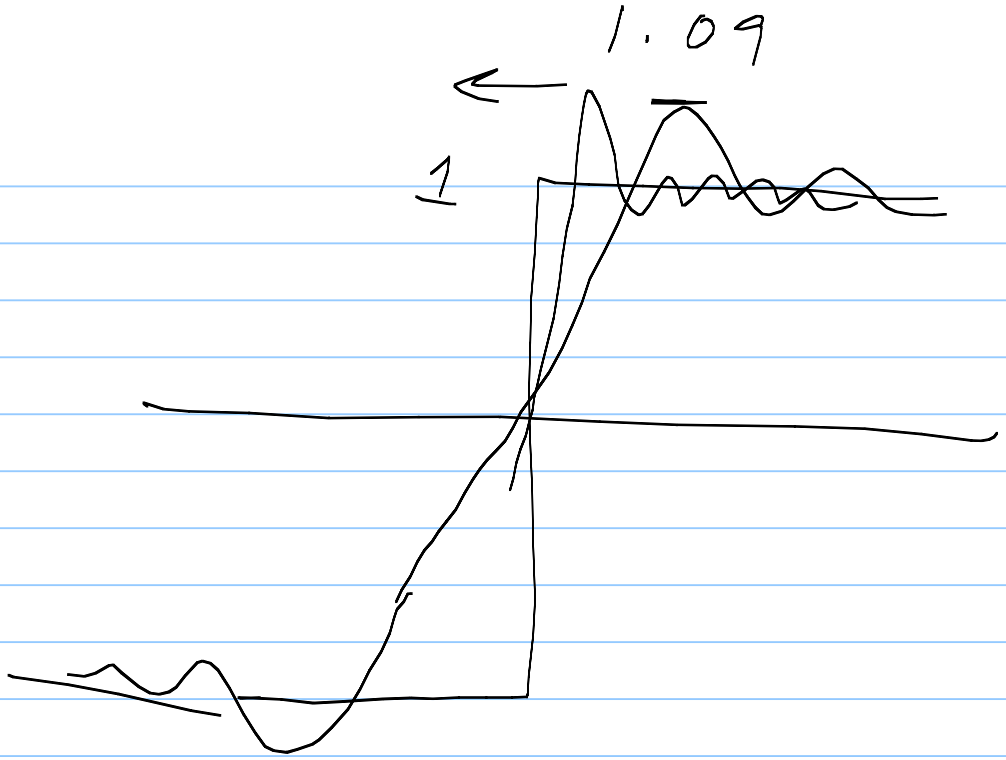
$$a_\ell = \langle x(t), \phi_\ell(t) \rangle$$

$$\langle p(t), q(t) \rangle = \int_{R} p(t)\, q^{*}(t)\, dt$$

For the Fourier basis $\langle e^{jk\Omega_0 t}, e^{jl\Omega_0 t} \rangle = T\delta_{kl}$

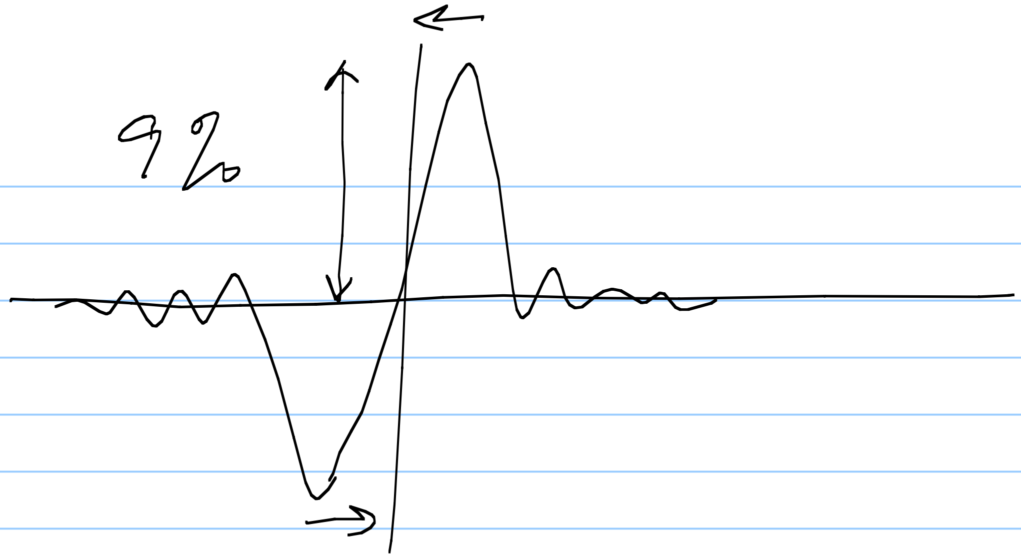
<u>Convergence</u> Does FS expansion converge?

How does it converge?

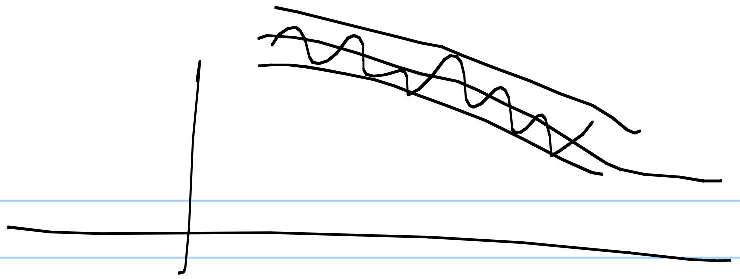


9% overshoot

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\Omega_0 t}$$



$x_N(t)$ as $N \rightarrow \infty$, will not converge to $x(t)$ pointwise



$$e(t) \xcancel{\longrightarrow} 0 \quad \forall t$$

$$\frac{1}{T} \int_T |e(t)|^2 \, dt \longrightarrow 0$$

Mean square convergence

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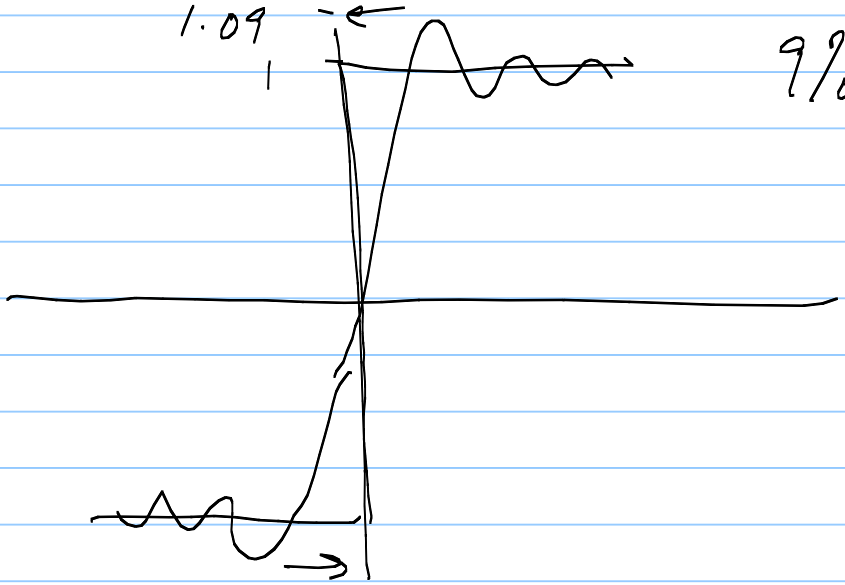


9% overshoot

Gibbs phenomenon

The Mathematics of Circuit Analysis – E.A. Guillemin

pp. 485 – 496

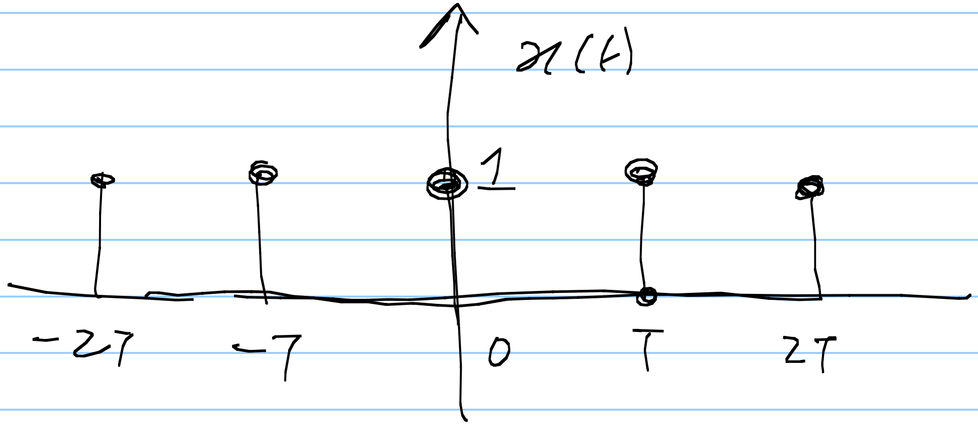
$$x_N(t) = \sum_{k=-N}^{N}{}' a_k e^{jk\Omega_0 t}$$

Note Title 06-09-2011

$$x(t) - x_N(t) = e_N(t) \;\cancel{\longrightarrow}\; 0 \text{ as } N \to \infty$$

$$\frac{1}{T}\int_T |e_N(t)|^2 dt \longrightarrow 0 \text{ as } N \to \infty$$

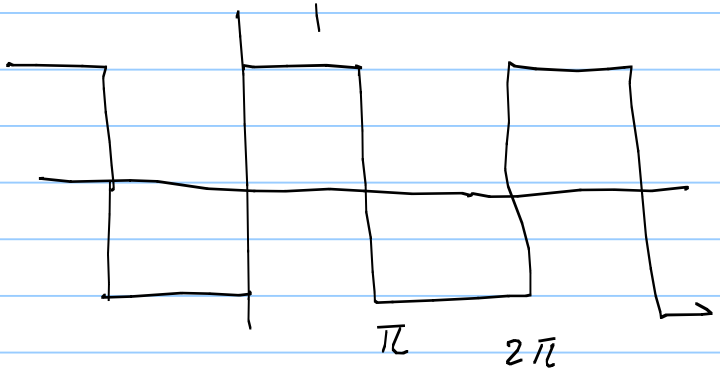
$\Rightarrow$ Convergence is in the mean-square sense



$a_k = 0$

Lookup Dirichlet's conditions

$$\longleftrightarrow \quad a_1 = \frac{1}{2j} \; ; \quad a_{-1} = a_1^*$$



$$\longleftrightarrow \quad \frac{4}{\pi}\left\{ \sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots \right\}$$

"On Bandwidth" – David Slepian

Proc. IEEE 1977?

$$x[n] = x[n+N]$$

$$e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N}$$

$$= e^{jk\frac{2\pi}{N}n}$$

$$\phi_k[n] = e^{jk\frac{2\pi}{N}n}$$

$$\boxed{x[n] = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}}$$

$$= \phi_{k+N}[n]$$

$$\langle x[n], \phi_l[n] \rangle = \sum_{k=0}^{N-1} a_k e^{j\frac{k2\pi}{N}n} \cdot e^{-jl\frac{2\pi}{N}n}$$

$$\sum_{n=0}^{N-1} x[n]\, \overset{\downarrow}{e}^{-j\frac{2\pi l k}{n}} = N\, a_l$$

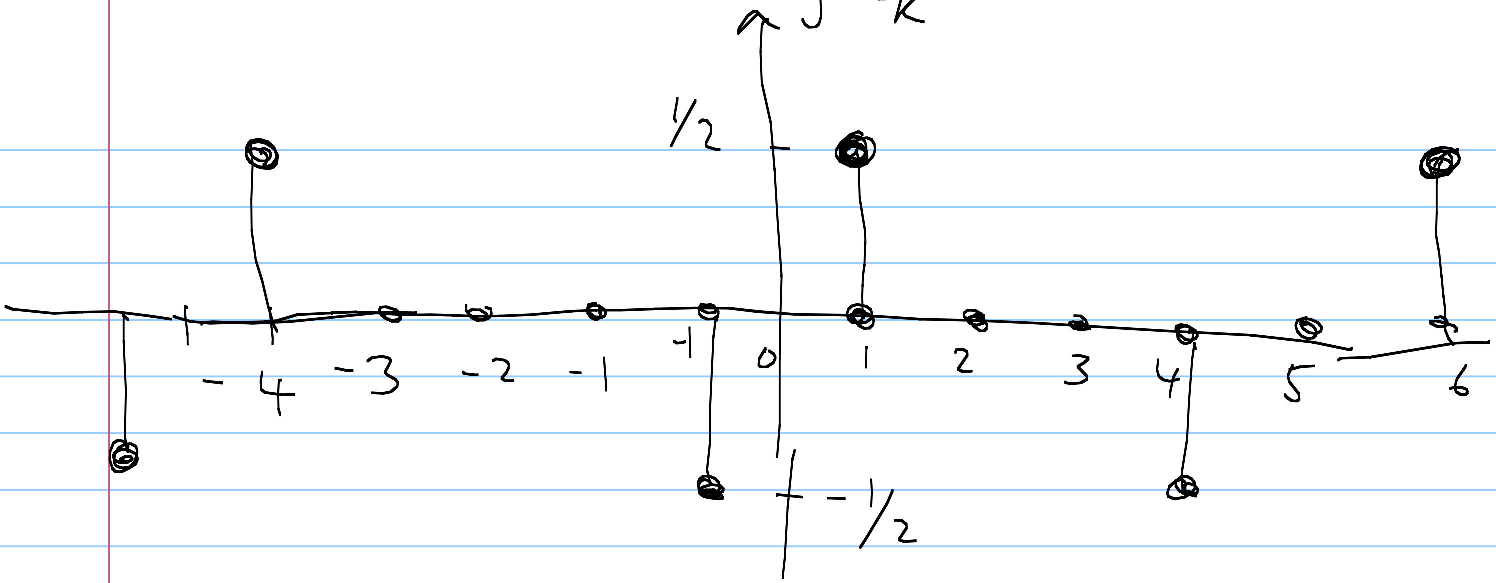
$$a_l = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j\, l \frac{2\pi}{N} n}$$

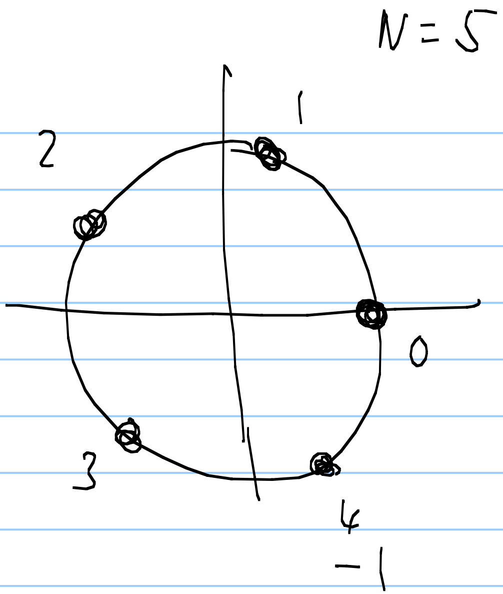
$$= a_{l+N}$$

Eg

$$\sin \frac{2\pi}{N} n \longleftrightarrow a_1 = \frac{1}{2j} = a_{-1}^{*} \Rightarrow a_{-1} = \frac{-1}{2j}$$

$N = 5$

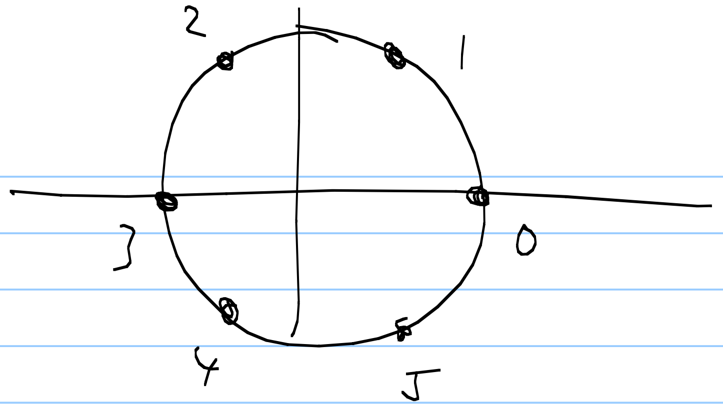




Eg 1 $\quad \sin 3 \frac{2\pi}{5} n = \sin \left(1 - 2/5\right) \frac{2\pi}{5} n$

$$= -\sin 2 \frac{2\pi}{5} n$$

$N = 6$



Properties

(1) Linearity

(2) $x[n - n_0] \xleftrightarrow{DTFS} a_k e^{-jk\frac{2\pi}{N}n_0}$

(3) $x[n]\, y[n] \longleftrightarrow a_k \circledast_N b_k$

(4) $x[n] \circledast_N y[n] \longleftrightarrow N \cdot a_k \cdot b_k$

(5) $e^{jl\frac{2\pi}{N}n} x[n] \longleftrightarrow a_{k-l}$

(6) $\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |a_k|^2$

Think about time scaling.

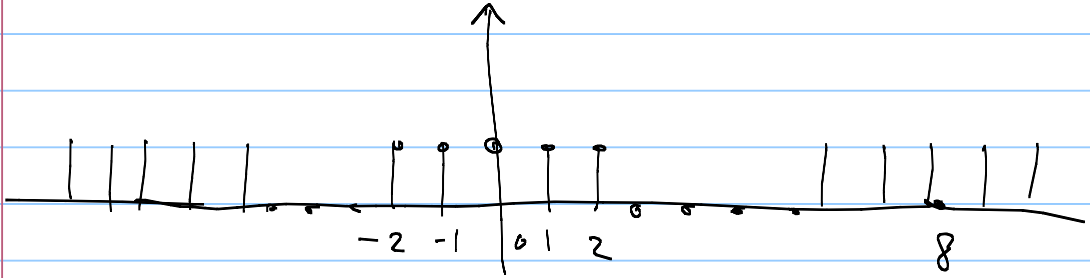
EC 5142 7^{th} Sep. 2011

$x[2n]$ & $x[n]$

Their FS coefficients are not related, in general.

$$y[n] = \begin{cases} x\left[\frac{n}{m}\right] & n = 0, \pm M, \pm 2M, \ldots \\ 0 & \text{otherwise} \end{cases}$$

What is the relationship between FS coeffs. of $x[n]$ & $y[n]$?



$$a_k = \frac{1}{8}\sum_{k=-2}^{2} 1 . e^{-j k \frac{2\pi}{8} n}$$

$$= \frac{1}{8} \frac{\sin \frac{5\pi k}{8}}{\sin \frac{\pi k}{8}}$$

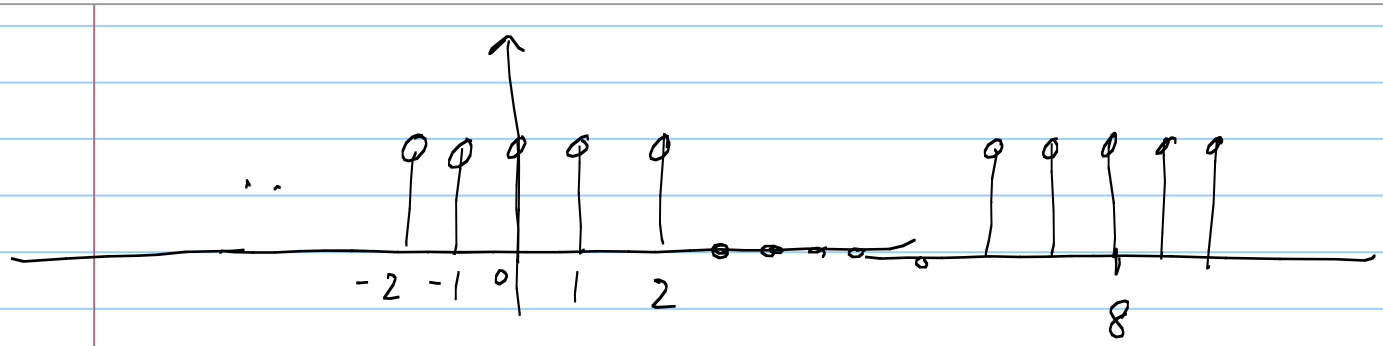
For $N = 32$

$$a_k = \frac{1}{32} \frac{\sin 5\pi k/32}{\sin \pi k/32}$$

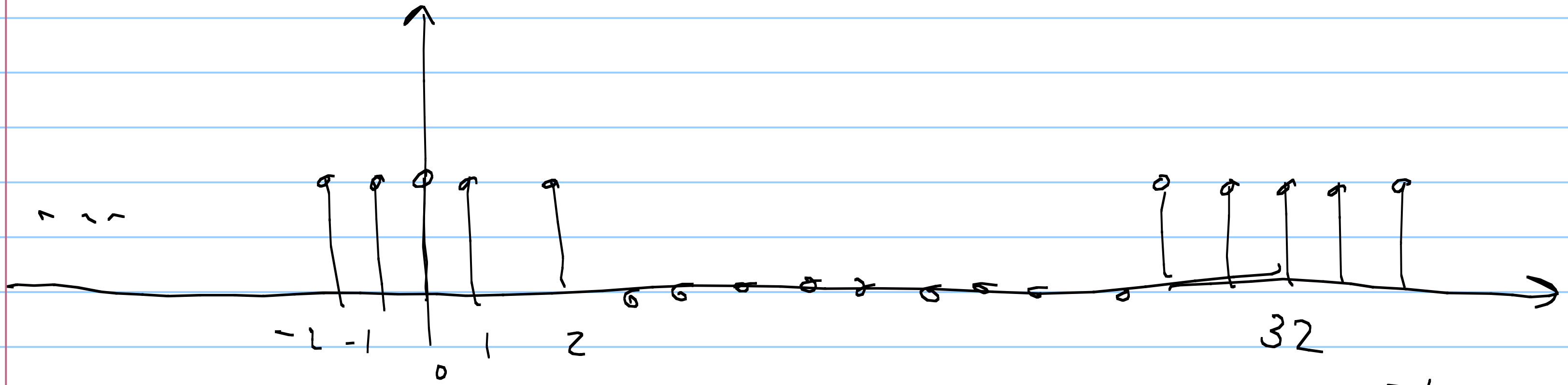
Plot a_k vs k

a_k vs. frequency

EC 5142 Sep. 8 2011

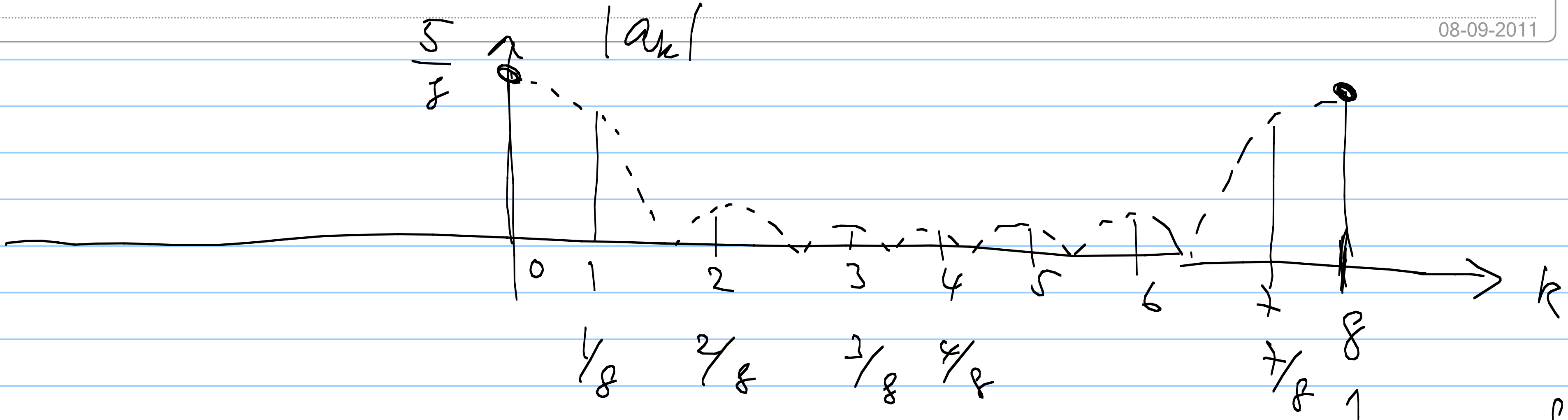


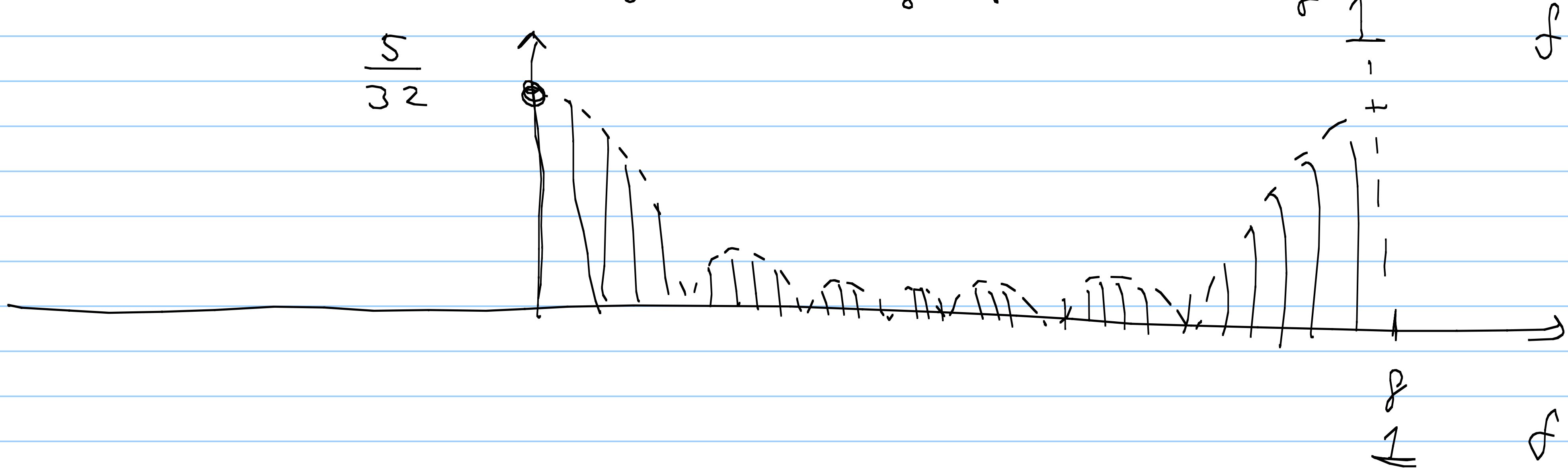
$$a_k = \frac{1}{8} \frac{\sin \frac{5\pi k}{8}}{\sin \frac{\pi k}{8}}$$



$$a_k = \frac{1}{32} \frac{\sin \frac{5\pi k}{32}}{\sin \frac{\pi k}{32}}$$

Note Title 08-09-2011





DFT

$x[n]$ given for $n = 0, 1, \ldots, N-1$

$$X[k] \overset{\text{def}}{=} \sum_{n=0}^{N-1} x[n]\, e^{-j 2\pi k n / N} \qquad \xrightarrow{\text{DFT}}$$

k^{th} DFT Coefficient

$$X[k+N] = X[k]$$

$X[k]$ need to be considered only for $k = 0, 1, \ldots, N-1$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi k n}{N}} \qquad \text{IDFT}$$

$$x[n+N] = x[n]$$

Recall:

$$x[n] = \sum_{n=0}^{N-1} a_k \, e^{j\frac{2\pi k n}{N}}$$

Where $x[n+N] = x[n]$

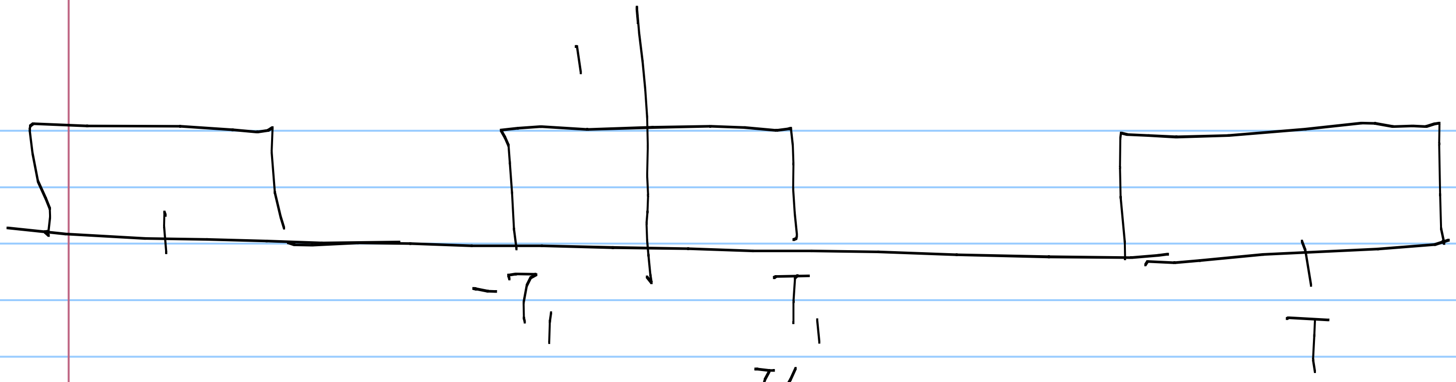
$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$$

$$N a_k = \sum_{n=0}^{N-1} x[n] e^{-j 2\pi k n/N} = X[k]$$

$$x(t+T) = x(t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\Omega_0 t} \qquad \Omega_0 = 2\pi/T$$

$$a_k = \frac{1}{T} \int_{T} x(t) e^{+jk\Omega_0 t} dt$$



$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} 1 . e^{-jk\frac{2\pi}{T}t} \, dt$$

$$= \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\frac{2\pi}{T}t} \, dt = \frac{1}{T} \frac{\sin \frac{2\pi k T_1}{T}}{k \ \pi/T}$$

DFT & DTFS (Cont'd)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \qquad k = 0, 1, \ldots, N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

We can also define with $\frac{1}{\sqrt{N}}$ scale factor for

Note Title 14-09-2011

both DFT & IDFT

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \, e^{-j k \frac{2\pi}{N} n}$$

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] \, e^{+j k \frac{2\pi}{N} n}$$

DTFS $$x[n+N] = x[n]$$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N} n}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}}$$

$$X[k] = N a_k$$

$$y[n] = X[n]$$

Find FS coeffs of $y[n]$ & relate them to $x[n]$

$\Rightarrow$ Duality property

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} \qquad k = 0, 1, \ldots, N-1$$

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & e^{-j1\cdot\frac{2\pi}{N}2} & \cdots & e^{-j1\cdot\frac{2\pi}{N}(N-1)} \\ & & \uparrow & & \\ \longleftarrow & & e^{-jk\frac{2\pi}{N}n} & \xrightarrow{n\,:\,0\to N-1} & \\ & & \downarrow k=0\to(N-1) & & \\ 1 & e^{-j(N-1)\frac{2\pi}{N}1} & \cdots & & e^{-j(N-1)\frac{2\pi}{N}(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} X[0] \\ X[1] \\ \\ \vdots \\ X[N-1] \end{bmatrix}$$

$N X N$

$$\underline{W}_N \, \underline{x} = \underline{X} \qquad W_N = e^{-j\frac{2\pi}{N}}$$

$$N \times N \quad N \times 1 \quad N \times 1$$

$$e^{-jk\frac{2\pi}{N}n} = W_N^{nk}$$

$$\underline{x} = \underline{W}_N^{-1} \, \underline{X} = \frac{1}{N} W_N^{H} \, \underline{X}$$

EC 5142 15th Sep. 2011

Note Title 15-09-2011

$$\underset{N\times 1}{\underline{X}} = \underset{N\times N}{\underline{W}}\ \underset{N\times 1}{\underline{x}}$$

$$\underline{W} \equiv \underline{W}_N$$

$$\underline{x} = \underline{W}^{-1}\,\underline{X}$$

$$\underline{W}^{-1} = \frac{1}{N}\,\underline{W}^{H}$$

$$\underline{W} = \left[\begin{array}{ccc} & \uparrow & \\ \longleftarrow & e^{-jk\frac{2\pi}{N}n} & \xrightarrow[n]{} \\ & \downarrow k & \end{array}\right]$$

$$\underline{x}^H \underline{x} = \sum_{n=0}^{N-1} |x[n]|^2$$

$$\underline{X}^H \underline{X} = (\underline{W}\,\underline{x})^H (\underline{W}\,\underline{x})$$

$$= \underline{x}^H \underline{W}^H \underline{W}\,\underline{x}$$

$$= \underline{x}^H N \underline{W}^{-1} \underline{W}\,\underline{x}$$

$$= N\,\underline{x}^H \underline{x}$$

$$\underline{x}^H \underline{x} = \frac{1}{N} \underline{X}^H \underline{X}$$

In DTFS, for real-valued $x[n]$

$$a_{-k} = a_k^*$$

$$a_{-k} = a_{N-k}$$

For DFT,
$$X[-k] = X^*[k]$$

$$X[-k] = X[N-k]$$

Notation to show periodicity explicitly:

$$x[n] \equiv x[\langle n \rangle_N]$$

$$\langle n \rangle_N \equiv n \bmod N$$

$$x_k \equiv x[k]$$

$$x[0] = x[N] = x[-N] = x[2N] = x[-2N] = \ldots$$

$$x[-1] = x[N-1]$$

$$X[0] = X[N]$$

$$X[-1] = X[N-1]$$

$$x[-n] = x[N-n]$$

$$x[n] \circledast h[n] = \sum_{k=0}^{N-1} x[k]\, h[n-k] = y[n]$$

$$y_0 = x_0 h_0 + x_1 \underset{\equiv h_{N-1}}{h_{-1}} + x_2 \underset{\equiv h_{N-2}}{h_{-2}} + \cdots + x_{N-1} \underset{\equiv h_1}{h_{-(N-1)}}$$

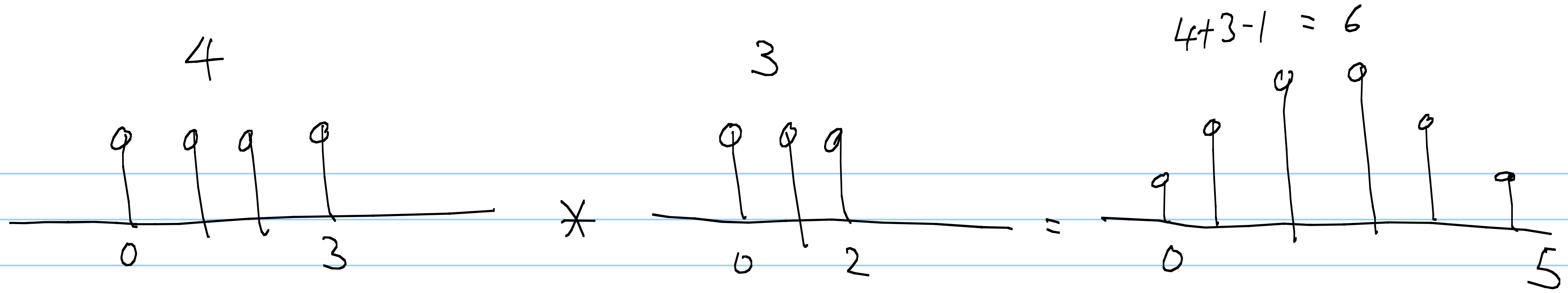
$$\begin{bmatrix} h_0 & h_{N-1} & h_{N-2} & \cdots & h_1 \\ h_1 & h_0 & h_{N-1} & \cdots & h_2 \\ & & & & \\ & & & & \\ h_{N-1} & h_{N-2} & & & h_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ , \\ \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \\ \\ \\ y_{N-1} \end{bmatrix}$$

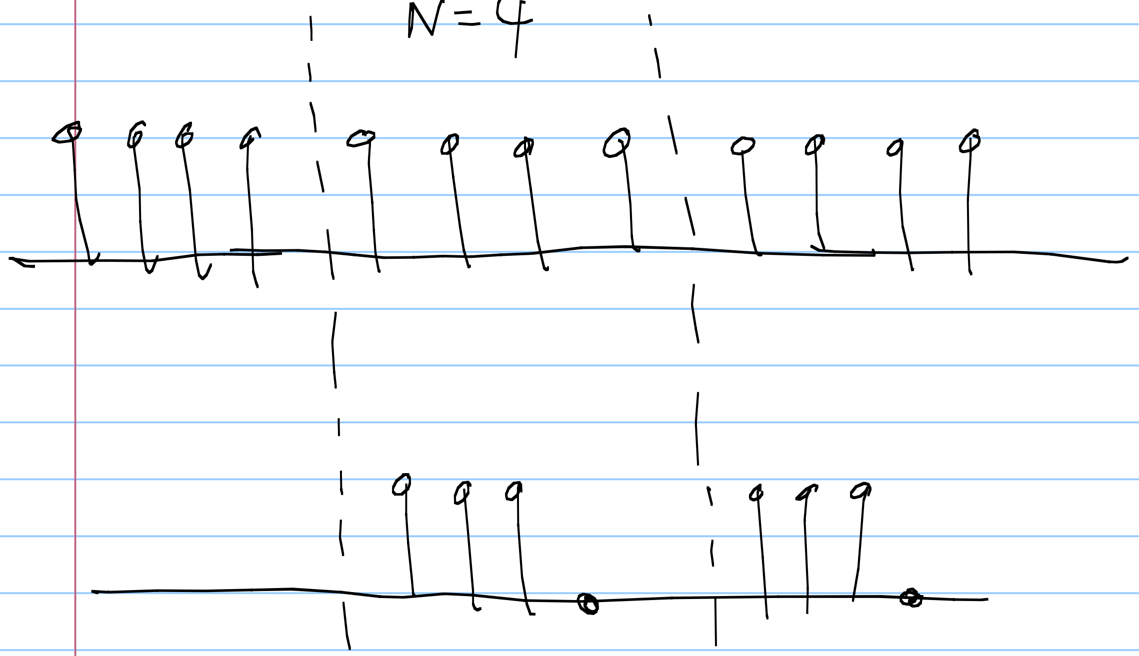
2^{nd} row is a circular shift of first row

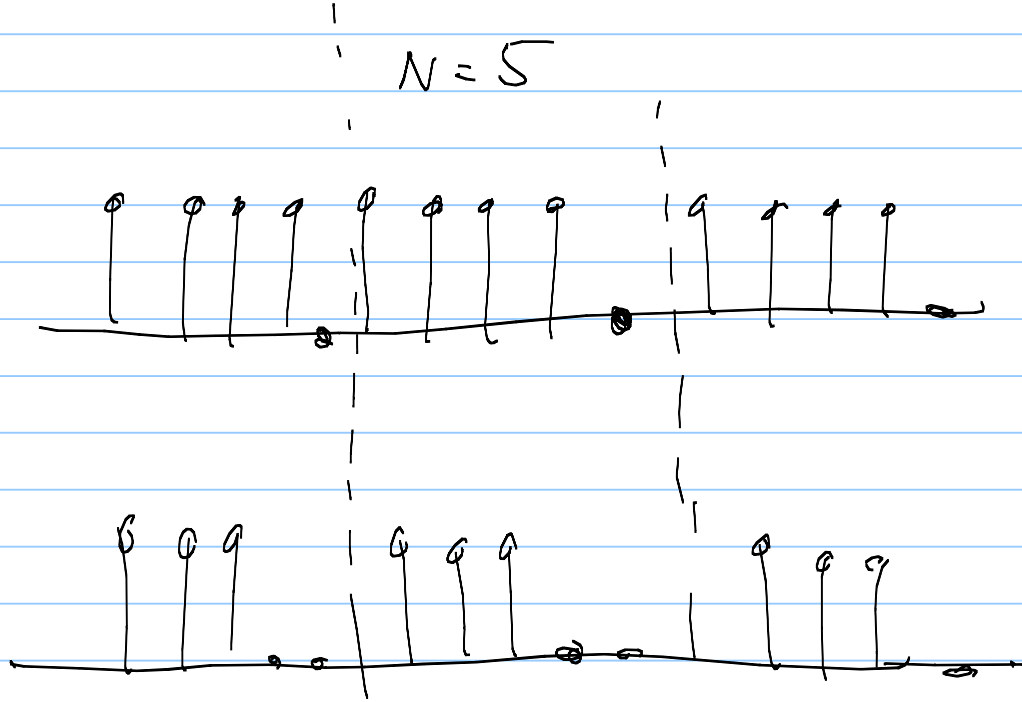
To get $x[n] \circledast h[n]$, the indirect approach:

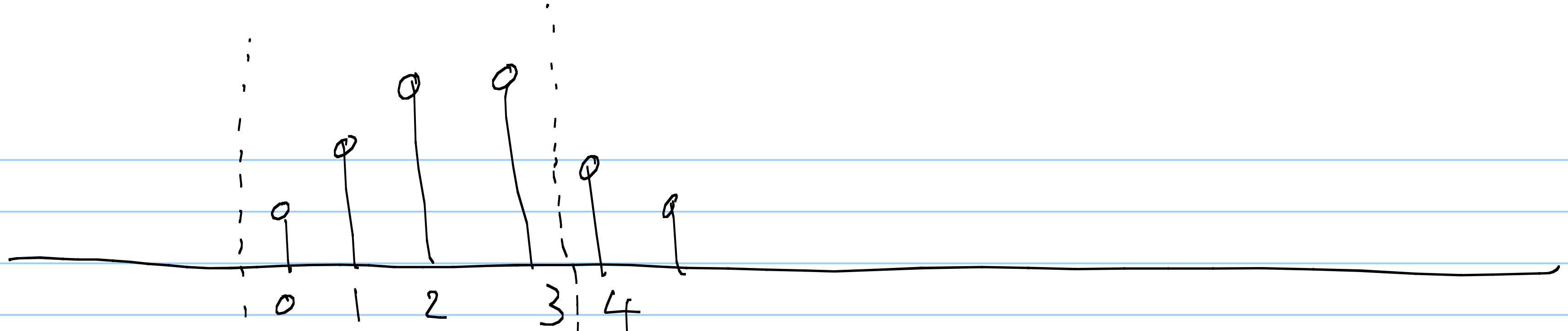
take the DFTs of $x[n]$ & $h[n]$, multiply in the transform domain, & then take inverse transform

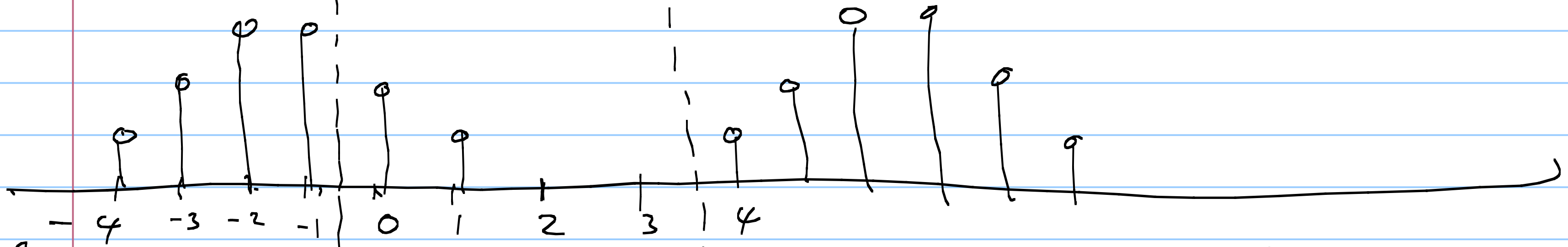
$$x[n] \circledast h[n] \longleftrightarrow X[k]\,H[k]$$


4
0
3
×
3
0
2
=
4+3-1 = 6
0
5


N=4


N=5

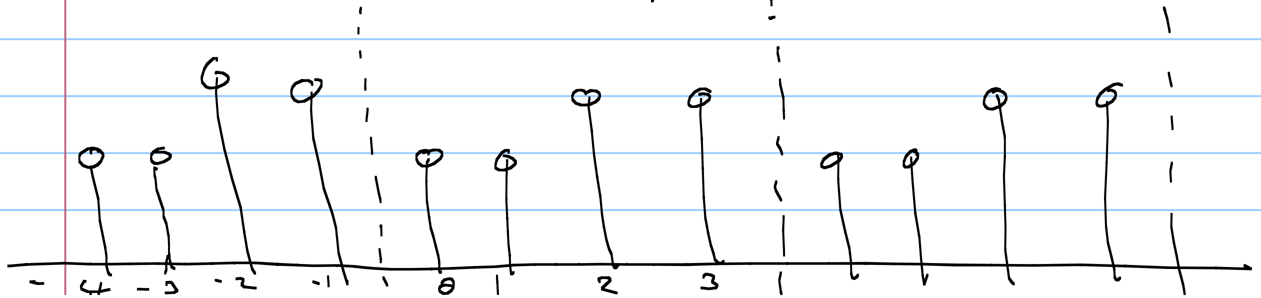




first repetition to the left.

Tail interferes with the first 2 samples

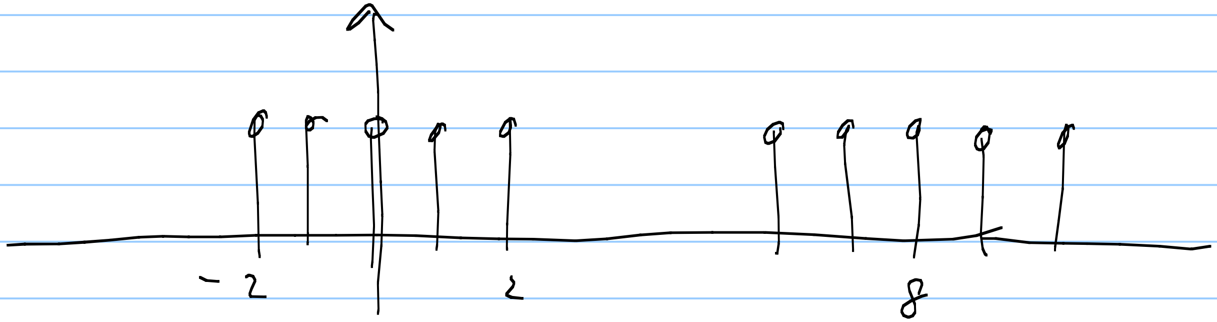
first repetition to the right – does not interfere with values in [0, 3]



result of circular convolution for N = 4

If the repetitions should not overlap then we must choose $N \geqslant P + Q - 1$. In such a situation, over $[0, N-1]$, the results of linear & circular convolutions are identical. If $N = P+Q-1$, the repetitions abut each other. If $N > P+Q-1$, then zeros occur between the repetitions.

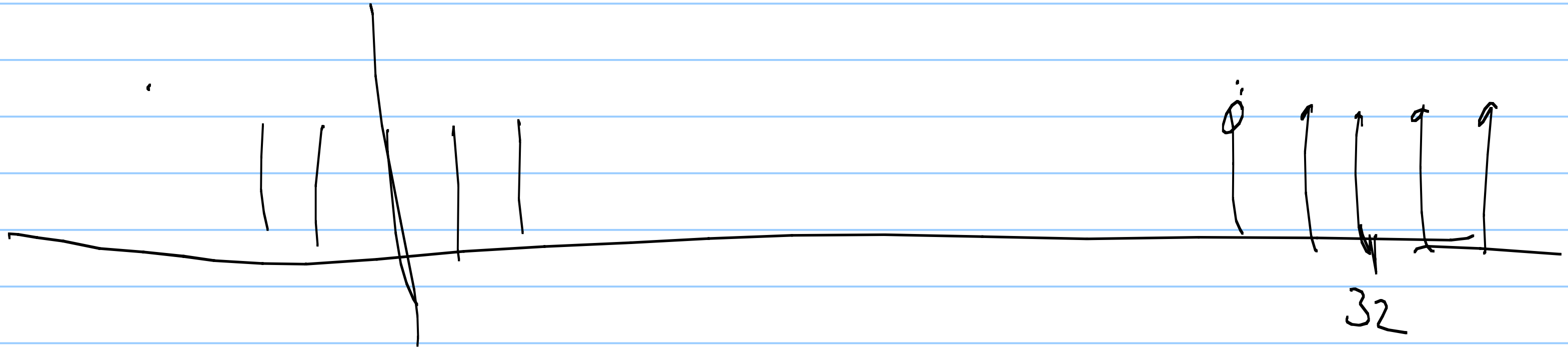
EC 5142 Sep 16 2011



$$a_k = \frac{1}{8}\,\frac{\sin\frac{5\pi k}{8}}{\sin\frac{\pi k}{8}}$$

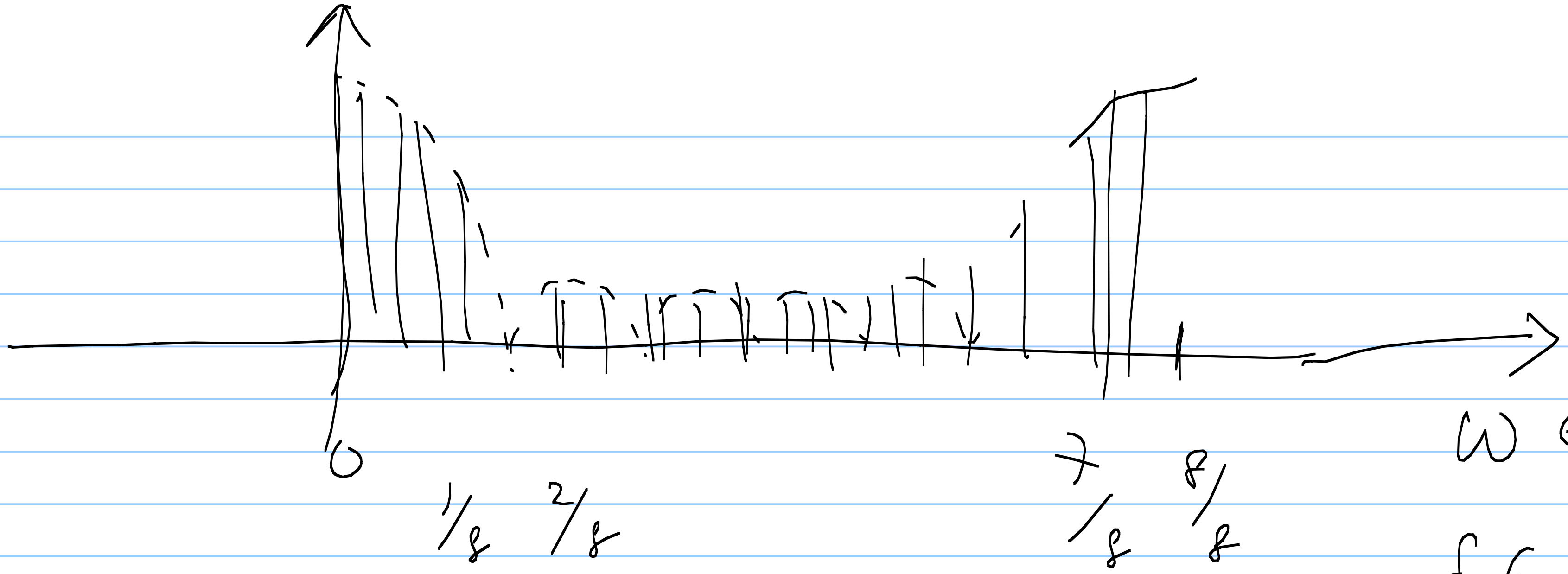
$k = 0, 1, \ldots, 7$

$\underline{N a_k}$



$k = 0, 1, \ldots, 31$

$$a_k = \frac{1}{32}\,\frac{\sin\frac{5\pi k}{32}}{\sin \pi k/32}$$



$\omega \in [0, 2\pi)$

$f \in [0, 1)$

$$e^{j 2\pi \underbrace{\frac{k}{N}}_{f} n}$$

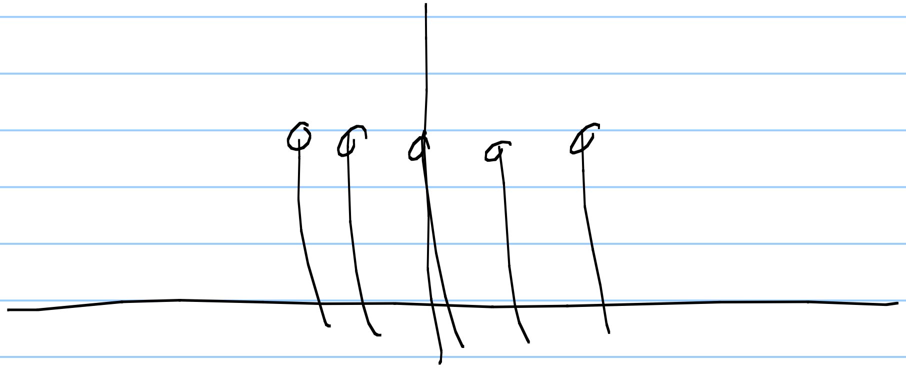
$$\Omega_0 = \frac{2\pi}{N}$$

$$k\Omega_0 = 2\pi \frac{k}{N} \rightarrow 0 \text{ to } N-1$$

$N a_k = X[k]$

8 point DFT

32 point DFT



$$y[n] = \begin{cases} x[n] & 0 \le n \le N-1 \\ 0 & N \le n \le L-1 \end{cases}$$

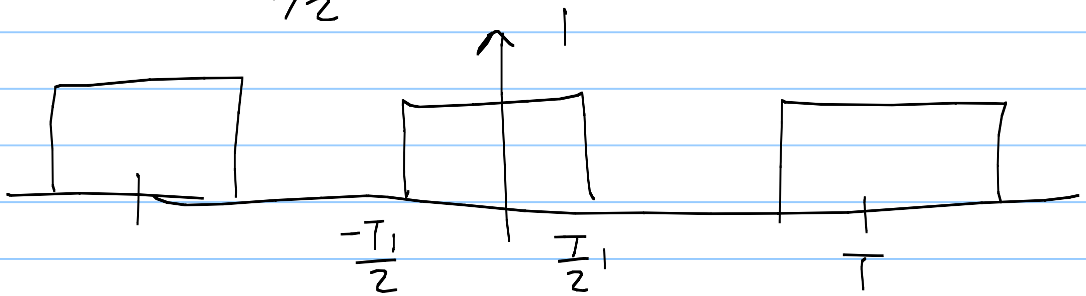
L-point DFT of $y[n]$

$$\sum_{n=0}^{L-1} y[n] e^{-j\frac{k2\pi}{L}n}$$

$$Y[l] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{k2\pi}{L}n} \qquad l = 0, 1, \ldots, L-1$$

EC 5142 Sep. 19 2011

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$



$$x(t) = \begin{cases} 1 & -T_1/2 \le t \le T_1/2 \\ 0 & T_1/2 < |t| < T/2 \end{cases}$$

$$a_k = \frac{1}{T} \quad \frac{\sin \pi k T_1/T}{\pi k/T}$$

$$\frac{1}{T} \frac{\sin \Omega T_1/2}{\Omega/2}\Bigg|_{\Omega = k\frac{2\pi}{T}}$$

$$a_k T = \frac{\sin \pi k T_1/T}{\frac{\pi k}{T} T_1} T_1$$

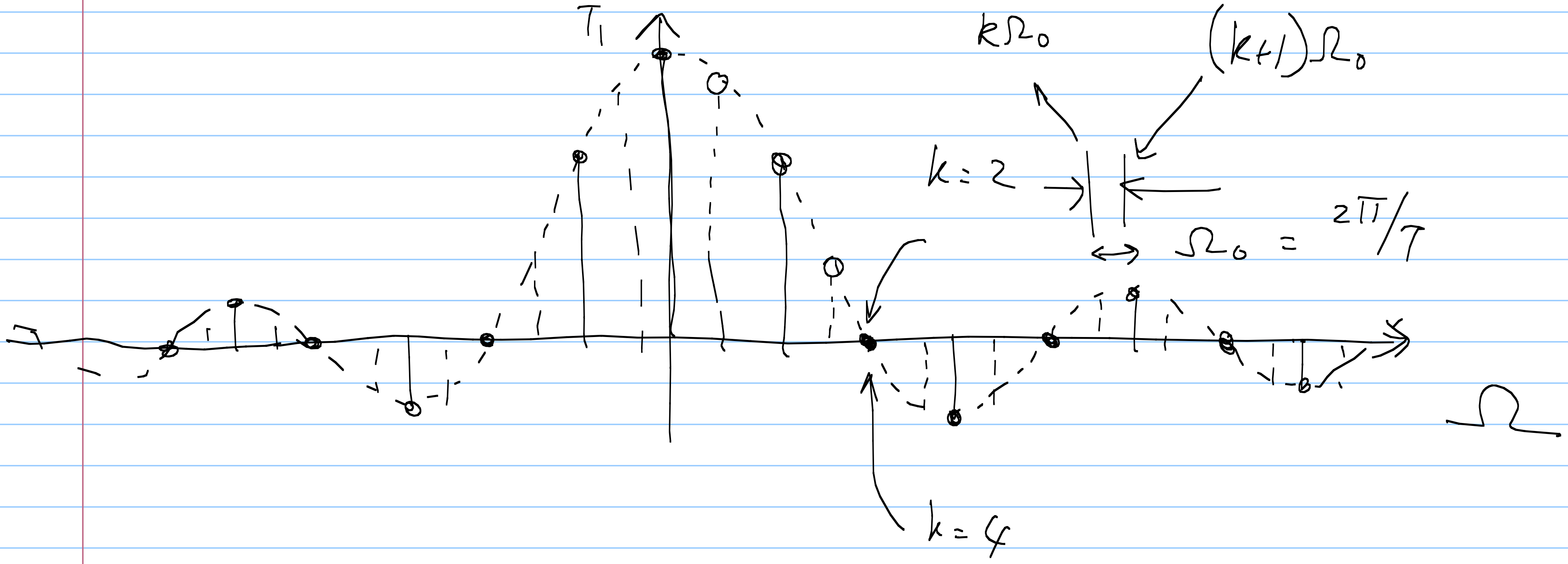
As $T \rightarrow \infty$, $a_k \longrightarrow 0$

Let us consider $\frac{T_1}{T} = \frac{1}{2}$

$$\frac{T_1}{T} = \frac{1}{4}$$

Case 1) $a_k T = \frac{\sin \pi k/2}{k\pi/2} \cdot T_1$

Case 2) $a_k T = \frac{\sin \pi k/4}{k\pi/4} \cdot T_1$



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j \underbrace{k \frac{2\pi}{T}}_{} t}\, dt$$

As $T \to \infty$, $a_k \to 0$, but $a_k T$ tends to a limit

$$a_k T \longrightarrow \int_{-\infty}^{\infty} \underbrace{x}_{}(t)\, e^{-j\Omega t}\, dt = X(\Omega)$$

$$X(\Omega) \overset{\text{def}}{=} \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt \qquad \text{CTFT}$$

$$e^{j\Omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow e^{j\Omega_0 t} \quad H(\Omega_0)$$

$$H(\Omega) = \int_{-\infty}^{\infty} h(t) e^{-j\Omega t} \, dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

$$\Omega_0 = \frac{2\pi}{T}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} (a_k T) e^{jk\Omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (a_k T) e^{jk\Omega_0 t} \Omega_0$$

$$\longrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega \quad \text{as } \Omega_0 \to 0$$

EC 5142 Sep. 20 2011

$$\boxed{X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt} \equiv X(j\Omega)$$

$$\mathrm{Lt}_{T\to\infty}\; a_k T = \mathrm{Lt}_{T\to\infty} \int_{-T/2}^{T/2} x(t)\, e^{-j\overbrace{k\Omega_0}^{\Omega_k} t}\, dt$$

$$x(t+T) = x(t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\Omega_0 t}$$ Synthesis eqn.

As $T \to \infty$ $\qquad \Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} a_k T \, e^{j\overbrace{k\Omega_0}^{\Omega_k} t}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} (a_k T) \, e^{j\Omega_k t} \, \Omega_0$$

$$\Omega_0 = (k+1)\Omega_0 - k\Omega_0 \longrightarrow \Delta\Omega$$

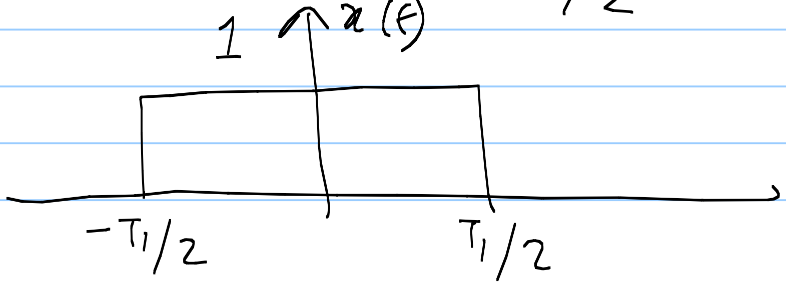
$$x(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} (a_k T) \, e^{j\Omega_k t} \, \Delta\Omega$$

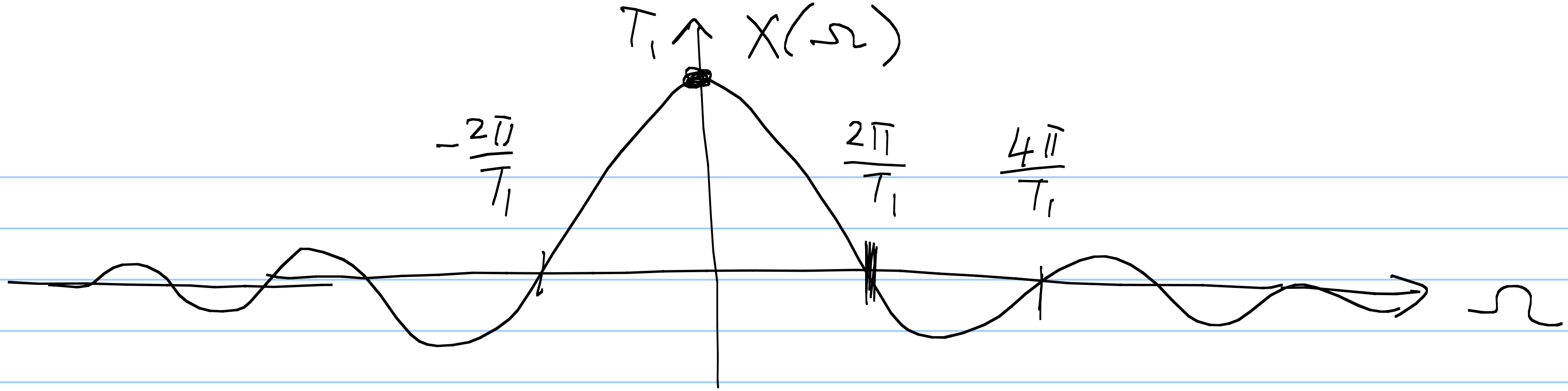
As $T \to \infty$,

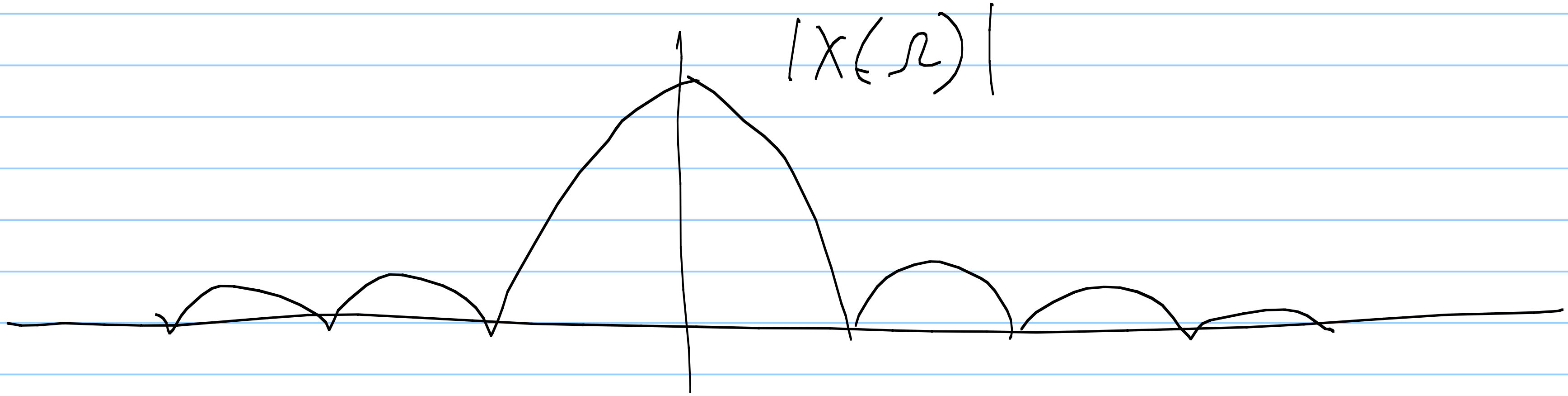
$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)\, e^{j\Omega t}\, d\Omega = x(t)}$$

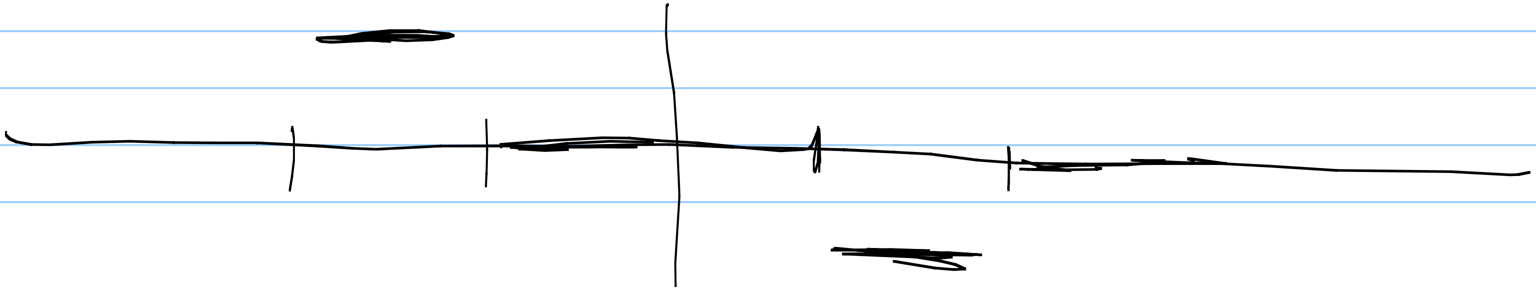
If $x(t) = \begin{cases} 1 & |t| < T_1/2 \\ 0 & \text{otherwise} \end{cases}$

$$X(\Omega) = \int_{-T_1/2}^{T_1/2} 1 .\, e^{-j\Omega t}\, dt = \frac{\sin \Omega T_1/2}{\Omega/2}$$




T_1
$X(\Omega)$
$-\frac{2\pi}{T_1}$
$\frac{2\pi}{T_1}$
$\frac{4\pi}{T_1}$
Ω


$|X(\Omega)|$



$x(t) = x^*(t)$ real-valued

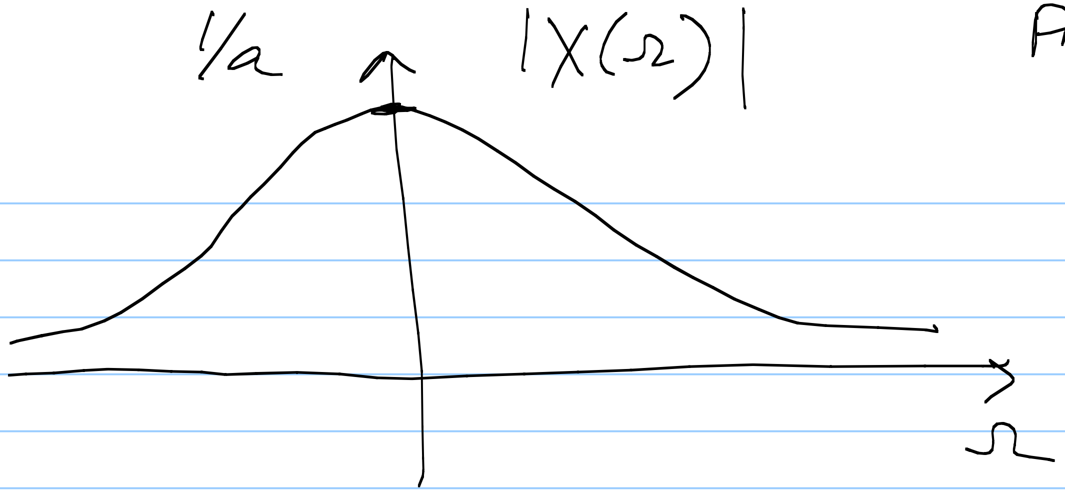
$X(\Omega) = X^*(-\Omega)$

$\Rightarrow$ $|X(\Omega)|$: even function of Ω mag. spectrum

$\angle X(\Omega)$: odd " " phase spectrum

$$\mathcal{F}\left\{e^{-at}u(t)\right\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\Omega t}\,dt$$

$$= \frac{1}{a+j\Omega} \qquad \mathrm{Re}\{a\} > 0$$



For real a,

$$|X(\Omega)| = \frac{1}{\sqrt{a^2+\Omega^2}}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a+j\Omega} e^{j\Omega t}\, d\Omega$$

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)\, d\Omega$$

$$x(t)\Big|_{t=0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{a}{a^2+\Omega^2} - j\,\frac{\Omega}{a^2+\Omega^2} \right] d\Omega$$

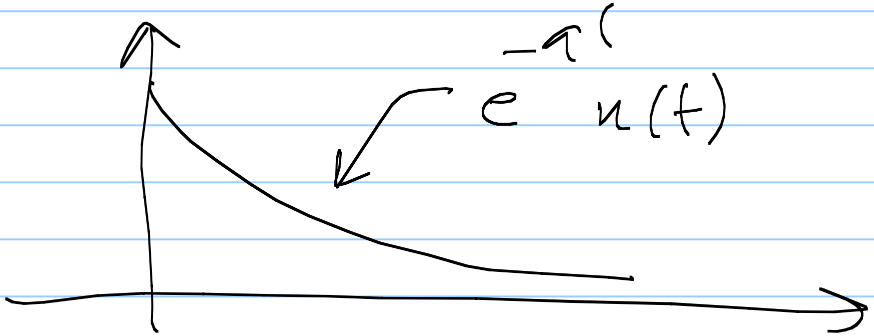
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{a^2}{a^2+\Omega^2}\, d\Omega - \underbrace{\frac{j}{2\pi} \int_{-\infty}^{\infty} \frac{\Omega}{a^2+\Omega^2}\, d\Omega}_{=\,0}$$

$$\underset{T\to\infty}{Lt} \int_{-T}^{T} \frac{\Omega}{a^2+\Omega^2}\, d\Omega = 0$$

Cauchy Principal Value

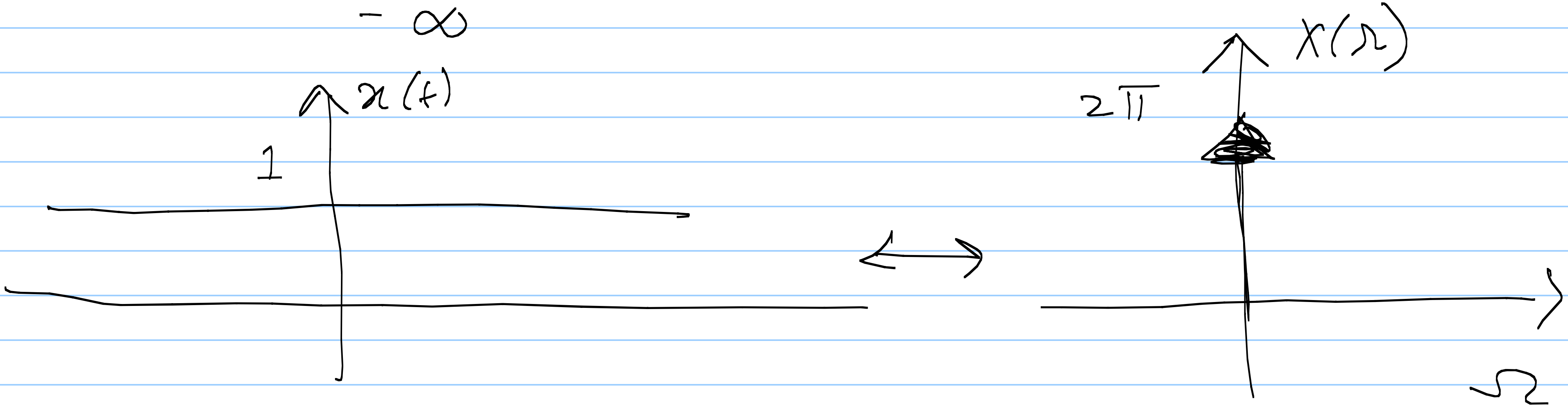
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+\Omega^2}\, d\Omega = \frac{1}{2\pi} \tan\left(\Omega/a\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi}\,\pi = \frac{1}{2}$$



$$X(\Omega) = 2\pi\,\delta(\Omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\Omega)\,e^{j\Omega t}\,d\Omega = 1 \quad \forall t$$



$$X(\Omega) = \int_{-\infty}^{\infty} 1 \cdot e^{-j\Omega t} dt = 2\pi\,\delta(\Omega)$$

$$1 \longleftrightarrow 2\pi\,\delta(\Omega)$$

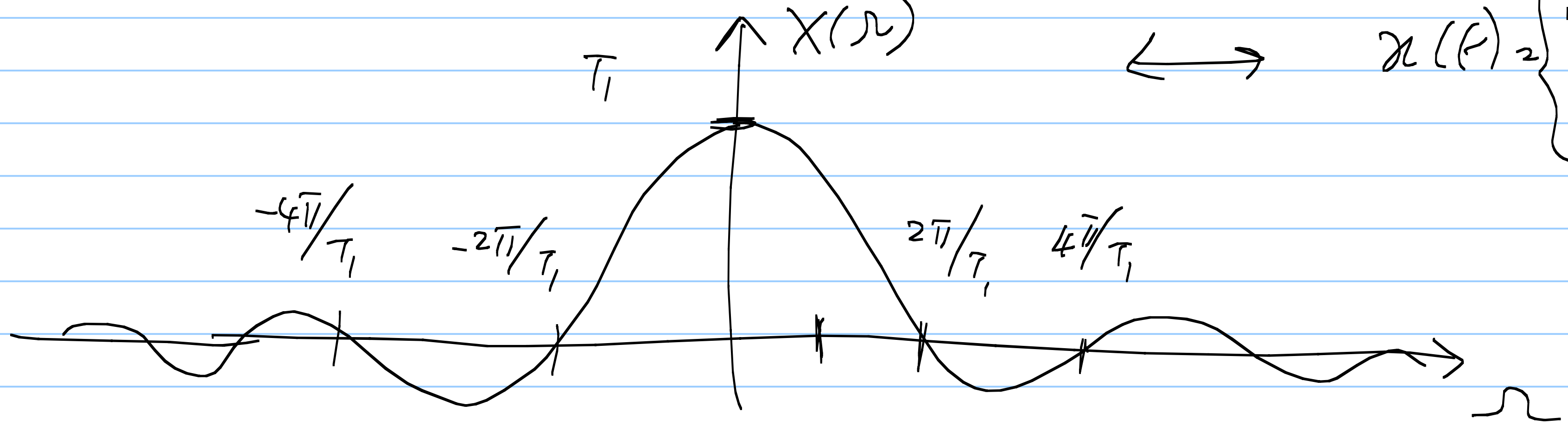
$$x(t-t_0) \longleftrightarrow e^{-j\Omega t_0} X(\Omega)$$

$$e^{j\Omega_0 t} x(t) \longleftrightarrow X(\Omega - \Omega_0)$$

$$\downarrow$$

$$e^{j\Omega_0 t} \cdot 1 \longleftrightarrow 2\pi\,\delta(\Omega - \Omega_0)$$

$$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(\Omega) + a_2 X_2(\Omega)$$



$$\longleftrightarrow \quad x(t) = \begin{cases} 1 & |t| < T_1/2 \\ 0 & \text{otherwise} \end{cases}$$

What is the DC value of $x(t)$?

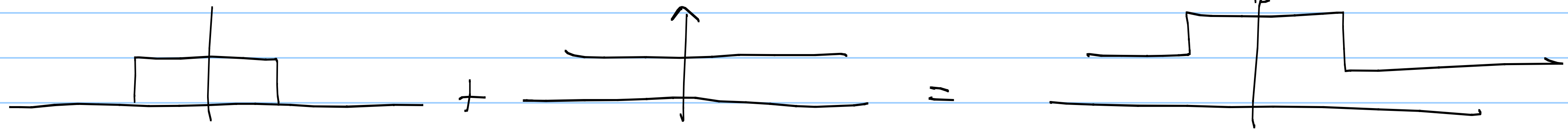
$$\overline{x(t)} = \operatorname{Lt}_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt$$

$$= \operatorname{Lt}_{T \to \infty} \frac{1}{T} \int_{-T_1/2}^{T_1/2} 1 \, dt$$

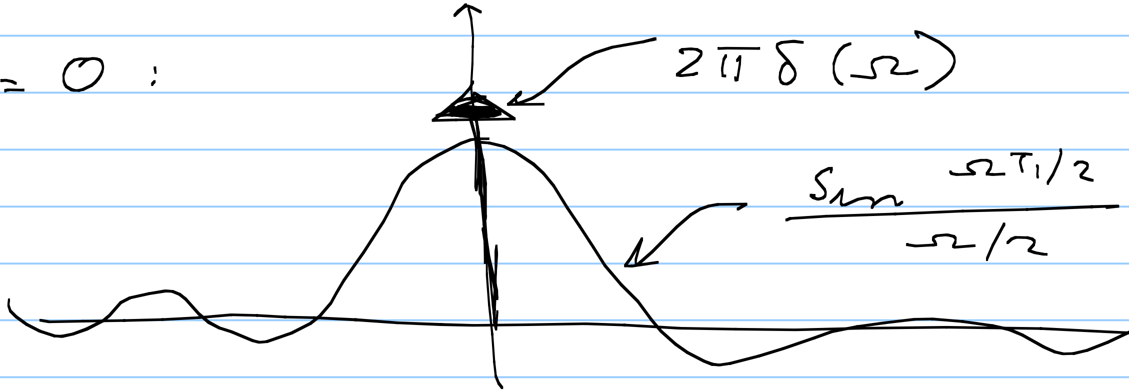
$$= \operatorname{Lt}_{T \to \infty} \frac{T_1}{T} = 0$$

DC value is actually zero. Hence if $X(0)$ is non zero, it doesn't mean a pure tone of that frequency is present.

If DC component is present, signal will be



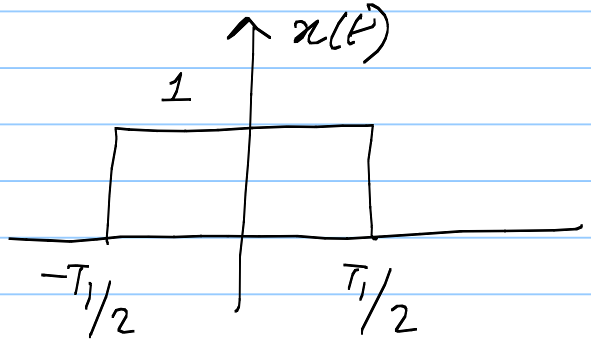
Spectrum will now contain an extra impulse at $\Omega = 0$:



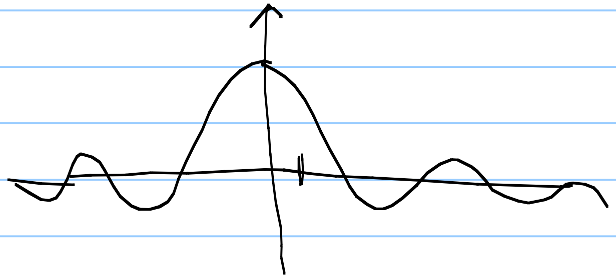
$$X(\Omega) = 2\pi\delta(\Omega) + \frac{\sin \Omega T_1/2}{\Omega/2}$$

Note Title 21-09-2011

EC 5142 21 Sep. 2011

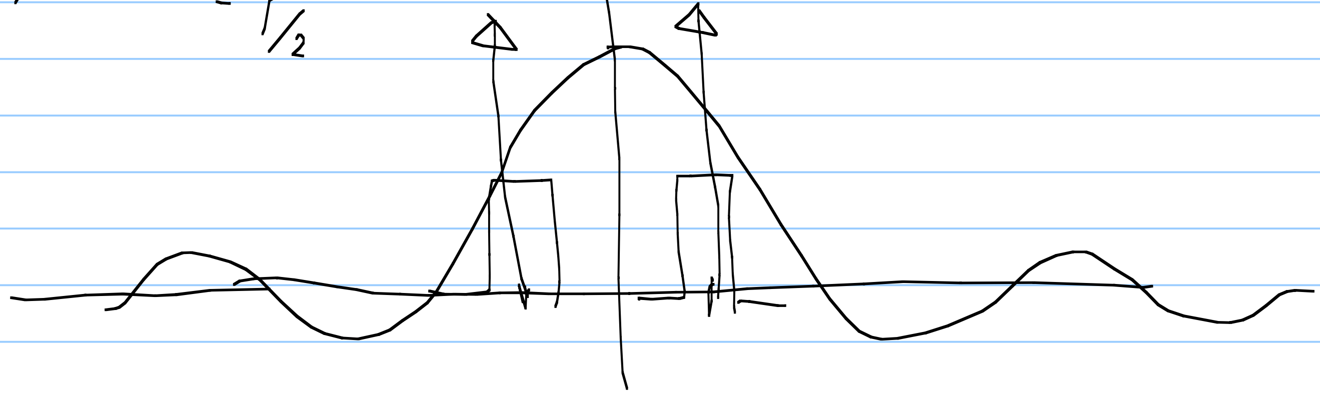


$\longleftrightarrow$ $$\frac{\sin \Omega \, T_1/2}{\Omega/2}$$



$X(0) = T_1$

$$\bar{x}(t) = \underset{T \to \infty}{Lt} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\,dt = 0$$

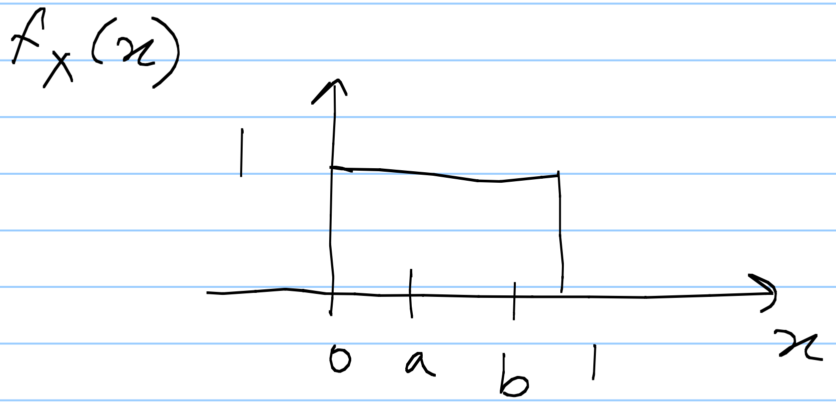


Note Title 21-09-2011

$$e^{j\Omega_0 t} \longleftrightarrow 2\pi\,\delta(\Omega-\Omega_0)$$

$$\cos\Omega_0 t \longleftrightarrow \pi\left[\delta(\Omega-\Omega_0) + \delta(\Omega+\Omega_0)\right]$$

$$\sin\Omega_0 t \longleftrightarrow \frac{\pi}{j}\left[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)\right]$$



$$0 \leq a < b \leq 1$$

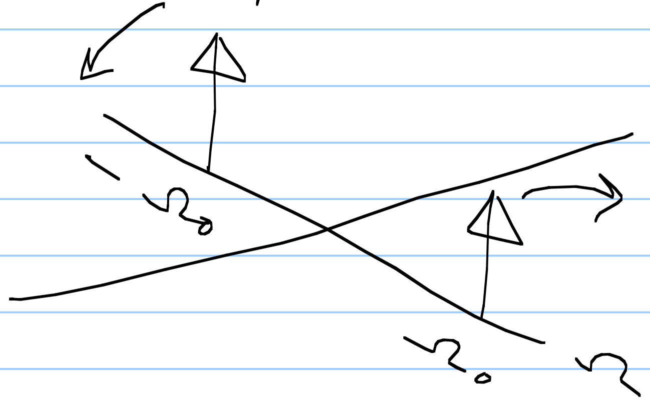
$$P\left[a \le X \le b \right] = b - a$$

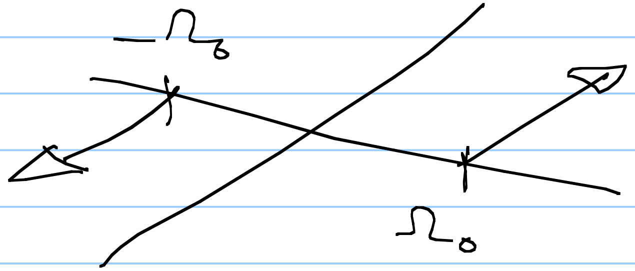
$$P\left[X = a \right] = 0$$

Properties

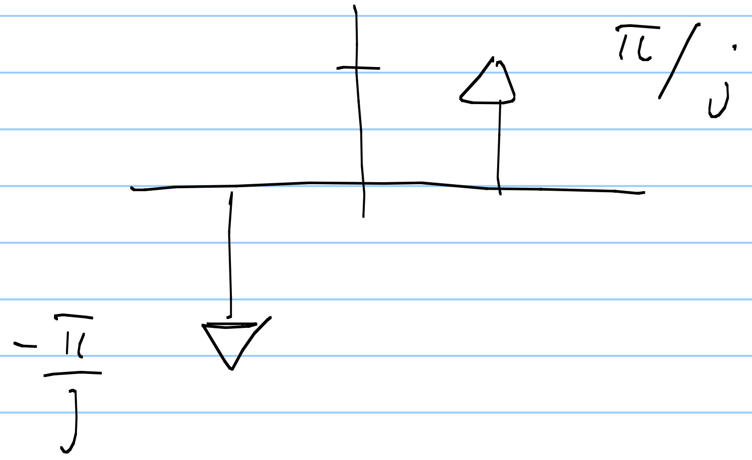
1) Linearity

2) Time Shift : $x(t - t_0) \longleftrightarrow e^{-j\Omega t_0} X(\Omega)$

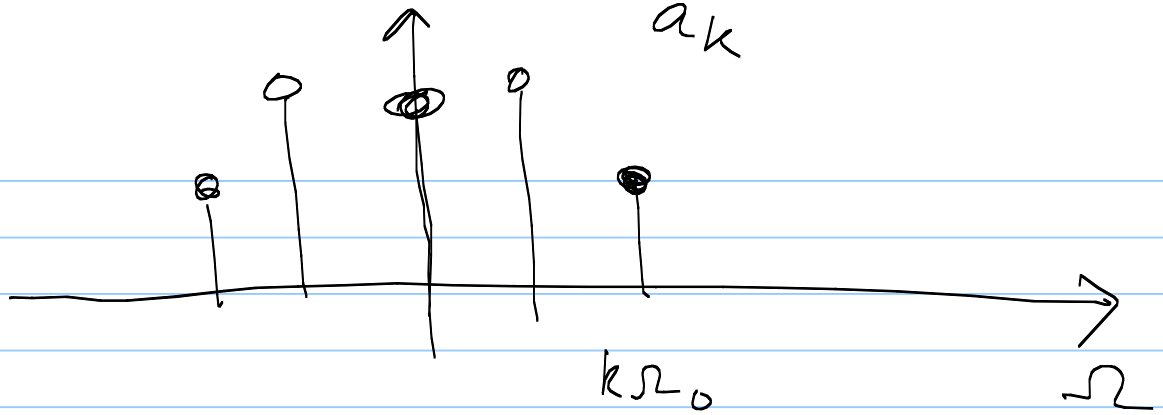


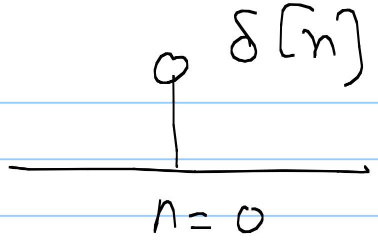


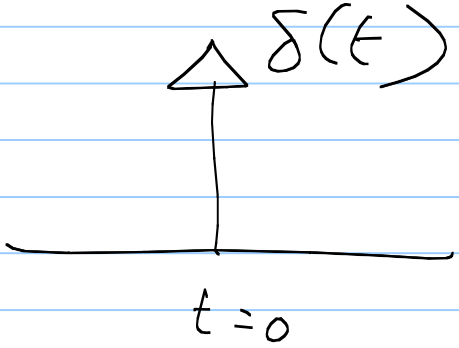
$$\cos \Omega_0 t \longleftrightarrow \pi\left[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)\right]$$

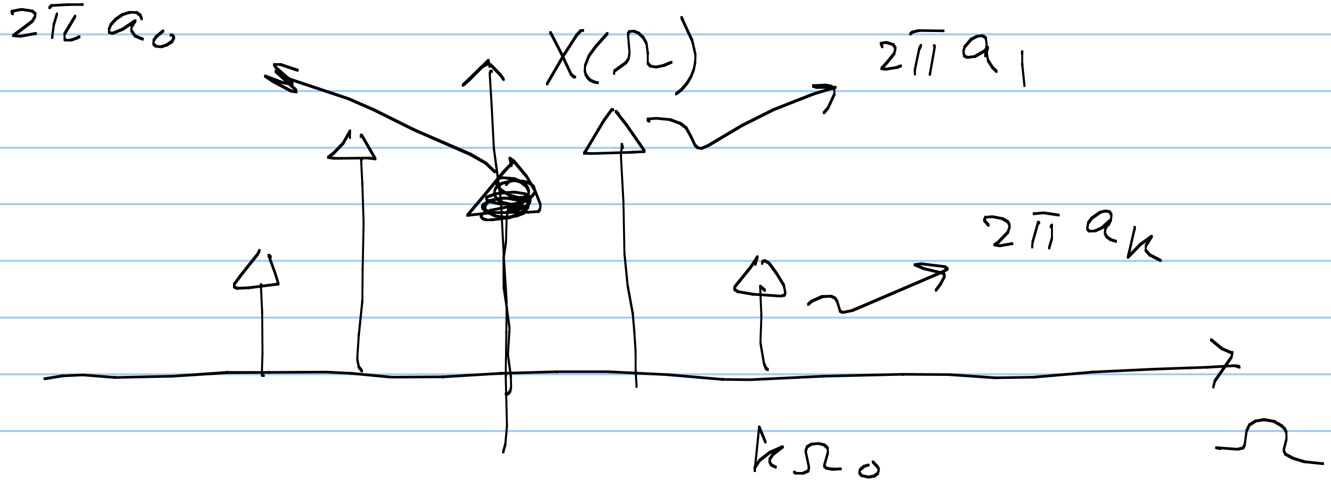


$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \longleftrightarrow \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0)$$







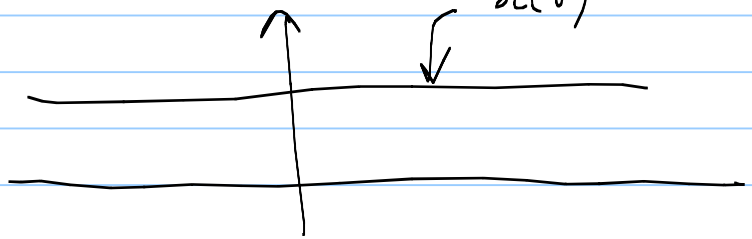


3) Modulation $e^{j\Omega_0 t} x(t) \longleftrightarrow X(\Omega - \Omega_0)$

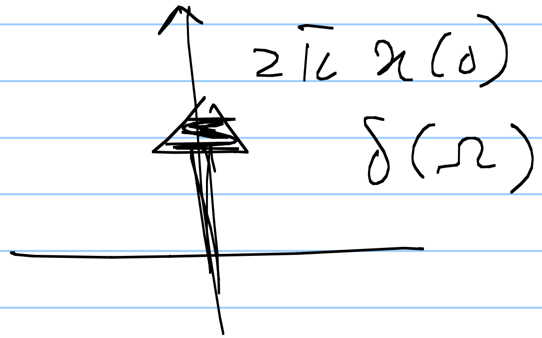
4) $y(t) = x(at) \longleftrightarrow \frac{1}{|a|} X(\Omega/a)$

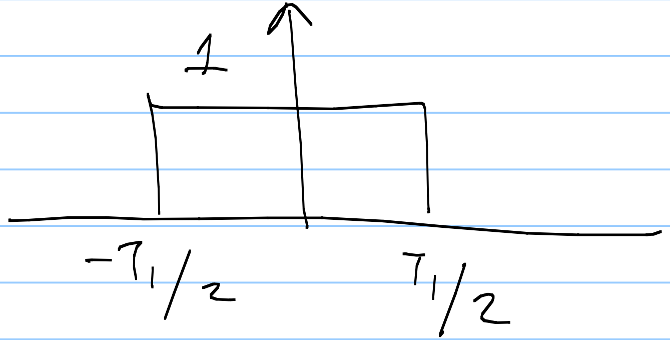
For $a = 0$

$y(t) = x(0) \quad \forall t$

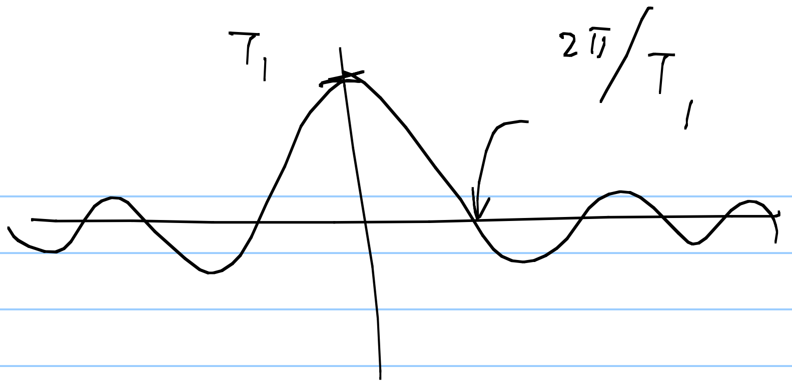


$\longleftrightarrow$





$\longleftrightarrow \dfrac{\sin \Omega T_1/2}{\Omega/2}$



As $T_1 \to \infty$, $x(t) \to 1 \ \forall t$

$$\frac{\sin \Omega T_1/2}{\Omega/2} \longrightarrow 2\pi\,\delta(\Omega)$$

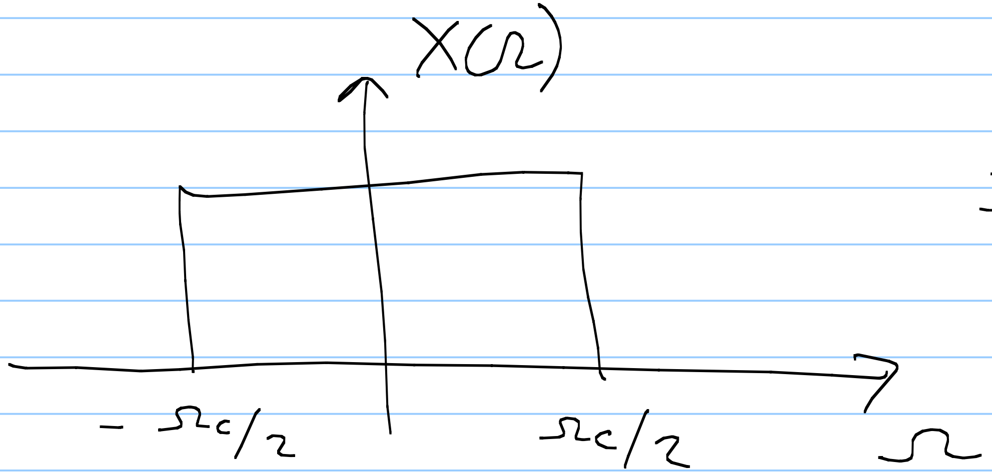
$$\frac{\sin \pi t}{\pi t} = \mathrm{Sinc}(t)$$

5) Duality

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

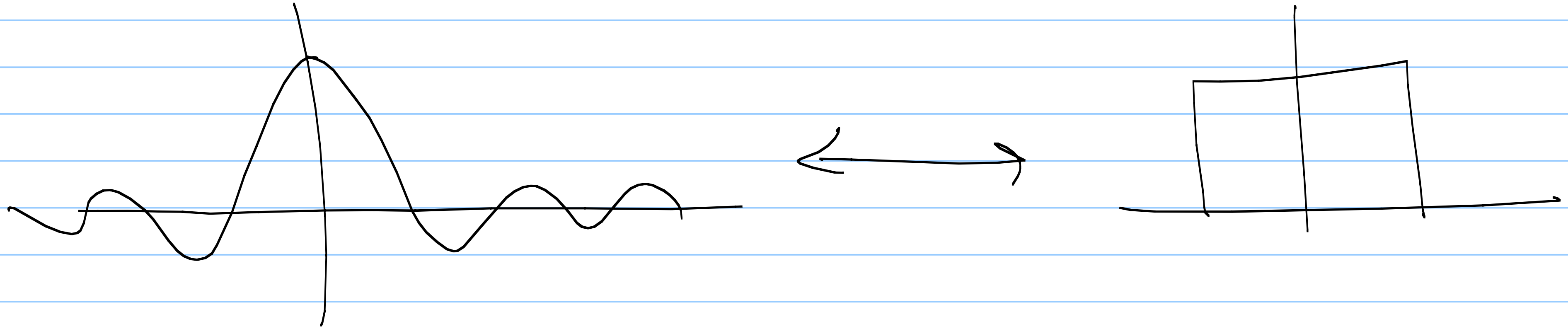
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\, e^{j\Omega t}\, d\Omega$$

$X(\Omega)$



$$\frac{\Omega_c}{2\pi} = F_c$$

$$x(t) = \frac{1}{2\pi} \int_{-\Omega_c/2}^{\Omega_c/2} 1 \, . \, e^{j\Omega t} \, d\Omega = F_c \, \mathrm{Sinc}(F_c t)$$



$$x(t) \longleftrightarrow X(\Omega)$$

$$y(t) = X(t) \longleftrightarrow 2\pi\, x(-\Omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)\, e^{j\Omega t}\, d\Omega$$

$$Y(\Omega) = \int_{-\infty}^{\infty} \underbrace{y(t)}_{}\, e^{-j\Omega t}\, dt$$

↳ $$2\pi x(\lambda) = \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega\lambda} d\Omega$$

$$= \int_{-\infty}^{\infty} X(t) e^{j\lambda t} dt$$

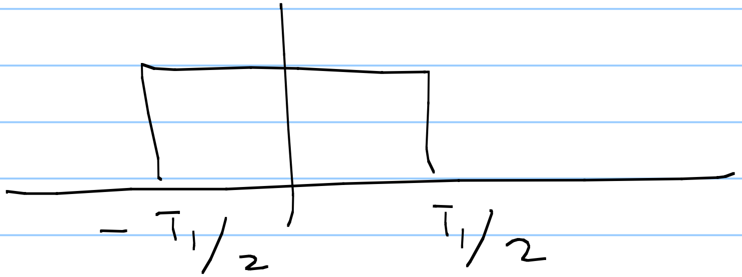
$$2\pi x(-\lambda) = \int_{-\infty}^{\infty} X(t) e^{-j\lambda t} dt$$

$$2\pi x(-\Omega) = \int_{-\infty}^{\infty} X(t) e^{-j\Omega t} dt$$

$$\mathcal{F}\left\{ X(t) \right\} = 2\pi\, x(-\Omega)$$

$$X(t) \longleftrightarrow 2\pi\, x(-\Omega)$$

$x(t)$



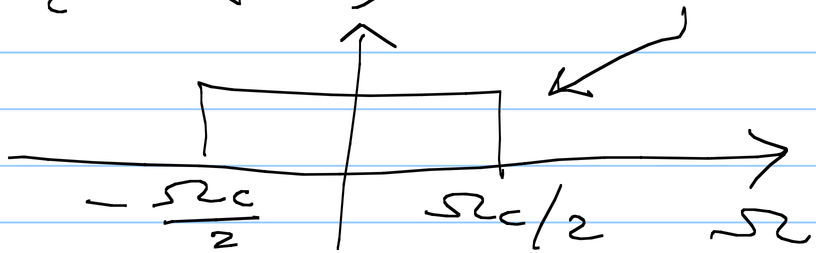
$$\longleftrightarrow \quad \frac{\sin \Omega\, T_1/2}{\Omega/2} = X(\Omega)$$

$$X(t) = \frac{\sin t\, T_1/2}{t/2} \equiv \frac{\sin t\, \Omega_c/2}{t/2}$$

$$\frac{X(t)}{2\pi} = \frac{\sin t\, \Omega_c/2}{\pi t}$$

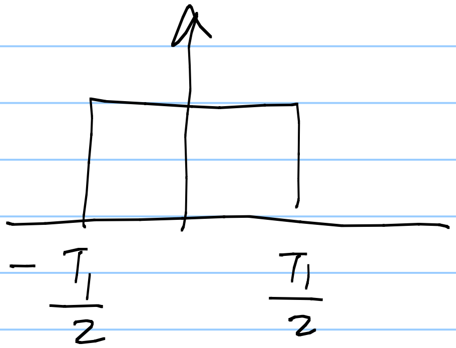
$$= \frac{\sin \pi F_c t}{\pi F_c t} \cdot F_c$$

$$= F_c \operatorname{sinc}(F_c t) \longleftrightarrow x(-\Omega)$$



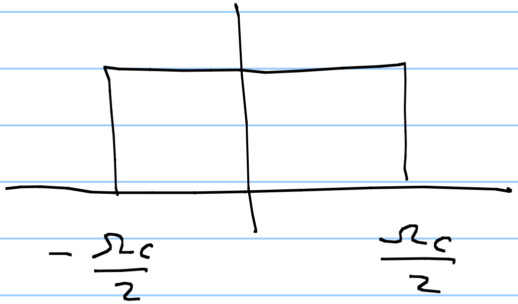
Ec 5142 26 Sep. 2011

$X(t) \longleftrightarrow 2\pi\, x(-\Omega)$



$\longleftrightarrow \quad \dfrac{\sin \Omega T_1/2}{\Omega/2}$

$F_c \operatorname{Sinc}(F_c t) \longleftrightarrow$



For real $x(t)$, $x(t) = x^*(t)$

$$X(\Omega) = X^*(-\Omega)$$

$$X(\Omega) = X_R(\Omega) + j X_I(\Omega)$$

$$X^*(-\Omega) = X_R(-\Omega) - j X_I(-\Omega)$$

$\Rightarrow$ $X_R(\Omega) = X_R(-\Omega)$ even

$X_I(-\Omega) = -X_I(\Omega)$ odd

$|X(\Omega)|$: even

$\angle X(\Omega)$: odd

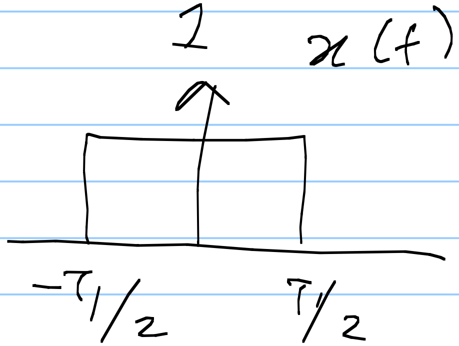
Differentiation:

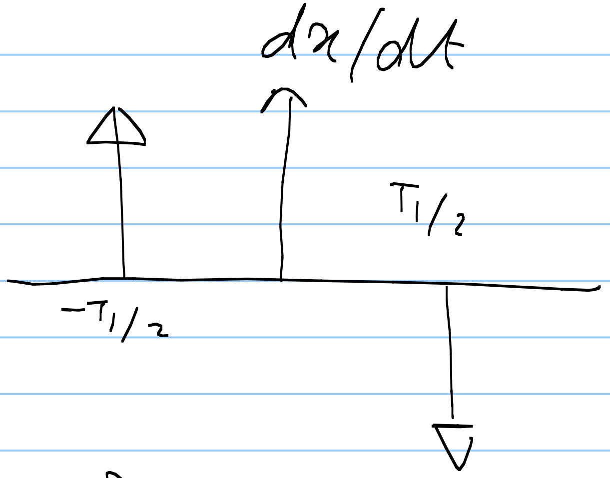
$$x(t) \longleftrightarrow X(\Omega)$$

$$x'(t) \longleftrightarrow j\Omega X(\Omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

$$\frac{dx}{dt} \overset{?}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} j\Omega X(\Omega)\, e^{j\Omega t}\, d\Omega$$





$$\delta(t+T_1/2) - \delta(t-T_1/2)$$

$$e^{j\Omega T_1/2} - e^{-j\Omega T_1/2} \longleftrightarrow 2j \, \text{Sin} \, \Omega T_1/2$$

$$x(t) \rightarrow \frac{\text{Sin} \, \Omega T_1/2}{\Omega/2}$$

$$x(at) \longleftrightarrow \frac{1}{|a|} X(\Omega/a)$$

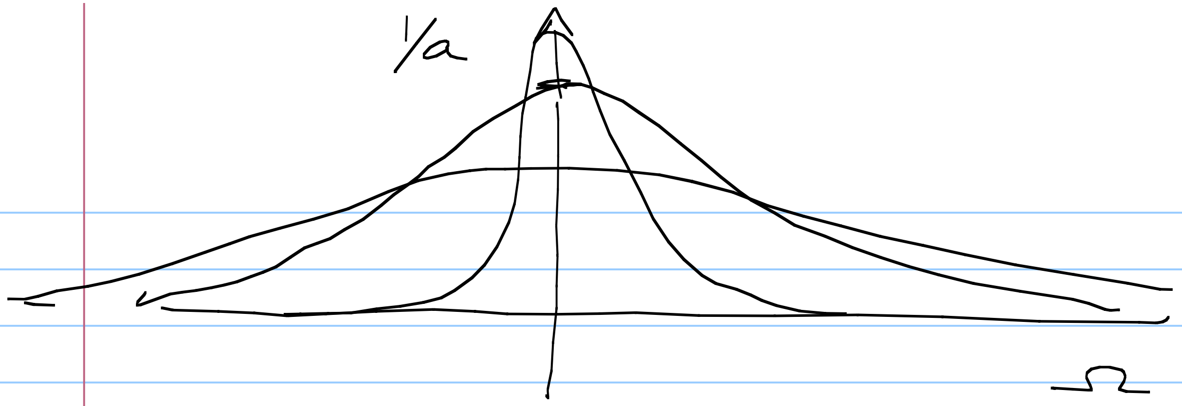
$$x(-t) \longleftrightarrow X(-\Omega)$$

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\Omega}$$

$$u(t) \longleftrightarrow \;?$$

$$e^{-at}u(t) \longleftrightarrow \frac{a}{a^2+\Omega^2} - \frac{j\Omega}{a^2+\Omega^2}$$

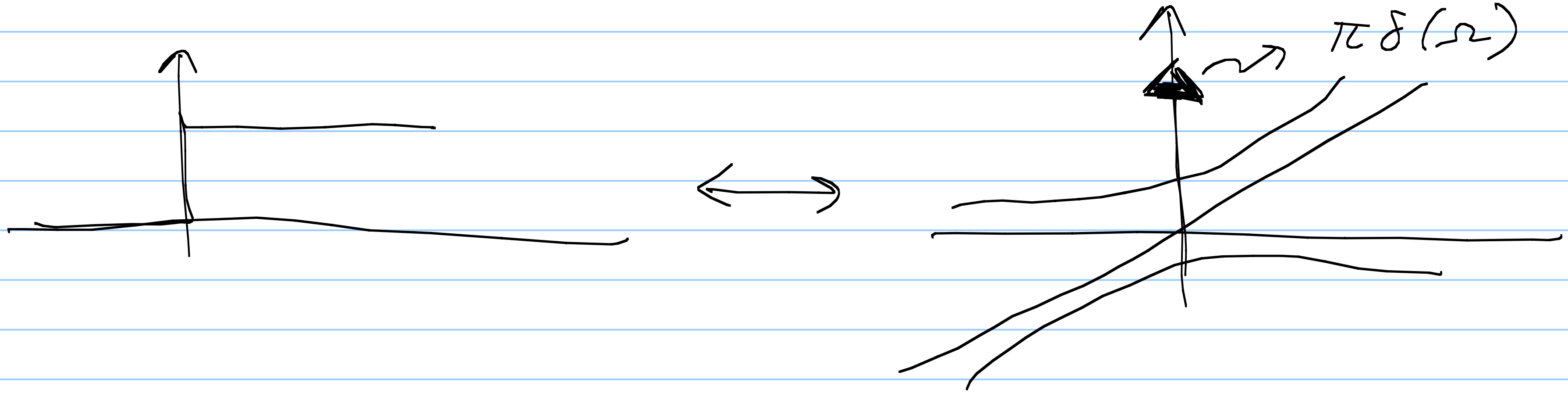
$$\lim_{a \to 0} \frac{a}{a^2+\Omega^2} - \frac{j\Omega}{a^2+\Omega^2} =$$



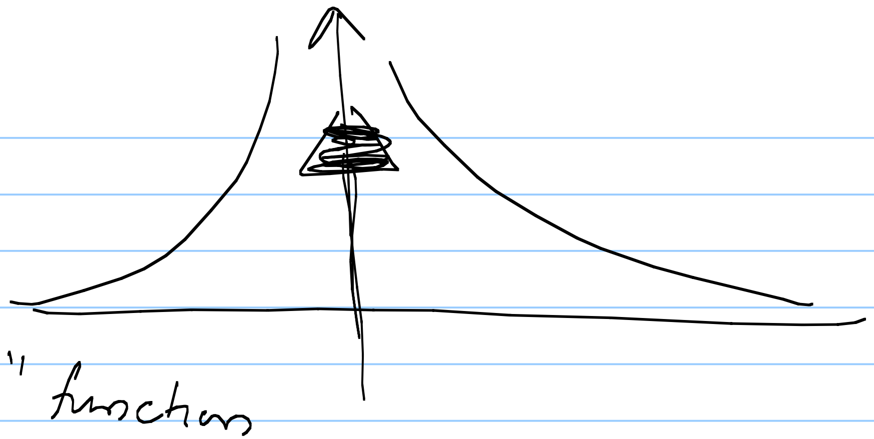
$$\int_{-\infty}^{\infty} \frac{a}{a^2+\Omega^2}\, d\Omega = \pi$$

indep. of a!

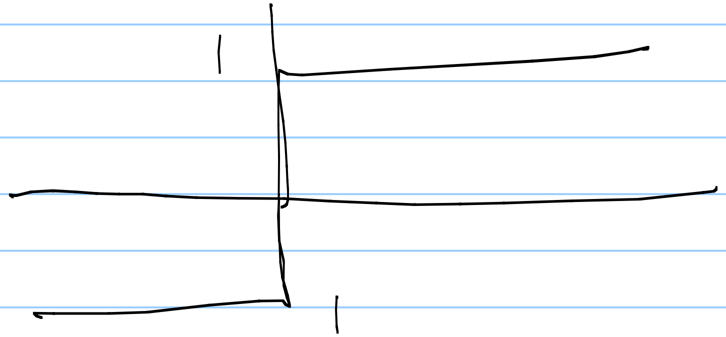
$$\lim_{a\to 0} \frac{a}{a^2+\Omega^2} - \frac{j\Omega}{a^2+\Omega^2} \longrightarrow \pi\,\delta(\Omega) + \frac{1}{j\Omega}$$



$|X(\Omega)|$



sgn (t) = "signum" function

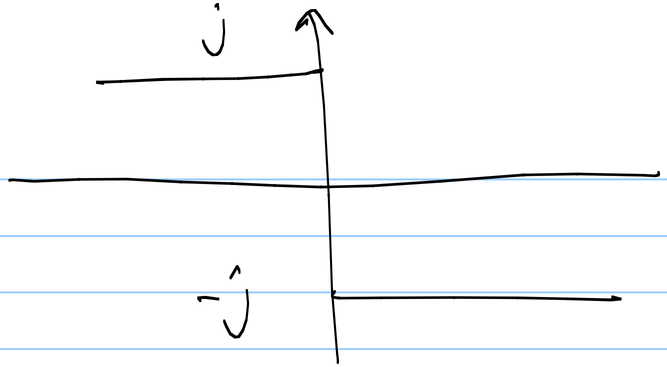


$= 2\left[u(t) - \frac{1}{2}\right]$

$= 2u(t) - 1$

$$\longleftrightarrow \quad 2\pi\delta(\Omega) + \frac{2}{j\Omega} - 2\pi\delta(\Omega) = \frac{2}{j\Omega}$$

$$x(t) = \frac{1}{\pi t} \longleftrightarrow$$



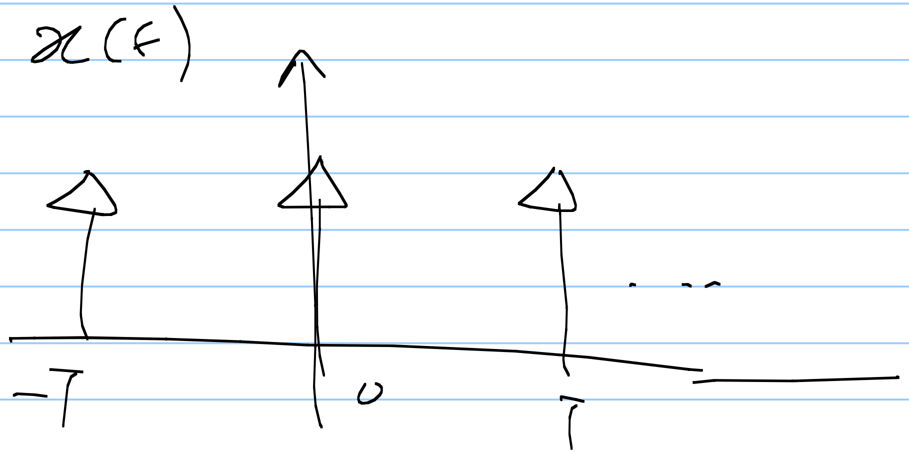
Other forms of $X(\Omega)$:

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi Ft}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(F)\, e^{j2\pi Ft}\, dF$$

$$\mathcal{F}\left\{\mathcal{F}\{x(t)\}\right\} = ?$$

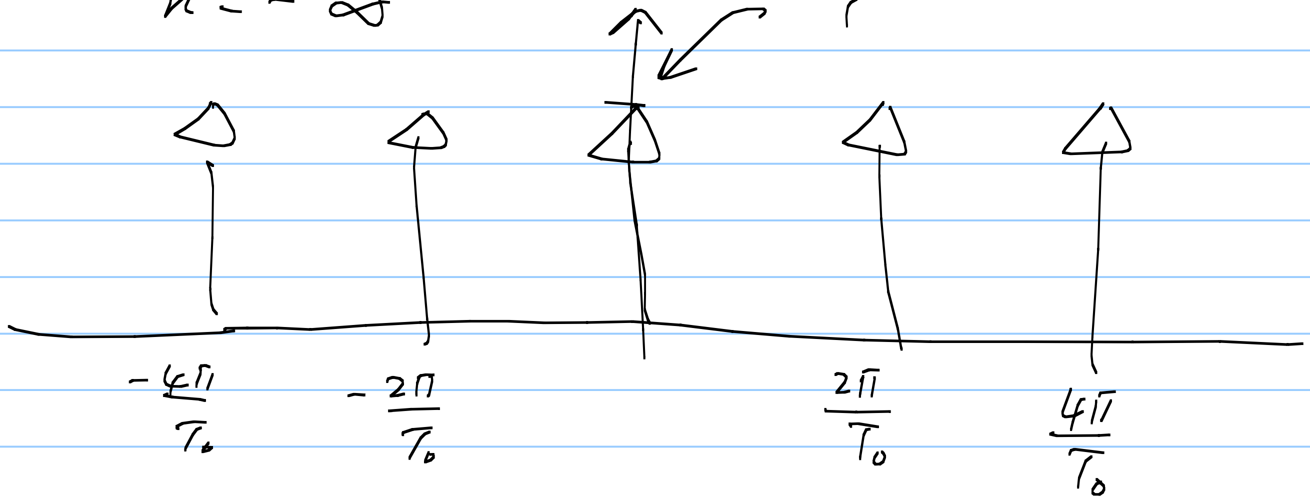
$$\mathcal{F}\mathcal{F}\mathcal{F}\mathcal{F}\{x(t)\} = ?$$



$$a_k = \frac{1}{T}$$

$$x(t) \quad \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\Omega_0 t}$$

$$\longleftrightarrow \quad \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\Omega - k\Omega_0)$$



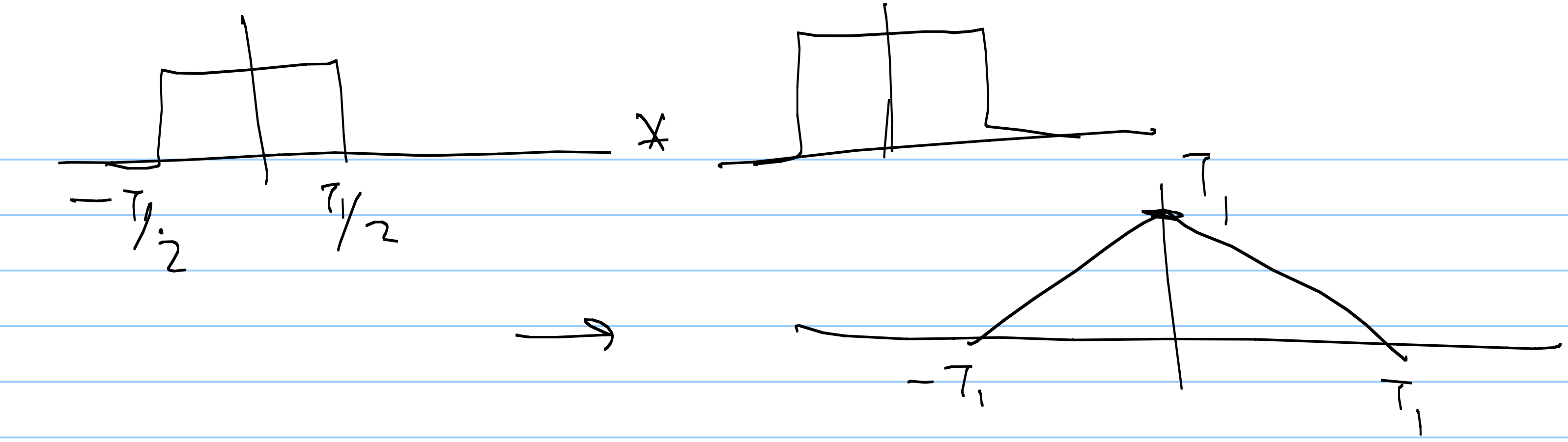
$$x(t) = e^{-t^2/2} \longleftrightarrow ?$$

$$\frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}}\,dt = 1$$

Convolution

$$x(t) * y(t) \longleftrightarrow X(\Omega)Y(\Omega)$$

$$x(t)\cdot y(t) \longleftrightarrow \frac{1}{2\pi} X(\Omega) * Y(\Omega)$$



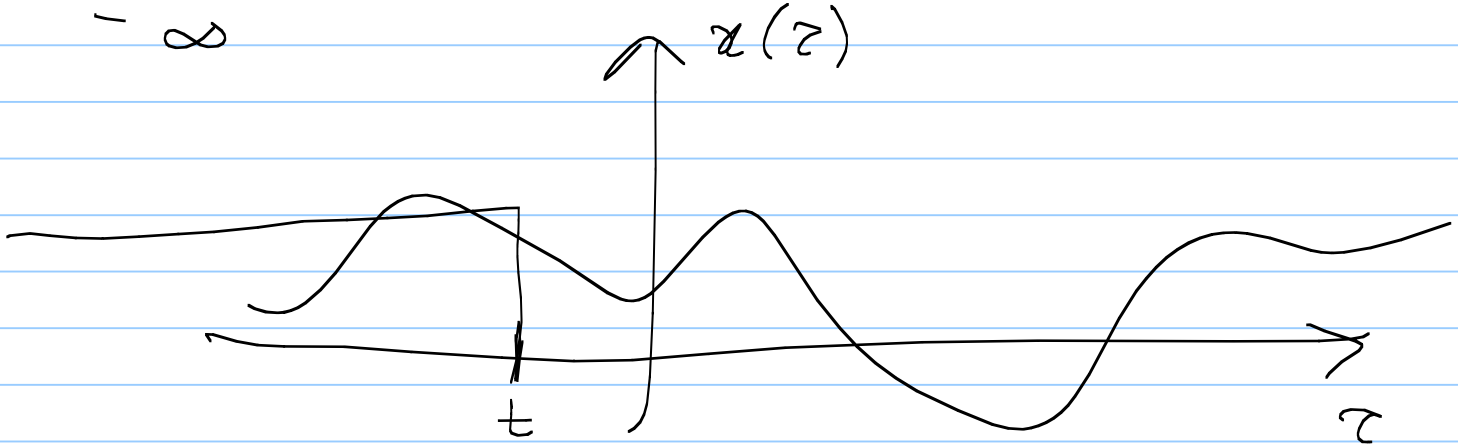
$$\longleftrightarrow \left[\frac{\sin \Omega T_1/2}{\Omega/2}\right]^2$$

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\, Y^*(\Omega)\, d\Omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega$$

Integration Property

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$$



$$\longleftrightarrow \quad X(\Omega)\ U(\Omega)$$

$$X(\Omega)\left[\pi\delta(\Omega) + \frac{1}{j\Omega}\right]$$

$$= \pi X(0)\delta(\Omega) + \frac{X(\Omega)}{j\Omega}$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \pi X(0)\delta(\Omega) + \frac{X(\Omega)}{j\Omega}$$

$$\frac{dy}{dt} = x(t)$$

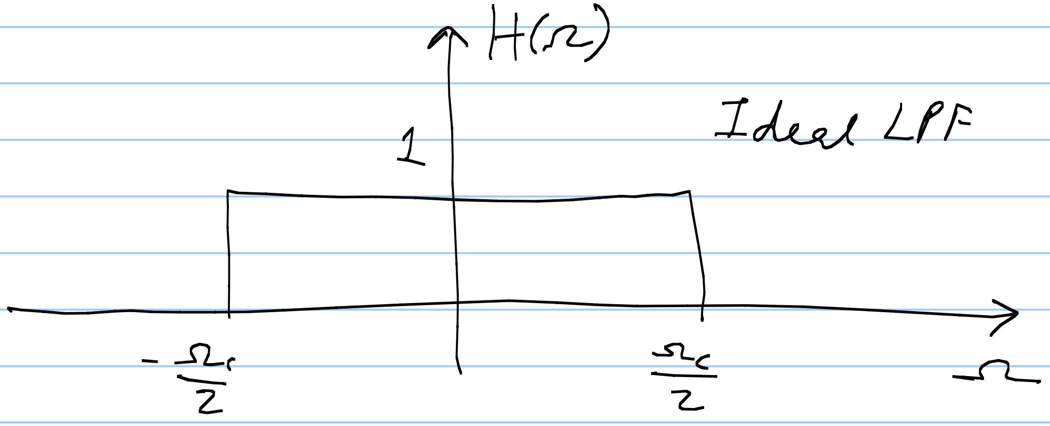
$$j\Omega\left[\pi X(0)\delta(\Omega)+\frac{X(\Omega)}{j\Omega}\right]$$

$$=\underbrace{j\Omega\,\delta(\Omega)}_{0}\,\pi X(0)+X(\Omega)$$

$= X(\Omega)$ as expected.

EC5142 Sep. 27 2011

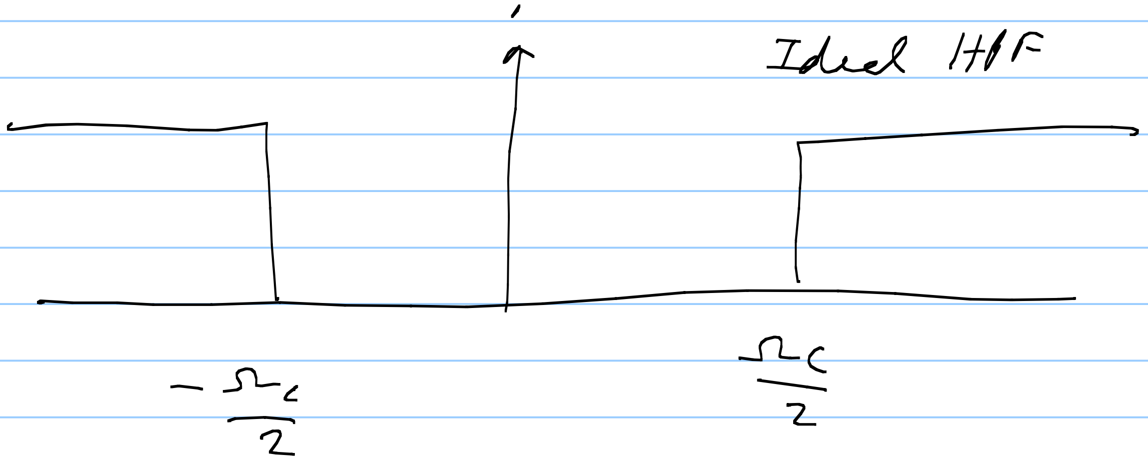
Ideal Filters

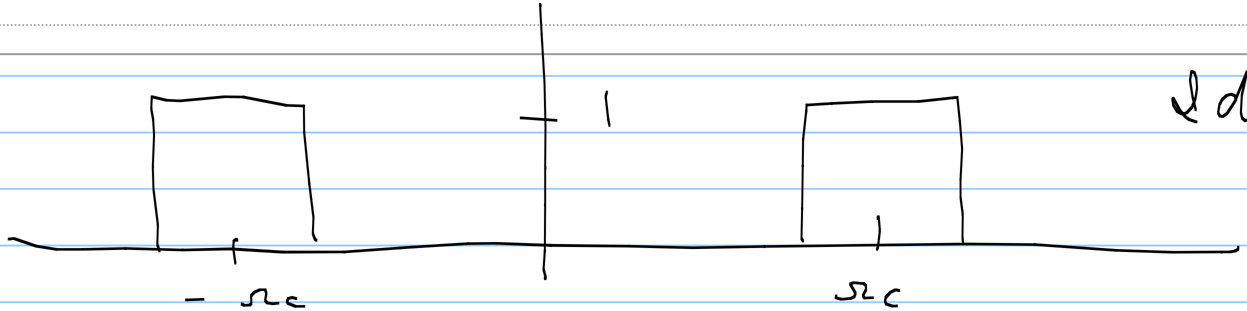


$\longleftrightarrow \quad F_c \, \text{Sinc}(F_c t)$

$\downarrow$

non-causal





Ideal BPF

Ideal BSF

Ideal notch filter

$$x[n+N] = x[n]$$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n}$$

$$N a_k = \sum_{n=\langle N \rangle} x[n] e^{-j \underbrace{k\omega_0}_{} n} \qquad \omega_0 = \frac{2\pi}{N}$$

As $N \rightarrow \infty$

$$N a_k \rightarrow \boxed{\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \equiv X(e^{j\omega})}$$

DTFT

↳ 2π periodic

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}$$

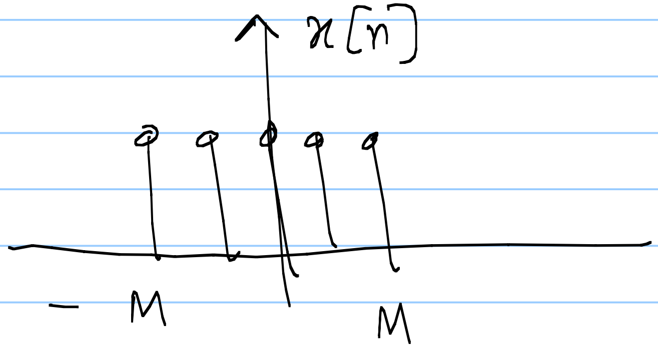
$$= \frac{1}{N} \sum_{k=0}^{N-1} (Na_k) e^{jk\frac{2\pi}{N}n} \qquad \omega_0 = \frac{2\pi}{N}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N-1} (Na_k) e^{jk\omega_0 n} \omega_0$$

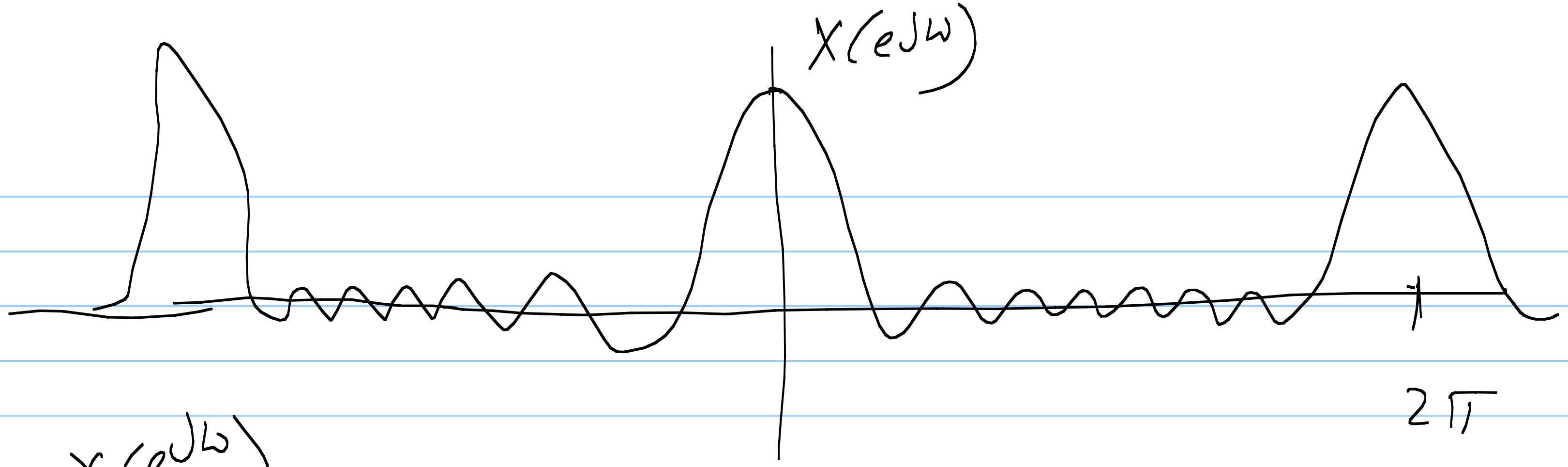
As $N \to \infty$ $\omega_0 \to 0$

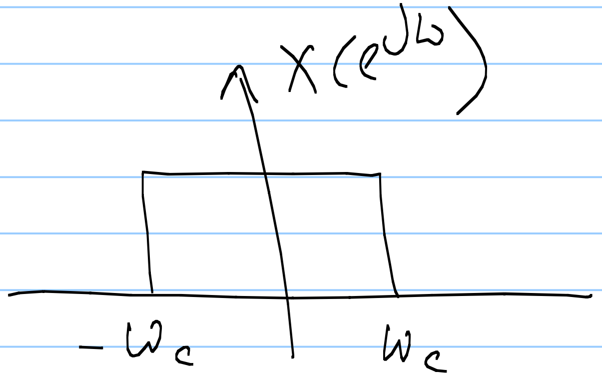
IDTFT $$\boxed{\frac{1}{2\pi}\int_{0}^{2\pi} X(e^{jw})\, e^{jwn}\, dw = x[n]}$$

$$\boxed{X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}}$$ DTFT

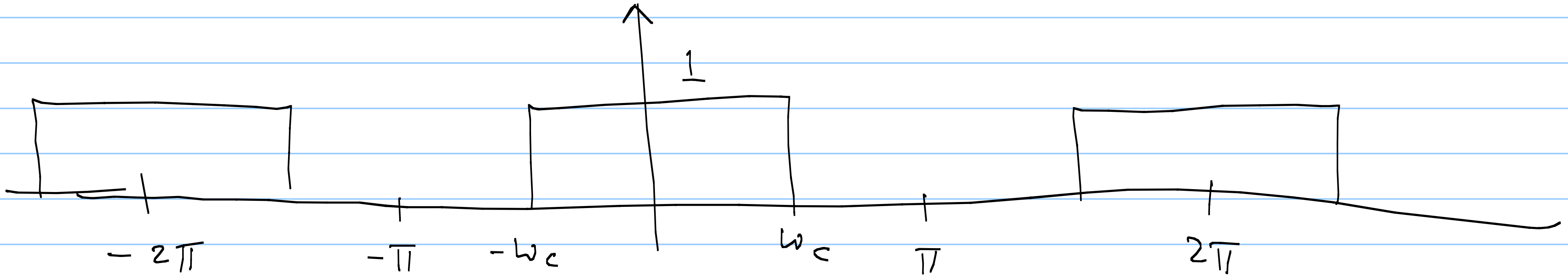


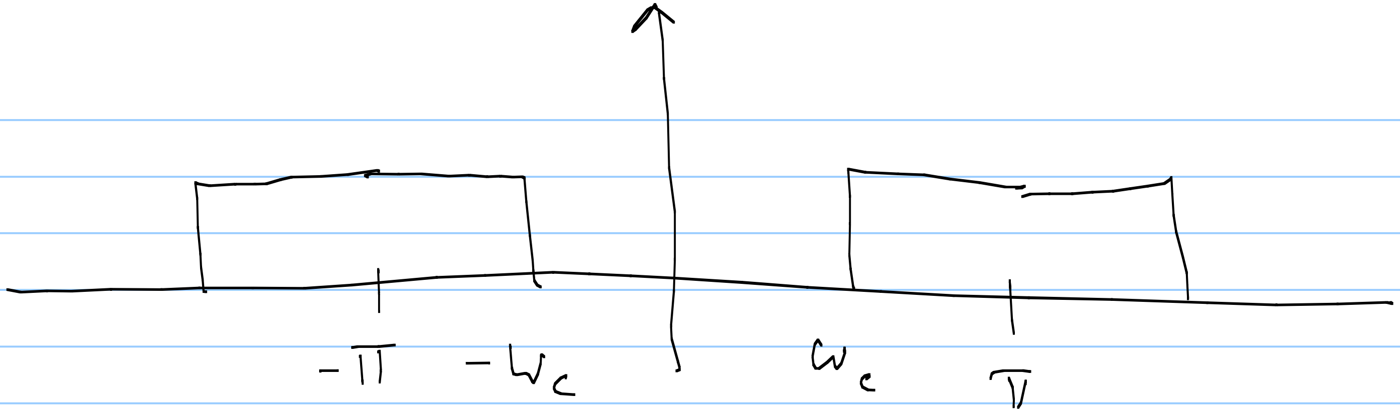
$$\longleftrightarrow \quad X(e^{jw}) = \frac{\sin (2M+1)\omega/2}{\sin \omega/2}$$





$$\longleftrightarrow \quad x[n] = \frac{\sin w_c n}{\pi n}$$




-π
-Wc
ωc
π

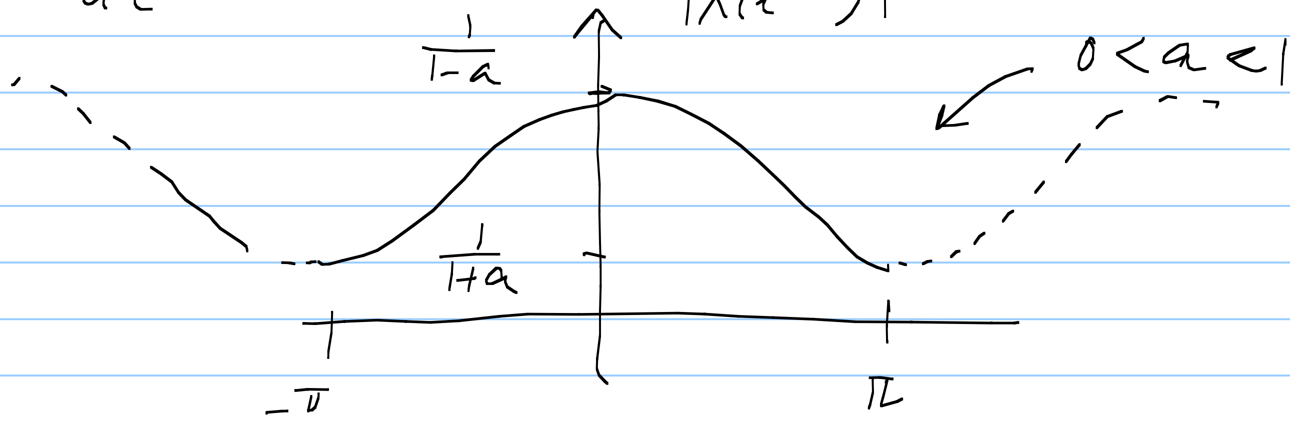
EC 5142 Oct. 03 2011

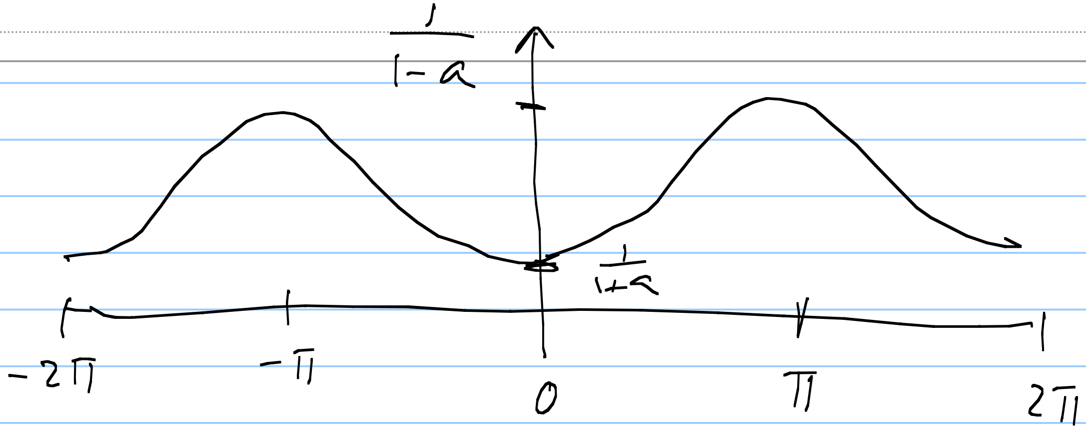
$$a^n u[n] \longleftrightarrow X(e^{j\omega})$$

$$\downarrow$$

$$\sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - a e^{-j\omega}} \qquad |a| < 1$$

$$(-1)^n a^n u[n] \longleftrightarrow \frac{1}{1 + a e^{-j\omega}}$$

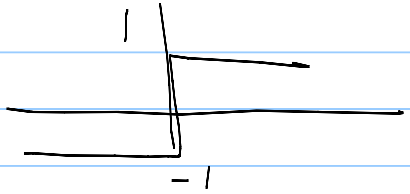




$u[n] \longleftrightarrow ?$

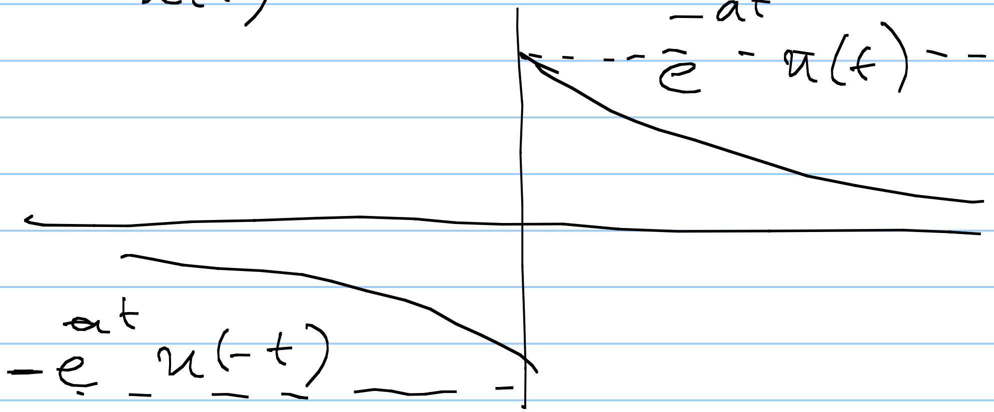
Recall:

$$\mathrm{Sgn}(t) \longleftrightarrow \frac{2}{j\Omega}$$



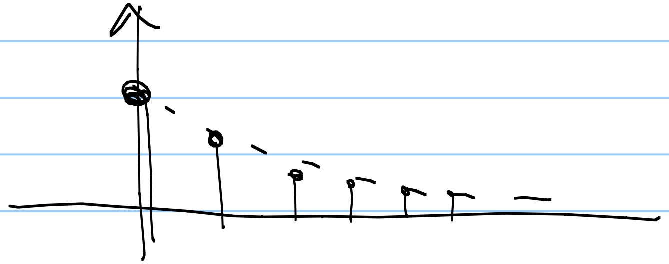
$$\frac{1}{2}\,\mathrm{sgn}(t) \longleftrightarrow \frac{1}{j\Omega}$$

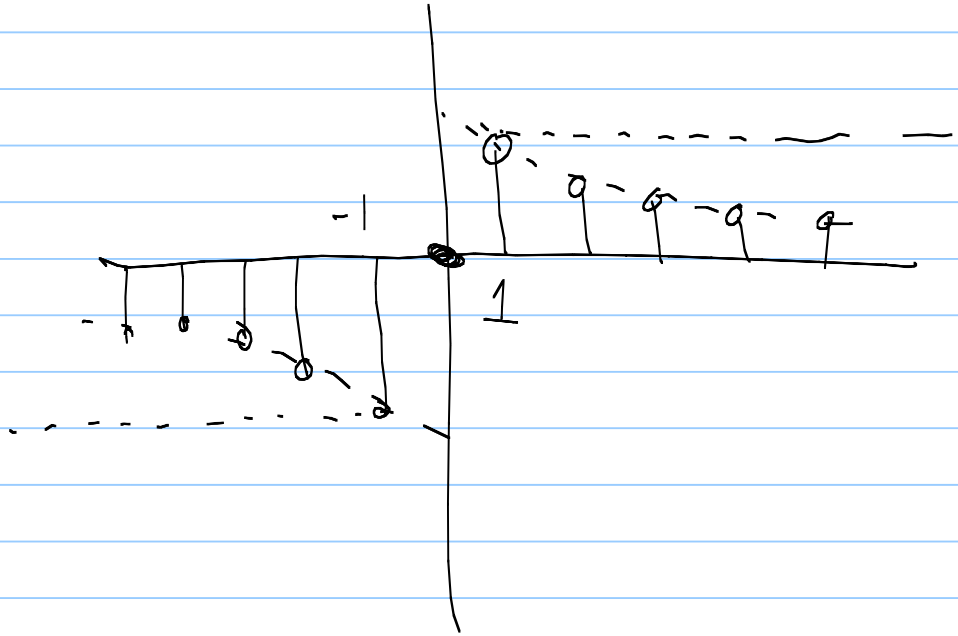
$$\underbrace{\frac{1}{2}\,\mathrm{sgn}(t) + \frac{1}{2}}_{u(t)} \longleftrightarrow \frac{1}{j\Omega} + \pi\,\delta(\Omega)$$



Take limit as $a \rightarrow 0$

$$a^{n} u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$$



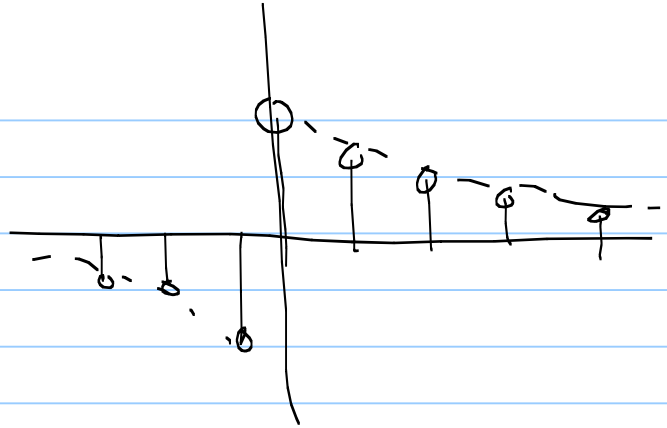


$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

$$a^{-n} u[-n] \longleftrightarrow \frac{1}{1-ae^{j\omega}}$$

$$a^{-(n+1)} u[-n-1] \longleftrightarrow \frac{e^{jw}}{1-ae^{jw}}$$



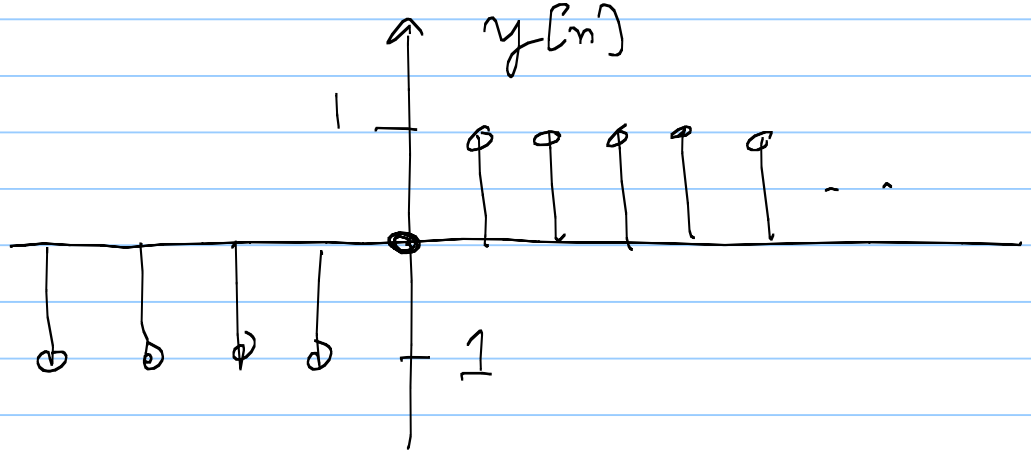
$$x[-n-n_0] \longleftrightarrow e^{-jwn_0} X(e^{jw})$$

$$a^{-n} u[-n-1] \longleftrightarrow \frac{ae^{jw}}{1-ae^{jw}}$$

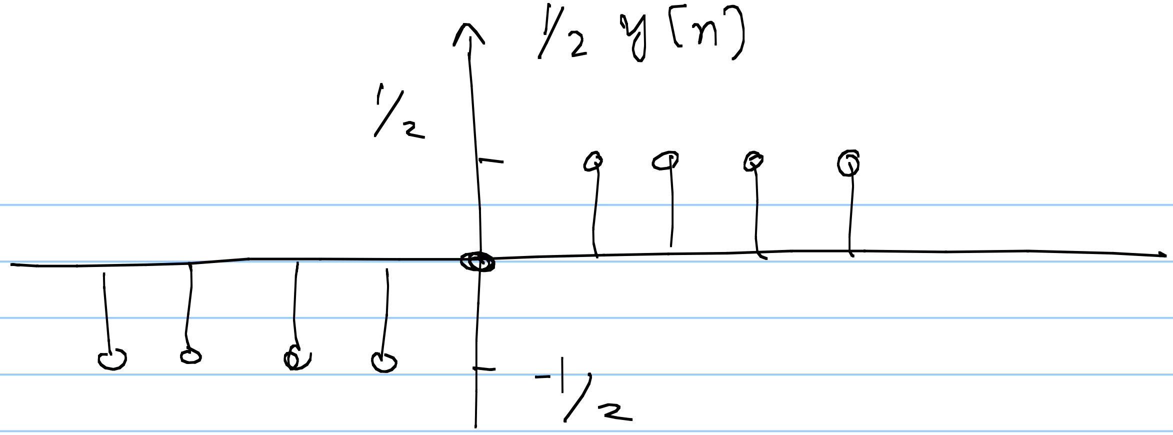
$$a^{n} u[n] - a^{-n} u[-n-1] \longleftrightarrow \frac{1}{1-ae^{-jw}} - \frac{ae^{jw}}{1-ae^{jw}}$$

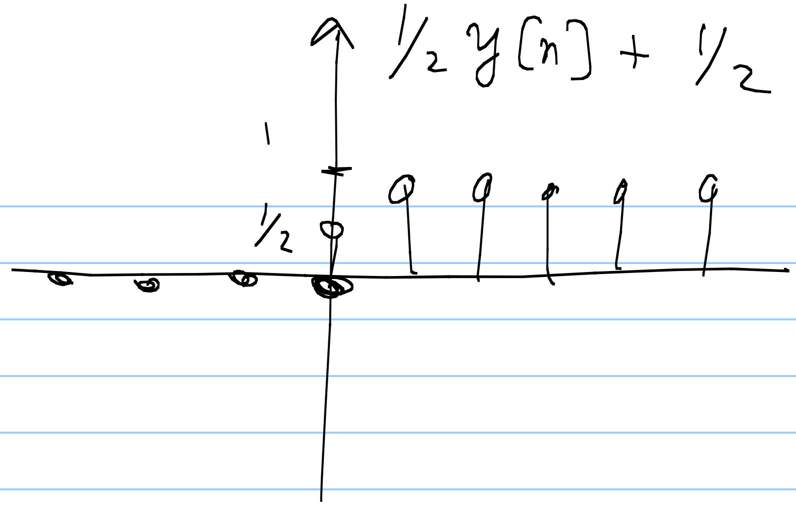
$$a^{n}u[n] - a^{-n}u[-n-1] - \delta[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}} - \frac{ae^{j\omega}}{1-ae^{j\omega}} - 1$$

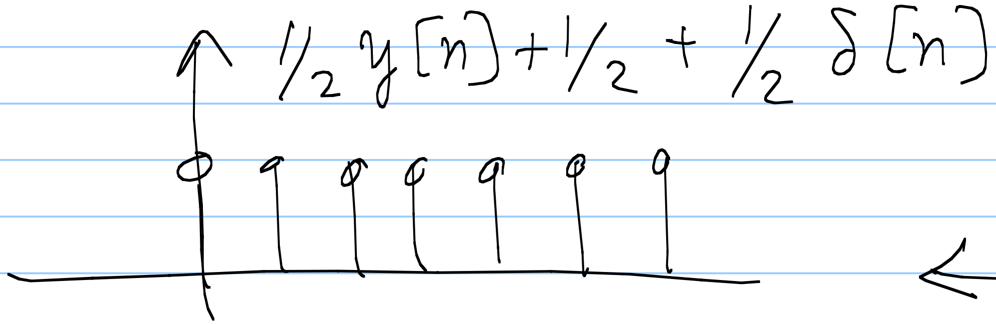
Take Limit as $a \to 1$ & simplify



$$\longleftrightarrow \frac{-j\sin\omega}{1-\cos\omega}$$







$$\longleftrightarrow \quad \frac{1}{1-e^{-j\omega}} + \pi\delta(\omega)$$

$$-\pi \leq \omega < \pi$$

$$\longleftrightarrow \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\,\delta(\omega - 2\pi k)$$

$$X(e^{j\omega}) \equiv X(\omega)$$

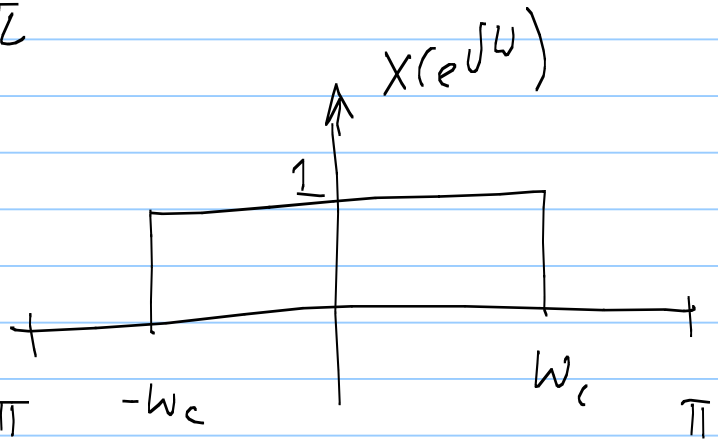
Properties (cont'd)

$$x[n] * y[n] \longleftrightarrow X(e^{j\omega})\,Y(e^{j\omega})$$

$$x[n]\,y[n] \longleftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)\,Y(\omega-\theta)\,d\theta$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

$$\frac{\sin \omega_c n}{\pi n} \longleftrightarrow$$



$$\sum_{n=-\infty}^{\infty} x[n] = X(e^{j\omega})\Big|_{\omega=0}$$

$$\sum_{n=-\infty}^{\infty} \frac{\sin w_c n}{\pi n} = 1$$

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2 w_c n}{\pi^2 n^2} = \frac{w_c}{\pi}$$

EC5142 Oct. 04 2011

Note Title 04-10-2011

$$z_0^n \longrightarrow \underset{LTI}{\boxed{h[n]}} \longrightarrow y[n]$$

$$y[n] = \sum_{m=-\infty}^{\infty} z_0^{n-m} h[m]$$

$$= z_0^n \underbrace{\sum_{m=-\infty}^{\infty} h[m] z_0^{-m}}_{H(z_0)}$$

$$H(z) \equiv \sum_{n=-\infty}^{\infty} h[n]\, z^{-n} \qquad \text{Z-transform of } h[n]$$

$$h[n] \xleftrightarrow{Z} H(z)$$

Eg: $x(n) = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \frac{1}{1 - a z^{-1}} \qquad |a z^{-1}| < 1$$

$$|az^{-1}| < 1 = |a| < |z| = \underbrace{|z| > |a|}$$

Region of convergence (RoC)

Eg:

$$x[n] = -a^{n} u[-n-1]$$

$$X(z) = -\sum_{n=-\infty}^{-1} a^{n} z^{-n} = \frac{1}{1-az^{-1}} \qquad |z| < |a|$$

Recall: $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad s = \sigma + j\Omega$

$$e^{+at} u(t) \longleftrightarrow \frac{1}{s-a} \qquad \mathrm{Re}\{s\} > \mathrm{Re}\{a\}$$

$$-e^{+at} u(-t) \longleftrightarrow \frac{1}{s-a} \qquad \mathrm{Re}\{s\} < \mathrm{Re}\{-a\}$$

$$a^{n} u[n] \xleftrightarrow{\text{DTFT}} \frac{1}{1 - a e^{-j\omega}} \qquad |a| < 1$$

$2^n u[n]$ does <u>NOT</u> possess DTFT

$$2^n u[n] \xleftrightarrow{Z} \frac{1}{1 - 2z^{-1}} \qquad |z| > |2|$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} = X(w)$$

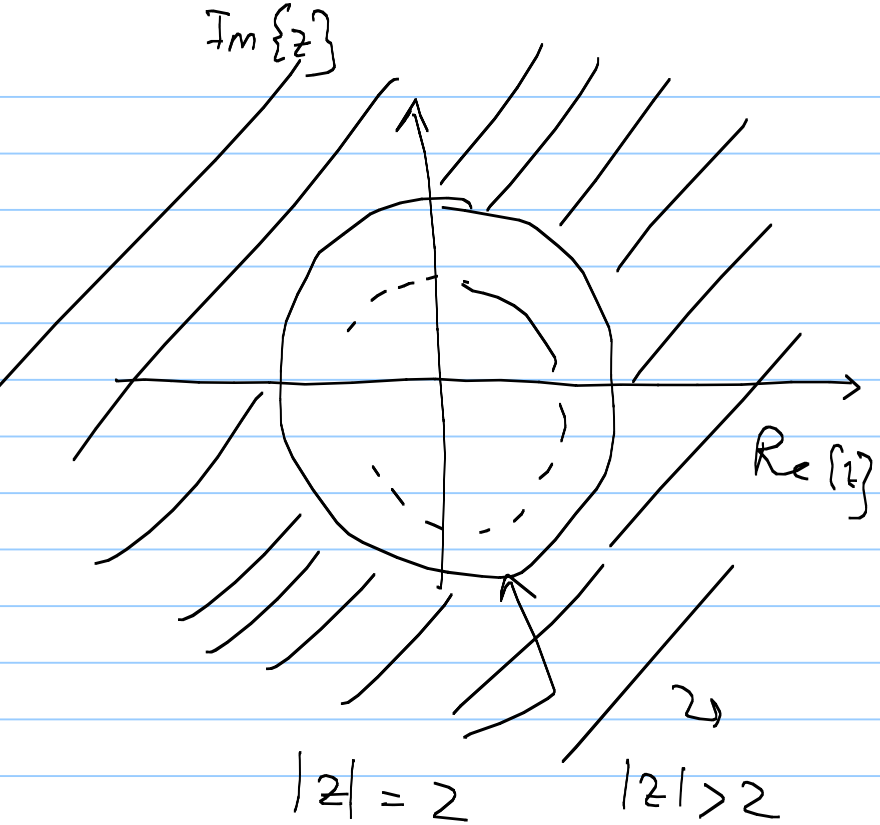
$$X(e^{jw}) = X(z)\Big|_{z=e^{jw}}$$

$$2^n u[n] \longleftrightarrow \frac{1}{1-2z^{-1}} \qquad |z|>2$$

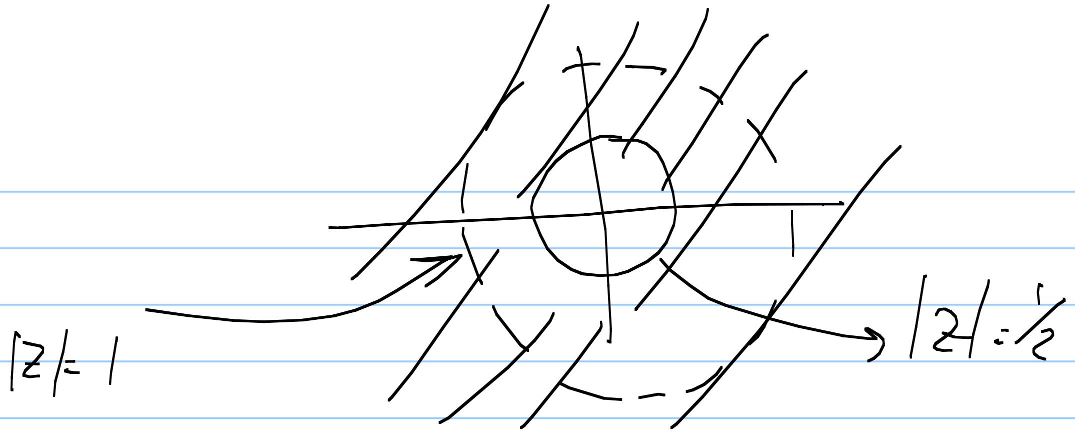
DTFT does not exist since

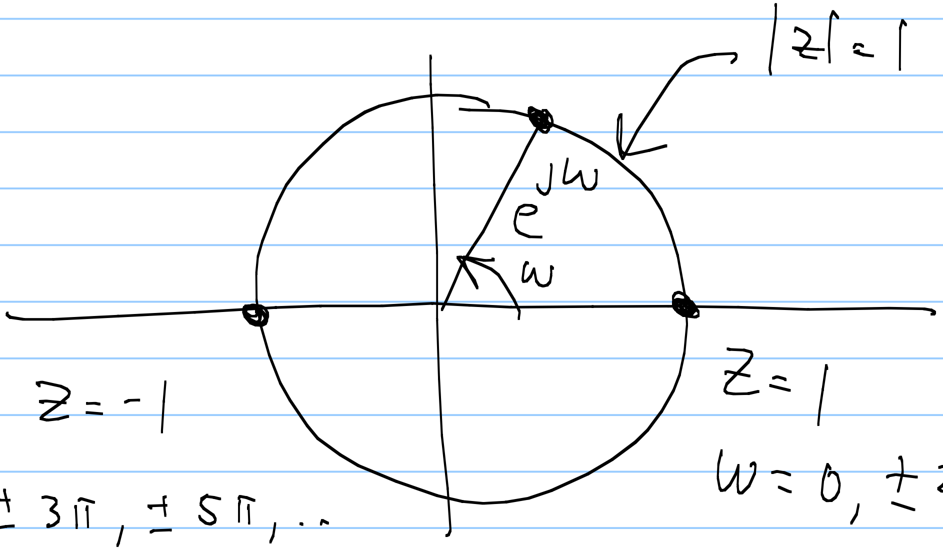
$$z = e^{jw} \notin RoC$$

$$\left(\tfrac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z|>\tfrac{1}{2}$$



$$\frac{1}{2}^{n} u[n] \longleftrightarrow \frac{1}{1-\frac{1}{2}e^{-jw}}$$





$w = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$

$w = 0, \pm 2\pi, \pm 4\pi, \ldots$

$$\sum_{n=-\infty}^{\infty} \underbrace{x[n]\, r^{-n}} e^{-j\omega n} \qquad z = r e^{j\omega}$$

$$2^n x[n] \longrightarrow \sum_{n=0}^{\infty} \left(\frac{2}{r}\right)^n e^{-j\omega n} \quad \text{exists for } r > 2$$

In general,

$$\sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\, r^{-n} e^{-j\omega n} = X(z)$$

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]\, r^{-n} e^{-j\omega n} \right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]|\, r^{-n} < \infty \Rightarrow \text{transform exists}$$

$$\sum_{n=0}^{\infty} \frac{|x[n]|}{r^{n}} + \sum_{n=-\infty}^{-1} \frac{|x[n]|}{r^{n}}$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{|x[n]|}{r^{n}}}_{\text{causal part}} + \underbrace{\sum_{n=1}^{\infty} |x[-n]|\, r^{n}}_{\text{anticausal part}}$$

Ⓐ Ⓑ

Suppose Ⓐ exist for $r = r_1$

then Ⓐ surely exist for all $r > r_1$

Suppose Ⓑ exist for $r = r_2$

then Ⓑ surely exists for all $r < r_2$

∴ The RoC is, in general of the form

$$r_1 < |z| < r_2$$

EC 5142 Oct. 10 2011

The RoC for a general $x[n]$ is of the form

$$r_1 < |z| < r_2$$

r_1 can be as small as zero

r_2 can be as large as ∞

For finite duration sequences, $X(z)$ is

$$x[-M]z^{M} + x[-M+1]z^{-M+1} + \dots + x[-1]z + x[0]$$
$$x[1]z^{-1} + x[2]z^{-2} + \dots + x[N]z^{-N}$$

$z = 0 \notin RoC \qquad z = \infty \notin RoC$

RoC = entire Z-plane except 0 and/or ∞

In general if $z = re^{j\omega} \in RoC$, then
$re^{j\omega} \in RoC \quad \forall\, \omega \in [-\pi, \pi)$

$$X(z) = \frac{P(z)}{Q(z)}$$ $P(z)$ & $Q(z)$ are polynomials in z

Then $X(z)$ is said to be rational

roots of $P(z)$ are called ZEROS of $X(z)$

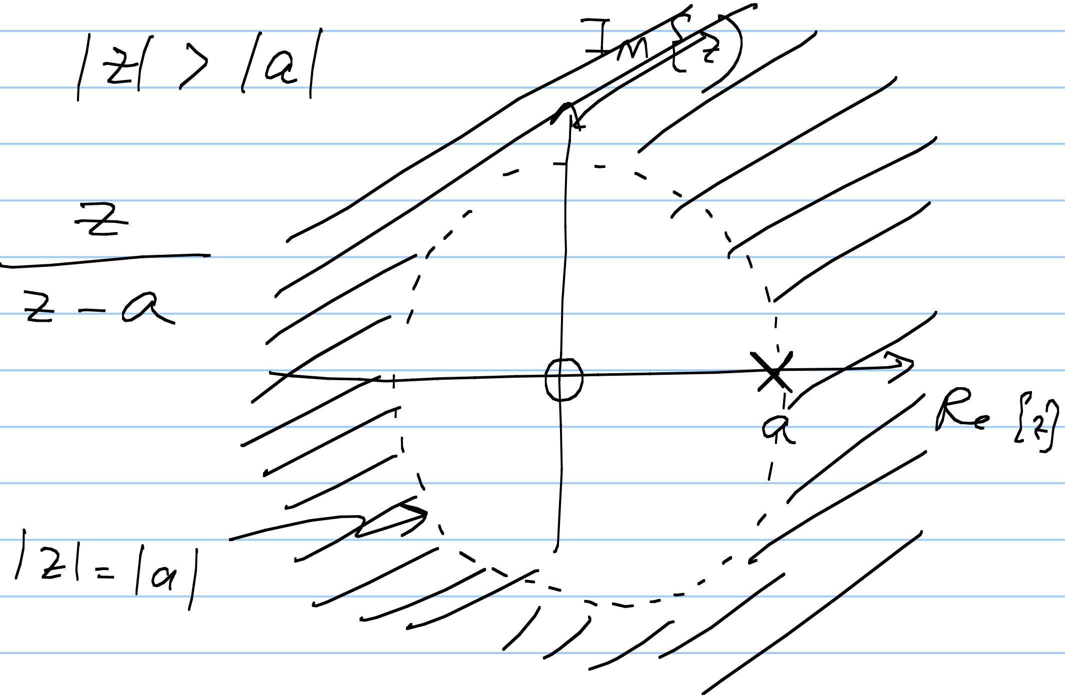
roots of $Q(z)$ are called POLES of $X(z)$

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

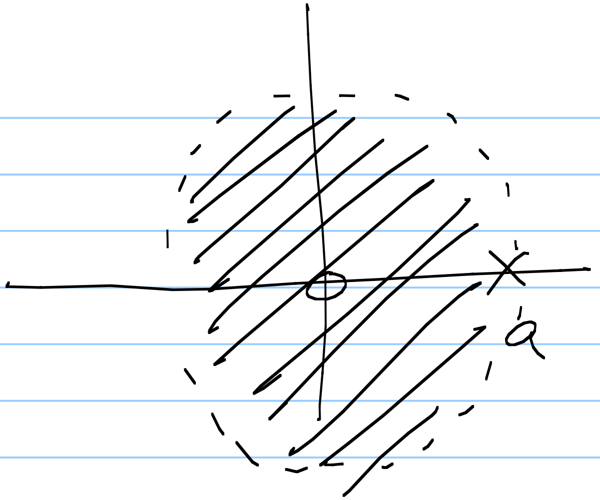
$$= \frac{z}{z-a}$$

pole: $z = a$

zero: $z = 0$



$$-a^n u[-n-1] \longleftrightarrow \frac{1}{1-az^{-1}} \quad |z|<|a|$$



RoC cannot contain poles

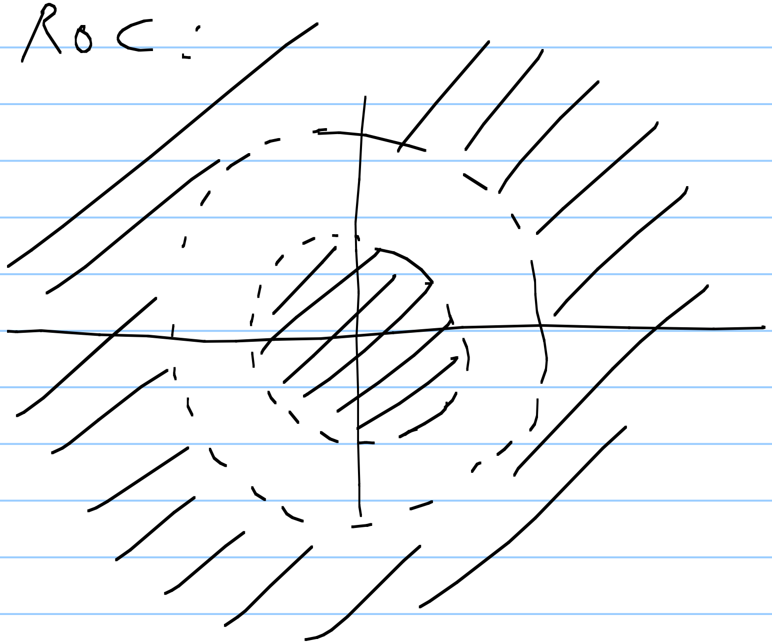
For rational $X(z)$:

(a) RoC is an annular region

(b) RoC cannot contain poles

(c) RoC must be a connected region

Invalid RoC:



(d) For finite duration sequences, RoC is the entire z-plane except possibly for 0 and/or ∞

(e) For right sided sequences, the RoC is a circle with radius R_1 s.t. RoC is of the form $R_1 < |z| \leq \infty$

(f) For left sided sequences, the RoC is bounded by a circle of radius R_2 s.t. the RoC is of the form $0 \leq |z| < R_2$

$X(z)$ is an analytic function in the RoC

$$z = x + jy$$

$$X(z) = U(x, y) + j\,V(x, y)$$

$$\left.\begin{array}{l} U_x = V_y \\ U_y = -V_x \end{array}\right\} \text{Cauchy-Riemann Equations}$$

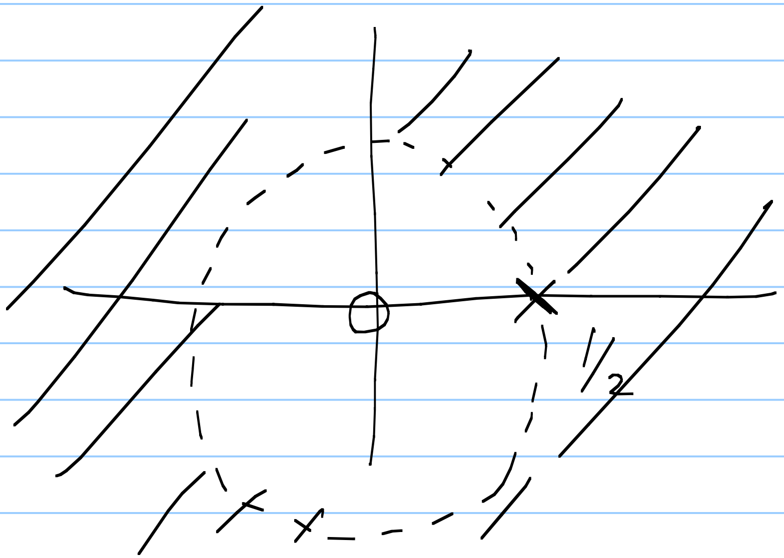
<u>Properties</u>

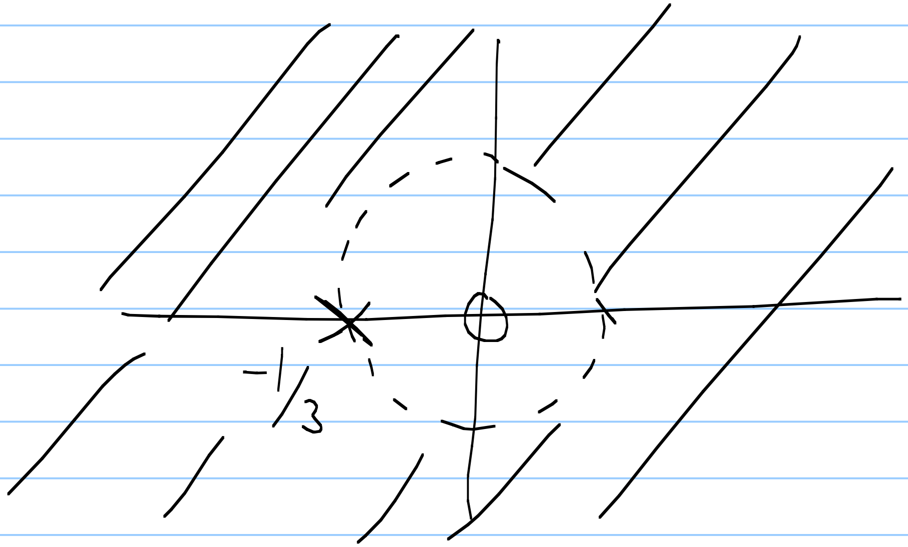
$$a_1 x_1[n] + a_2 x_2[n] \longleftrightarrow a_1 X_1(z) + a_2 X_2(z)$$

$$RoC \supseteq RoC_{x_1} \cap RoC_{x_2}$$

$$\left(\tfrac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1-\tfrac{1}{2}z^{-1}} \qquad |z| > \tfrac{1}{2}$$

$$\left(-\tfrac{1}{3}\right)^n u[n] \longleftrightarrow \frac{1}{1+\tfrac{1}{3}z^{-1}} \qquad |z| > \tfrac{1}{3}$$





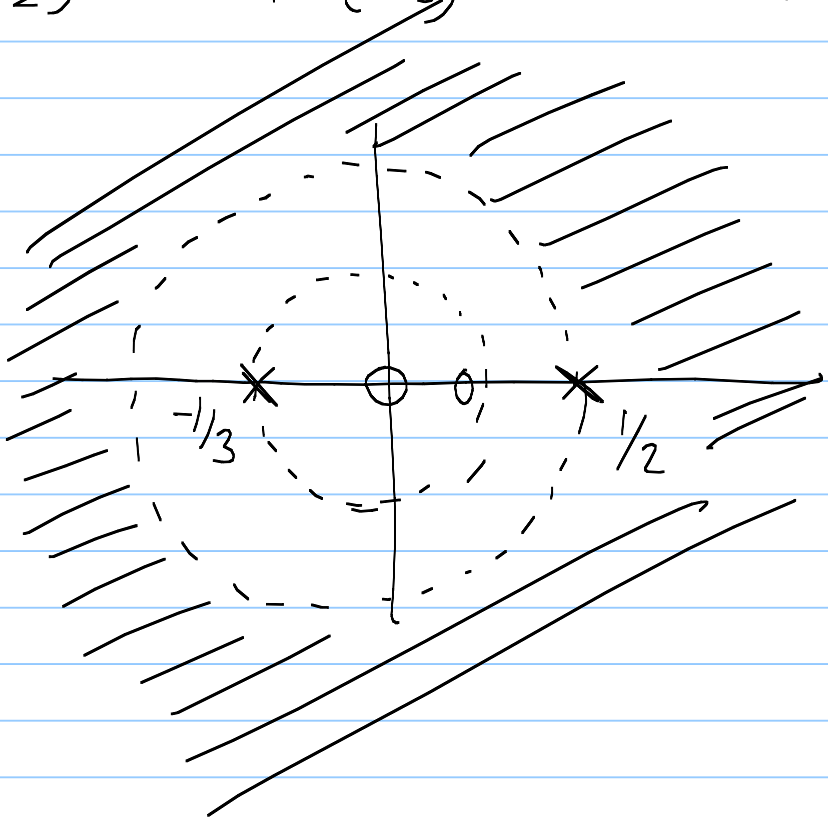
$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}$$

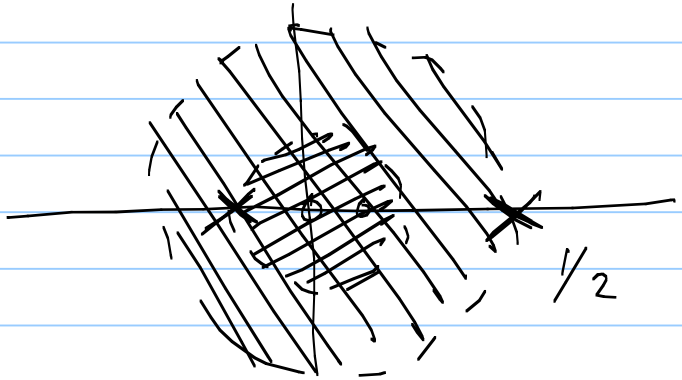
$$|z| > \tfrac{1}{2} \quad \cap \quad |z| > \tfrac{1}{3} \qquad |z| > \tfrac{1}{2}$$

$$\frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

EC 5142 Oct. 11, 2011

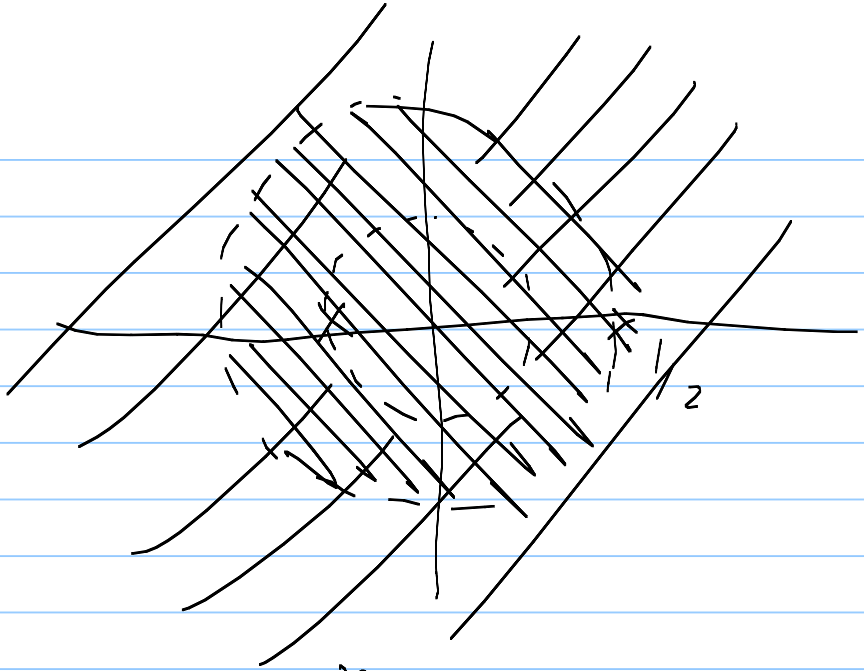
$\left(\frac{1}{2}\right)^n u[n] + \left(-1/3\right)^n u[n] \longleftrightarrow \dfrac{2\left(1 - 1/12\, z^{-1}\right)}{\left(1 - 1/2\, z^{-1}\right)\left(1 + 1/3\, z^{-1}\right)}$ $|z| > 1/2$





$$|z| < 1/3 \cap |z| < 1/2 = |z| < 1/3$$

$$-\left(-1/3\right)^n u[-n-1] \qquad -\left(1/2\right)^n u[-n-1]$$



$|z| > 1/3 \cap |z| < 1/2$

$= 1/3 < |z| < 1/2$

$(-1/3)^n u[n] \qquad -(1/2)^n u[-n-1]$

Time Delay

$$x[n-n_0] \longleftrightarrow z^{-n_0} X(z)$$

RoC is identical except possibly for addition or deletion of 0 and/or ∞

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

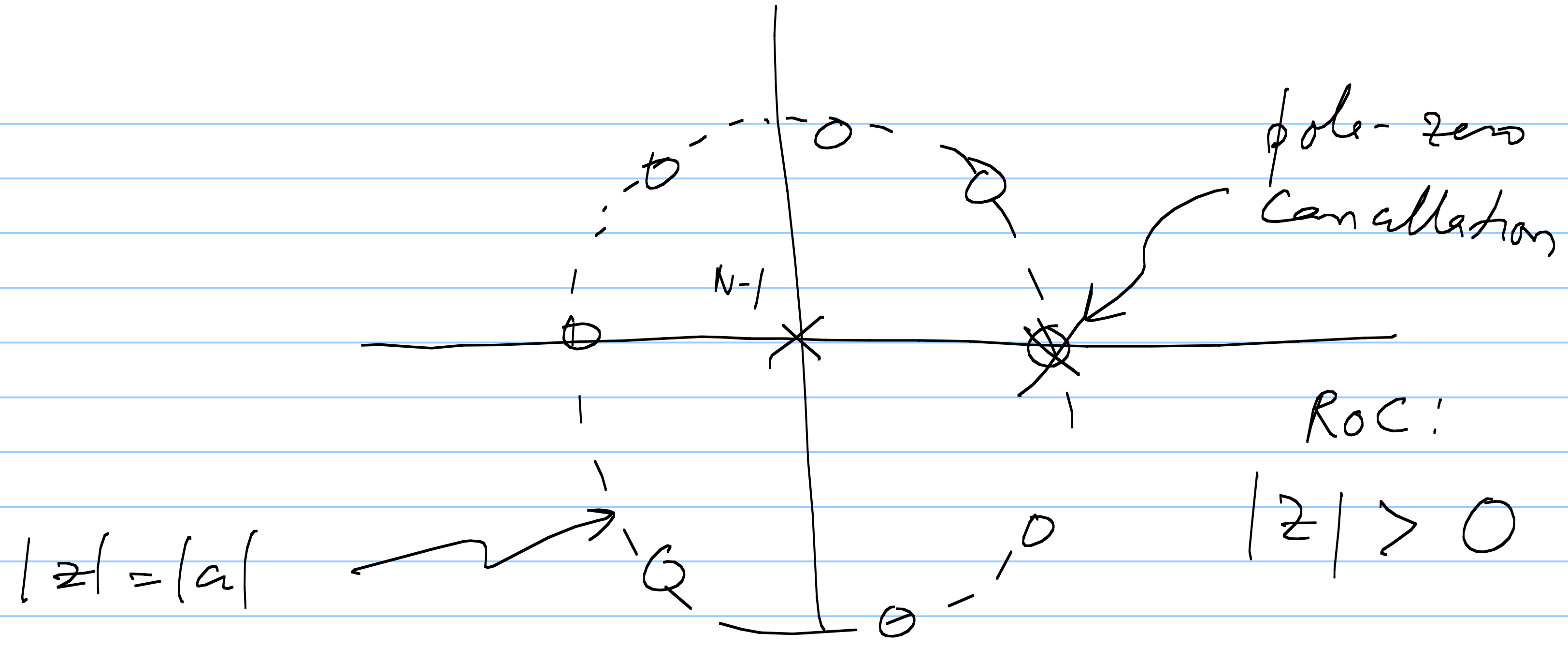
$$a^{n-N} u[n-N] \longleftrightarrow \frac{z^{-N}}{1-az^{-1}}$$

$$a^n u[n-N] \longleftrightarrow \frac{a^N z^{-N}}{1-az^{-1}} \qquad |z| > |a|$$

$$a^n \left[u[n] - u[n-N] \right] \longleftrightarrow \frac{1-a^N z^{-N}}{1-az^{-1}}$$

$$1, a, a^2, \dots, a^{N-1}$$

$$= \frac{z^N - a^N}{z^{N-1}(z-a)}$$



pole or zero at the origin: "trivial" pole/zero

Uncancelled nontrivial pole: $\frac{1}{1-az^{-1}}$

gives rise to either $a^n u[n]$ or $-a^n u[-n-1]$

(depends on RoC) $\Rightarrow$ ∞- duration sequence.

If there are only trivial poles, then

the sequence is finite duration

$$1, a, a^2, \ldots, a^{N-1} \longleftrightarrow 1 + a z^{-1} + a^2 z^{-2} + \cdots + a^{-(N-1)} z^{-(N-1)}$$

"All zero filter"

$$\frac{z^{N-1} + a z^{N-2} + \cdots + a^{-(N-1)}}{z^{N-1}}$$

N-1 zeros

N-1 trivial poles

$$X(z) = \frac{1}{1 + a_1 z^{-1} + \dots + a_{N-1} z^{-(N-1)}}$$

"All pole filter"

$$X(z) = \frac{P(z)}{Q(z)}$$ "pole zero filter"

$$\frac{1}{(1 - a z^{-1})(1 - b z^{-1})} = \frac{z^2}{(z-a)(z-b)}$$

$$\frac{z}{(z-a)(z-b)} \Rightarrow$$ poles: $z = a, b$

zero: $z = 0, \infty$

$$\frac{(z-a)(z-b)}{z}$$

zeros: $z = a, b$

poles: $z = 0, \infty$

Exponential Multiplication

$$x[n] \longleftrightarrow X(z) \qquad r_1 < |z| < r_2$$

$$a^n x[n] \longleftrightarrow \sum_{n=-\infty}^{\infty} x[n] (z/a)^{-n} \longleftrightarrow X(z/a)$$

$$r_1 < |z/a| < r_2$$

$$|a| r_1 < |z| < |a| r_2$$

Suppose z_0 is a zero of $X(z)$, i.e. $X(z_0) = 0$

$$y[n] = a^n x[n] \longleftrightarrow Y(z) = X(z/a)$$

$$Y(a z_0) = X(a z_0 / a) = X(z_0) = 0$$

$\Rightarrow$ $a z_0$ is a root of $Y(z)$

Note Title 12-10-2011

EC 5142 Oct 12 2011

$$\gamma^{n} x[n] \longleftrightarrow X(z/\gamma) \qquad |\gamma| r_1 < |z| < |\gamma| r_2$$

$$\sum_{n=-\infty}^{\infty} u[n] z^{-n} = \frac{1}{1-z^{-1}} = X(z) \quad |z| > 1$$

$$a^{n} u[n] \longleftrightarrow X(z/a) \qquad |z/a| > 1$$

$$\frac{1}{1-(z/a)^{-1}} = \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$e^{j\omega_0 n} x[n] \longleftrightarrow X\left(z / e^{j\omega_0} \right)$$ RoC is same

DTFT : $X(z)\Big|_{z = e^{j\omega}}$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{DTFT} X\left(e^{j\omega} / e^{j\omega_0} \right) = X\left(e^{j(\omega - \omega_0)} \right)$$

Differentiation in the Z-Domain

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\frac{dX}{dz} = \sum_{n=-\infty}^{\infty} x[n] \left(-n z^{-n-1}\right)$$

$$-z \frac{dX}{dz} = \sum_{n=-\infty}^{\infty} n x[n] z^{-n}$$

$$n\, x[n] \longleftrightarrow -z \frac{dX}{dz}$$

RoC is same except possibly for the boundary circle

Eg

$$x[n] = \begin{cases} \dfrac{1}{n^2} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

$X(z)$ — dilogarithm function $\quad |z| \geq 1$

$$n\,x[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

$X(z)$: RoC $|z| > 1$

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$n\,a^n u[n] \longleftrightarrow -z \frac{d}{dz} \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$= (-z)\ \frac{-(-a)(-z^{-2})}{(1-az^{-1})^2}$$

$$= \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > |a|$$

$$(n+1)\,a^{n+1}\,u[n+1] \longleftrightarrow \frac{a}{(1-az^{-1})^2} \qquad |z| > |a|$$

$$(n+1)\,a^{n}\,u[n+1] \longleftrightarrow \frac{1}{(1-az^{-1})^2} \qquad |z| > |a|$$

$$(n+1)a^n u[n] \longleftrightarrow \frac{1}{(1-az^{-1})^2} \qquad |z|>|a|$$

$$? \longleftrightarrow \frac{1}{(1-az^{-1})^m} \qquad |z|>|a|$$

Exponential mult.

$$\frac{1}{2} r^n e^{j\omega_0 n} u[n] \longleftrightarrow \frac{1/2}{1-\left(\frac{z}{re^{j\omega_0}}\right)^{-1}} \qquad |z|>r$$

$$= \frac{1/2}{1 - re^{j\omega_0}z^{-1}}$$

$$\tfrac{1}{2} r^n e^{-j\omega_0 n} u[n] \longleftrightarrow \frac{1/2}{1 - re^{-j\omega_0}z^{-1}} \qquad |z| > r$$

$$r^n \cos\omega_0 n \, u[n] \longleftrightarrow \frac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$$

$$\left(1 - re^{j\omega_0}z^{-1}\right)\left(1 - re^{-j\omega_0}z^{-1}\right) \longleftarrow$$

Suppose $x[n]$ is real-valued. $x[n] = x^*[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x^*[n]\, z^{-n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x[n] \left(z^*\right)^{-n} \right]^*$$

$$= X^*(z^*)$$

$$X^*(z) = X(z^*)$$

Suppose z_0 is a root of $X(z)$.

$$X(z_0) = 0$$

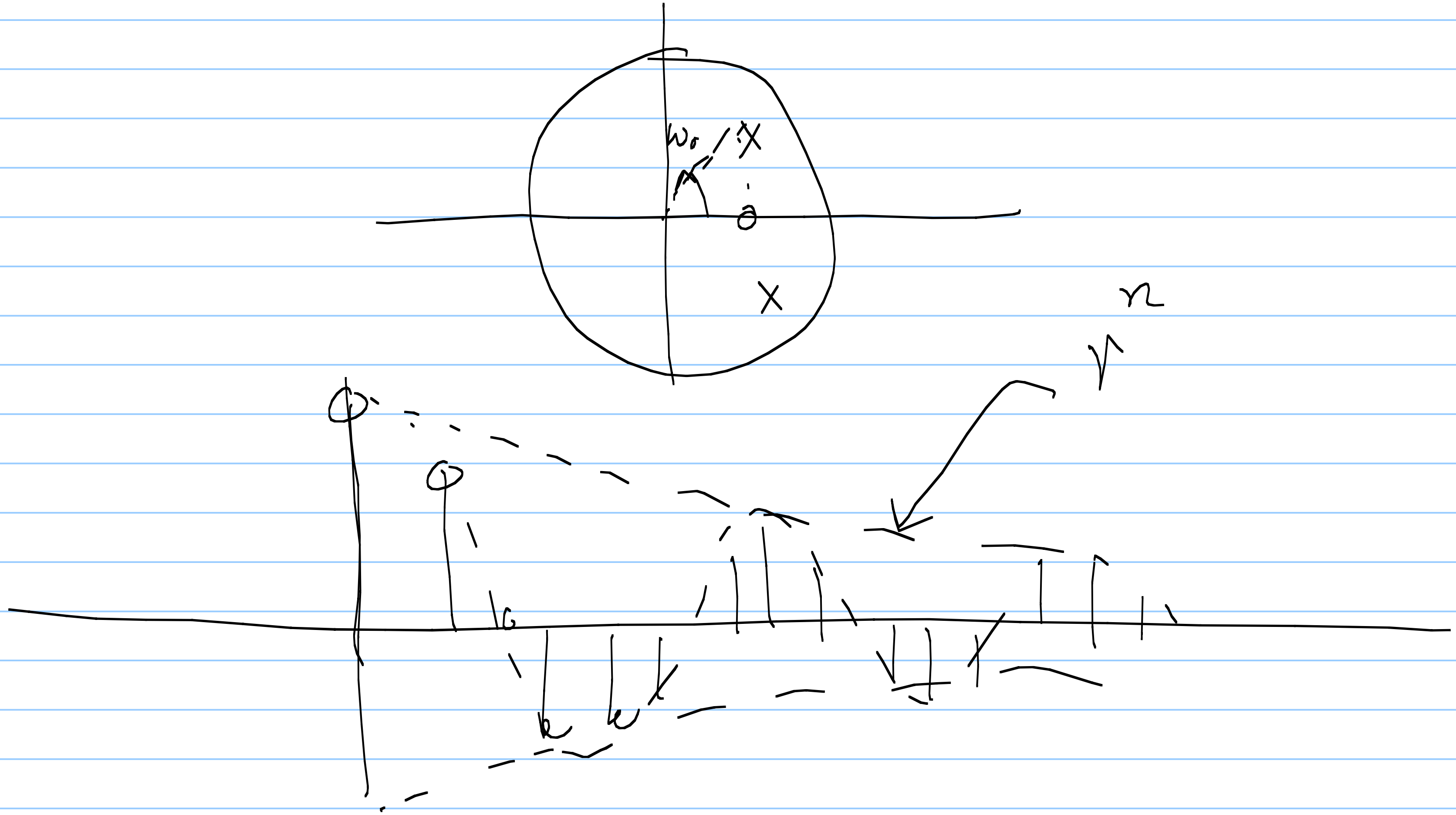
$$X^*(z_0) = 0$$

$$X(z_0^*) = 0$$

$\Rightarrow$ z_0^* is also a root

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$b_k, a_k \in \mathbb{R}$



Time Reversal $\quad x[n] \longleftrightarrow X(z) \quad r_1 < |z| < r_2$

$$x[-n] \longleftrightarrow X(z^{-1}) \quad 1/r_2 < |z| < 1/r_1$$

$$a^n u[n] \longleftrightarrow \frac{1}{1 - az^{-1}} \quad |z| > |a|$$

$$a^{-n} u[-n] \longleftrightarrow \frac{1}{1 - az} \quad |z| < 1/|a|$$

$$= \frac{-a^{-1} z^{-1}}{1 - a^{-1} z^{-1}}$$

$$a^{-n-1} u[-n-1] \longleftrightarrow \frac{-a^{-1}}{1 - a^{-1} z^{-1}} \qquad |z| < |a^{-1}|$$

$$-(a^{-1})^{n} u[-n-1] \longleftrightarrow \frac{1}{1 - a^{-1} z^{-1}} \qquad |z| < |a^{-1}|$$

$$x(-n) \longleftrightarrow X(z^{-1}) \qquad 1/r_2 < |z| < 1/r_1$$

$$(-1)^n x(n) \longleftrightarrow X(-z) \qquad \text{RoC is same}$$

Convolution in the Time Domain

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$

$$= x[n] * h[n]$$

$Y(z) = X(z)H(z) \qquad ROC \supseteq ROC_x \cap ROC_h$

Note Title 13-10-2011

EC 5142 Oct. 13 2011

Convolution in the time domain:

$$y[n] = x[n] * h[n]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} x[m] h[n-m] \right] z^{-n}$$

$$\overset{?}{=} \sum_{m=-\infty}^{\infty} x[m] \left[\sum_{n=-\infty}^{\infty} h[n-m] z^{-n} \right]$$

$$= \sum_{m=-\infty}^{\infty} x[m] \, H(z) \, z^{-m}$$

$$= H(z) \, X(z) \qquad RoC \supseteq RoC_x \cap RoC_h$$

$$a^n u[n] * (-b^n u[-n-1]) \longleftrightarrow \frac{1}{(1-az^{-1})(1-bz^{-1})}$$

$$\downarrow \qquad \downarrow$$

$$\frac{1}{1-az^{-1}} \qquad \frac{1}{1-bz^{-1}} \qquad |a| < |z| < |b|$$

$|z| > |a| \qquad |z| < |b|$

$$X(\theta) \equiv X(e^{j\theta})$$

Multiplication in the time Domain

$$x[n]\, h[n] \xleftrightarrow{DTFT} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta) H(\omega - \theta)\, d\theta$$

$$\sum_{n=-\infty}^{\infty} x[n]\, h[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta) e^{j\theta n}\, d\theta \right] h[n]\, e^{-j\omega n}$$

$$\overset{?}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta) \left[\sum_{n=-\infty}^{\infty} h[n] e^{-j(\omega-\theta)n} \right] d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta) H(\omega - \theta) d\theta$$ circular convolution

What about corresponding z-transform property?

$$x[n]\, h[n] \xleftrightarrow{Z} ?$$

Will turn out to be complex convolution

[Complex Variables & Laplace Transform for Engineers
— Wilbur LePage, Dover]

Initial Value Theorem $\quad x[n] = 0 \quad n < 0$

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots$$

$$\operatorname{Lt}_{z \to \infty} X(z) = x[0]$$

Final Value Theorem Let $v[n]$ be right sided,

i.e. $v[n] = 0 \quad n < M$

Let $x[n] = v[n] - v[n-1]$

$$X(z) = (1 - z^{-1}) V(z)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=M}^{\infty} \left(v[n] - v[n-1] \right) z^{-n}$$

$$\mathrm{Lt}_{z \to 1} X(z) = \sum_{n=M}^{\infty} \left(v[n] - v[n-1] \right)$$

$$= \mathrm{Lt}_{N \to \infty} \sum_{n=M}^{N} \left(v[n] - v[n \; 1] \right)$$

$$= \mathrm{Lt}_{N \to \infty} \Big[v[M] - v[M-1] + v[M+1] - v[M] + \ldots + v[N-1] + v[N] \Big]$$

$$= \underset{N\to\infty}{Lt} \left[\underset{\substack{\downarrow \\ 0}}{-v[M-1]} + v[N] \right]$$

$$= v[\infty]$$

$$= \underset{z\to 1}{Lt} \; X(z)$$

$$= \underset{z\to 1}{Lt} \; (1 - z^{-1}) V(z)$$

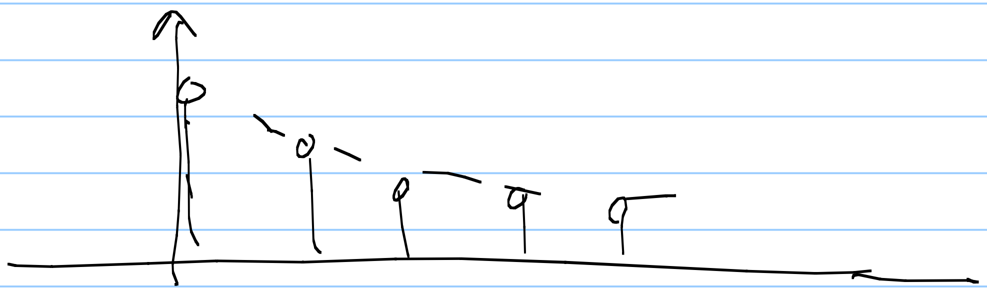
$$v[\infty] = \underset{z\to 1}{Lt} \; (1 - z^{-1}) V(z)$$

Eg: $x[n] = \left(\frac{1}{3}\right)^n u[n] + 2u[n] + \left(-\frac{1}{2}\right)^n u[n]$

$x[\infty] = 2$

Verify Final Value Theorem for above Example

$a^n u[n] \longleftrightarrow \dfrac{1}{1 - az^{-1}} \qquad |z| > |a|$



$a^{n} \overset{z}{\longleftrightarrow}$ does not exist

$1 \overset{z}{\longleftrightarrow}$ – do –

EC5142 Oct 17 2011

$x[n] * h[n] \longleftrightarrow X(z)\,H(z) \qquad RoC \supseteq RoC_x \cap RoC_h$

$x[n]\,h[n] \longleftrightarrow$ complex convolution

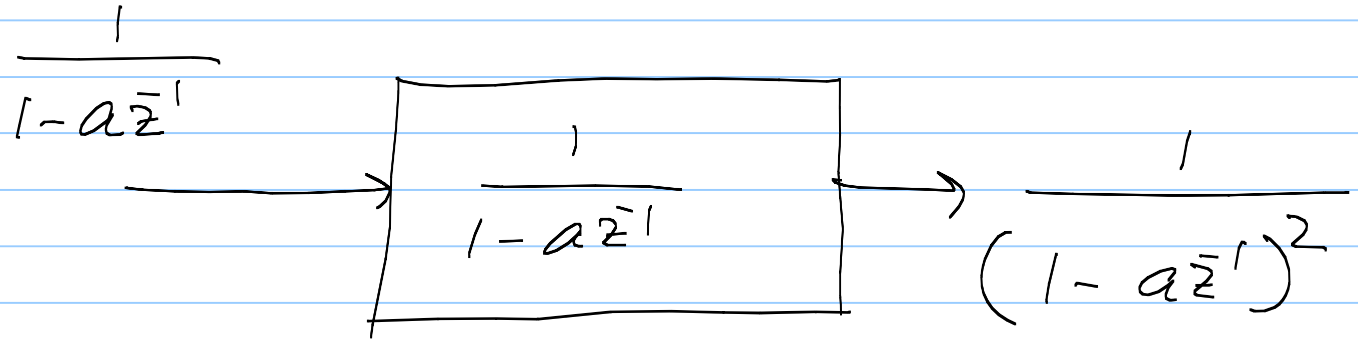
$a^n u[n] = x[n] \xleftrightarrow{z} \dfrac{1}{1-az^{-1}} \qquad |z| > |a|$

$h[n] = a^n u[n]$

$y[n] - a\,y[n-1] = x[n]$

$x[n] = b^n u[n]$

$$x(n) = a^n u(n)$$



Methods of Inversion

If $X(z)$ is rational, i.e $X(z) = \frac{\overset{M^{th}\text{ order}}{B(z)}}{\underset{N^{th}\text{ order}}{A(z)}}$

$$= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1\,\cancel{a_0} + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{b_0 \prod_{l=1}^{M} (1 - z_l z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

Suppose $b_0 = b_1 = b_2 = \cdots = b_{k-1} = 0$

$$X(z) = \frac{z^{-k} B_1(z)}{A(z)}$$

i.e. We can find the inverse transform of $\frac{B_1(z)}{A(z)}$

& shift the inverse transform appropriately.

Similarly for $A(z)$

Assume $M < N$, simple roots.

$$X(z) = \frac{B(z)}{\prod_{k=1}^{N} (1 - p_k z^{-1})} \longrightarrow \text{partial fractions}$$

$$= \sum_{k=1}^{N} \frac{A_k \longrightarrow \text{residues}}{1-p_k z^{-1}}$$

Matlab:

residue(b, a)

$$A_k = \left. \frac{B(z)}{\prod_{\substack{l=1 \\ l \neq k}}^{n} \left(1-p_l z^{-1}\right)} \right|_{z=z_k}$$

$$\frac{A_k}{1-p_k z^{-1}} \longrightarrow A_k (p_k)^n u[n]$$

or

$$-A_k (p_k)^n u[-n-1]$$

depending on RoC

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$= \frac{2}{1-z^{-1}} + \frac{-1}{1-\frac{1}{2}z^{-1}}$$

(a) $|z| < \frac{1}{2}$ (b) $\frac{1}{2} < |z| < 1$ (C) $|z| > 1$

<u>$M \geq N$</u>

$$X(z) = \frac{B(z)}{A(z)} = \sum_{r=1}^{M-N} C_r z^{-r} + \left.\frac{B_1(z)}{A(z)}\right\}$$ proper form rational

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 2 + \frac{B_1(z)}{A(z)}$$

proceed as before

Repeated Roots:

$$X(z) = \frac{G \prod_{i=1}^{M} (1 - a_i z^{-1})}{\prod_{q=1}^{Q} (1 - b_q z^{-1}) \prod_{l=1}^{R} (1 - \gamma_l z^{-1})^{\sigma_l}} \qquad M < N$$

$$N = Q + \sum_{l=1}^{R} \sigma_l$$

$$X(z) = \sum_{q=1}^{Q} \frac{A_q}{1 - b_q z^{-1}} + \sum_{l=1}^{R} \sum_{k=1}^{\sigma_l} \frac{c_{lk}}{(1 - \gamma_l z^{-1})^{k}}$$

γ_l : root , σ_l = multiplicity

$$A_q = X(z)(1 - b_q z^{-1}) \Big|_{z = b_q}$$

$$c_{lk} = \frac{1}{(-\gamma_l)^{\sigma_l - k} (\sigma_l - k)!} \frac{d^{\sigma_l - k}}{d\xi^{\sigma_l - k}} \left[X(\xi^{-1})(1 - \sigma_l \xi)^{\overset{\text{fixed}}{\sigma_l}} \right] \Bigg|_{\xi = \frac{-1}{\gamma_l}}$$

Eg

$$X(z) = \frac{12 - 22z^{-1} + 16z^{-2}}{(1-2z^{-1})^3}$$

$R = 1$

$\ell = 1$

$\sigma_\ell = \sigma_1 = 3$

$$C_{1,3} = \frac{1}{(-2)^0 \, 0!} \frac{d^0}{d\xi^0} \left[\frac{12 - 22\xi + 16\xi^2}{(1-2\xi)^3} (1-2\xi)^3 \right] \Bigg|_{\xi = 1/2} = 5$$

$$C_{1,2} = \frac{1}{(-2)^1 \, 1!} \frac{d^1}{d\xi^1} \left[\frac{12 - 22\xi + 16\xi^2}{(1-2\xi)^3} (1-2\xi)^3 \right] \Bigg|_{\xi = 1/2} = 3$$

$$C_{11} = \frac{1}{(-2)^2\, 2!} \frac{d^2}{d\xi^2}\left[\frac{12-22\xi+16\xi^2}{(1-2\xi)^3}(1-2\xi)^3\right]\Bigg|_{\xi=1/2} = 4$$

$$X(z) = \frac{4}{1-2z^{-1}} + \frac{3}{(1-2z^{-1})^2} + \frac{5}{(1-2z^{-1})^3}$$

$$a^n u(n) \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

$$(n+1)\,a^n u[n] \longleftrightarrow \frac{1}{(1-az^{-1})^2} \qquad |z|>|a|$$

$$? \longleftrightarrow \frac{1}{(1-az^{-1})^3} \qquad |z|>|a|$$

Power Series Expansion

$$X(z) = \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

$$= 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots\cdots$$

$$\longleftrightarrow \{\ldots, 0, 0, \underset{\uparrow n=0}{1}, a, a^2, a^3, \cdots\cdots\} = a^n u[n]$$

EC 5142 Oct. 18 2011

$$X(z) = \frac{1}{1-az^{-1}}$$

$$|az^{-1}| < 1 \Rightarrow |a| < |z|$$

$$= |z| > |a|$$

$$= 1 + az^{-1} + a^{2}z^{-2} + \dots$$

$$\longleftrightarrow \{0, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, a, a^{2}, \dots\} = a^{n}u[n]$$

For anticausal resp., i.e. $|z| < |a|$, we need powers of z, rather than z^{-1}

$$X(z) = \frac{-a^{-1}z}{1 - a^{-1}z}$$

$$= -a^{-1}z\left[1 + a^{-1}z + a^{-2}z^{2} + \ldots\right] \qquad |a^{-1}z| < 1$$

$$= |z| < |a|$$

$$= -a^{-1}z - a^{-2}z^{2} - a^{-3}z^{3} - \ldots$$

$n = -1$ $\quad$ $n = -2$ $\quad$ $n = 0$

$$\longleftrightarrow \left\{ \ldots, -a^{-3}, -a^{-2}, -a^{-1}, \overset{\downarrow}{0}, 0, 0 \right\}$$

$$-a^{n}u[-n-1]$$

Long Division

$$\begin{array}{r|l} & 1 + az^{-1} + \cdots\cdot \\ \hline 1 - az^{-1} & 1 \\ & \underline{1 - az^{-1}} \\ & az^{-1} \\ & \underline{az^{-1} - a^{2}z^{-2}} \\ & a^{2}z^{-2} \\ & \vdots \end{array}$$

causal response

Anticausal response:

$$\begin{array}{r|l} & -a^{-1}z - a^{-2}z^{2} - \\ \hline -a^{-1}z + 1 & 1 \\ & \underline{1 - a^{-1}z} \\ & a^{-1}z \\ & \underline{a^{-1}z - a^{-2}z^{2}} \\ & a^{-2}z^{2} \\ & \vdots \end{array}$$

$$a^{|n|} = x[n] = a^{n} u[n] + a^{-n} u[-n-1]$$

$$\downarrow$$

$$a^{n} u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

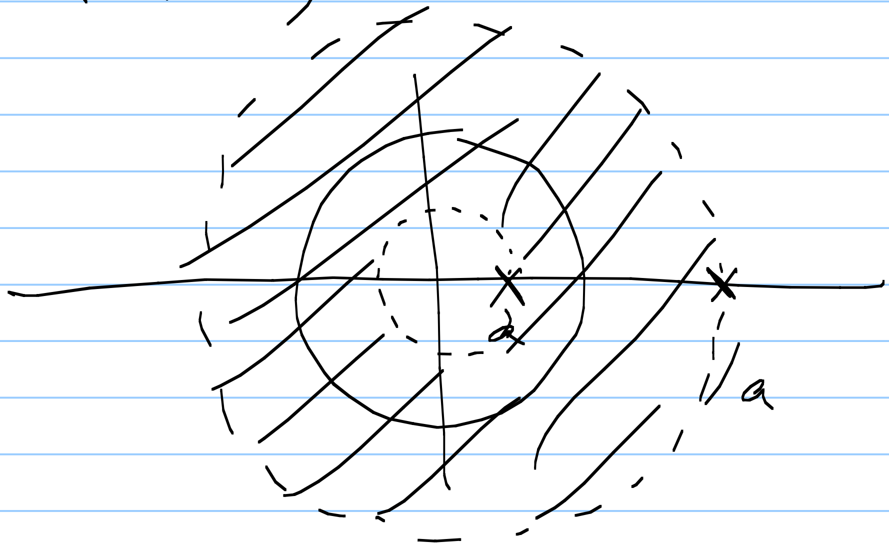
$$a^{-n} u[-n] \longleftrightarrow \frac{1}{1-az} \qquad |z| < \left|1/a\right|$$

$$a^{-n-1} u[-n-1] \longleftrightarrow \frac{z}{1-az} \qquad |z| < \left|1/a\right|$$

$$-a^{-n} u[-n-1] \longleftrightarrow \frac{az}{1-az} \qquad |z| < \left|1/a\right|$$

$$a^{|n|} \longleftrightarrow \frac{1}{1-az^{-1}} + \frac{az}{1-az} \qquad |a| < |z| < \frac{1}{|a|}$$

$$= \frac{1-a^2}{1+a^2-a(z+z^{-1})}$$



$$X(z) = b_{-q} z^{q} + b_{-(q-1)} z^{q-1} + \cdots + b_{-1} z + b_0 + b_1 z^{-1}$$
$$+ b_2 z^{-2} + \cdots + b_p z^{-p}$$

$$\longleftrightarrow \{ 0, 0, \underset{\overset{\uparrow}{n=-q}}{b_{-q}}, b_{-(q-1)}, \ldots, b_{-1}, \underset{\overset{\uparrow}{n=0}}{b_0}, b_1, \ldots, b_p, 0, \ldots \}$$

$$X(z) = z^{2} \left(1 - \tfrac{1}{2} z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

$x[n] =$

$$X(z) = e^{z} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \qquad |z| < \infty$$

$$\longleftrightarrow \left\{ \dots, \underset{n=-3}{\frac{1}{3!}}, \underset{n=-2}{\frac{1}{2!}}, \underset{n=-1}{\frac{1}{1!}}, \underset{\underset{n=0}{\uparrow}}{\frac{1}{1}}, 0, 0, \dots \right\} = x[n]$$

$$X(z) = e^{z^{-1}} \longleftrightarrow x[-n]$$

$$X(z) = \ln(1 + az^{-1}) \qquad |z| > |a|$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n z^{-n}}{n}$$

$$\longleftrightarrow$$

Can also get same answer using differentiation in the z-domain property

Contour Integral Method

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\frac{1}{2\pi j}\oint X(z) z^{k-1} dz = \frac{1}{2\pi j}\oint \left[\sum_{n=-\infty}^{\infty} x[n] z^{-n}\right] z^{k-1} dz$$

$$\overset{?}{=} \sum_{n=-\infty}^{\infty} x[n] \frac{1}{2\pi j}\oint z^{k-n-1} dz$$

$$\frac{1}{2\pi j}\oint z^{n}\,dz = \begin{cases} 1 & n=-1 \\ 0 & n\neq -1 \end{cases}$$

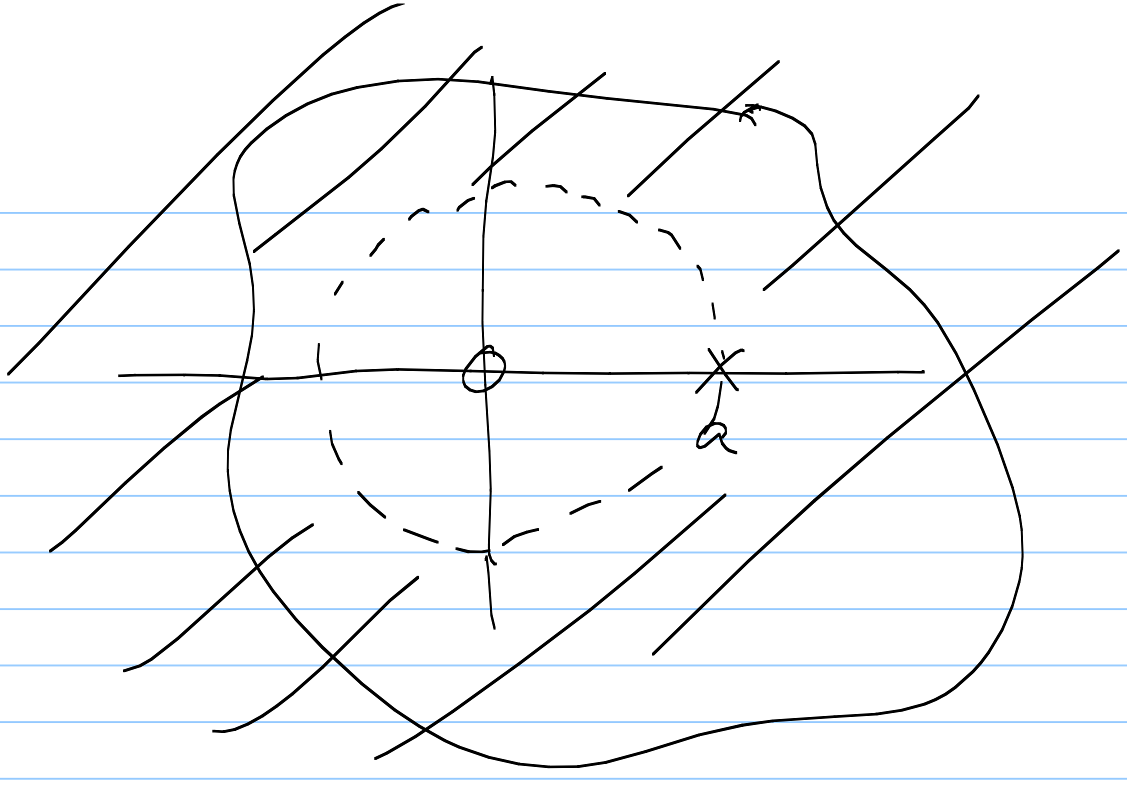
$$\frac{1}{2\pi j}\oint z^{k-n-1}\,dz = \begin{cases} 1 & k=n \\ 0 & k\neq n \end{cases}$$

$$\frac{1}{2\pi j}\oint X(z)\, z^{k-1}\,dz = x[k]$$

$$X(z) = \frac{1}{1-az^{-1}}$$

$$= \frac{z}{z-a}$$

$$|z| > |a|$$



$$\frac{1}{2\pi j} \oint \frac{z}{z-a} \, z^{n-1} \, dz$$

$$= \frac{1}{2\pi j} \oint \frac{z^{n}}{z-a} \, dz$$

Using Residue theorem:

$$\text{Res}\left[\frac{X(z)}{z-a} \right]_{z=a} = X(z)\Big|_{z=a} = a^{n} \qquad n \geq 0$$

$$x[n] = a^{n} \qquad n \geq 0$$

[Can also see Markushevich]

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]$$

$$a_0 = 1$$

$$y(n) = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{l=0}^{M} b_l x[n-l]$$

Applying z-transform,

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{l=0}^{M} b_l z^{-l} X(z)$$

$$Y(z)\overbrace{\left[1+\sum_{k=1}^{N} a_k z^{-k}\right]}^{A(z)} = X(z)\overbrace{\sum_{l=0}^{M} b_l z^{-l}}^{B(z)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{1+\sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

$H(z)$: system transfer function

rational transfer function

$$H(z) = \frac{B(z)}{A(z)} = \frac{Y(z)}{X(z)}$$

$$A(z)\,Y(z) = B(z)\,X(z)$$

EC5142 19 Oct 2011

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{l=0}^{M} b_l\, x[n-l]$$

$$Y(z)\underbrace{\left[1+\sum_{k=1}^{N} a_k z^{-k}\right]}_{A(z)} = X(z)\underbrace{\left[\sum_{l=0}^{M} b_l z^{-l}\right]}_{B(z)} \qquad a_0 = 1$$

$$\frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = H(z) \leftarrow \text{Transfer function}$$

$$\updownarrow$$

$$h[n] \leftarrow \text{impulse response}$$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} \leftarrow \text{Frequency response}$$

Roots of $B(z)$ are <u>zeros</u> of the system

" " $A(z)$ " <u>poles</u> "

BIBO stable: $$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Recall $$|H(z_0)| < \infty \Rightarrow \sum_{n=-\infty}^{\infty} |h[n]| \, |z_0|^{-n} < \infty$$

$$\Rightarrow z_0 \in \text{RoC}$$

Consider a BIBO stable system:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

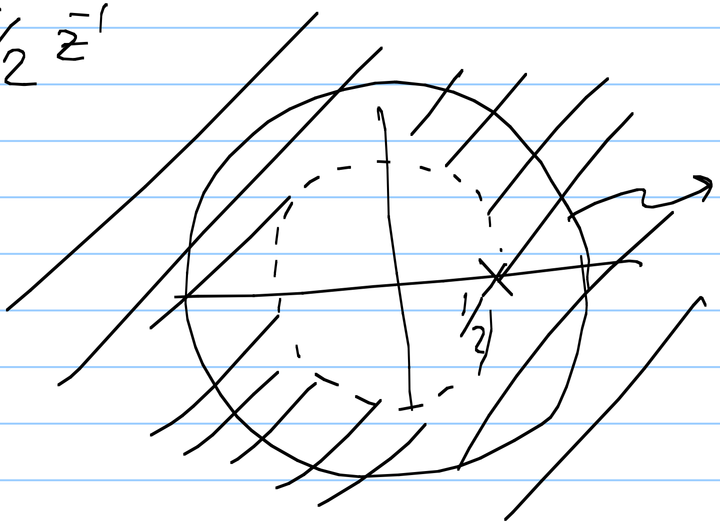
$$\sum_{n=-\infty}^{\infty} |h(n)| \, |1|^{-n} < \infty$$

$$\sum_{n=-\infty}^{\infty} |h[n]| \, |e^{j\omega}|^{-n} < \infty$$

$\Rightarrow \quad e^{jw} \in RoC$

Eg $h(n) = \left(\frac{1}{2}\right)^n u(n) \longleftrightarrow \dfrac{1}{1-\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$

$\Rightarrow \sum_{n=-\infty}^{\infty} |h(n)| < \infty$



$2^n u(n) \longleftrightarrow \dfrac{1}{1-2z^{-1}} \quad |z| > 2$

not stable

$$-2^{n}u[-n-1] \longleftrightarrow \frac{1}{1-2z^{-1}} \qquad |z| < 2$$

$e^{j\omega} \in ROC \Rightarrow$ stable

$$-(1/2)^{n}u(-n-1) \longleftrightarrow \frac{1}{1-1/2\,z^{-1}} \qquad |z| < 1/2$$

not stable

$$a^{|n|} \longleftrightarrow \frac{1-a^{2}}{1+a^{2}-a(z+z^{-1})} \qquad |a| < |z| < 1/|a|$$

Causality implies that RoC is a region outside a certain circle

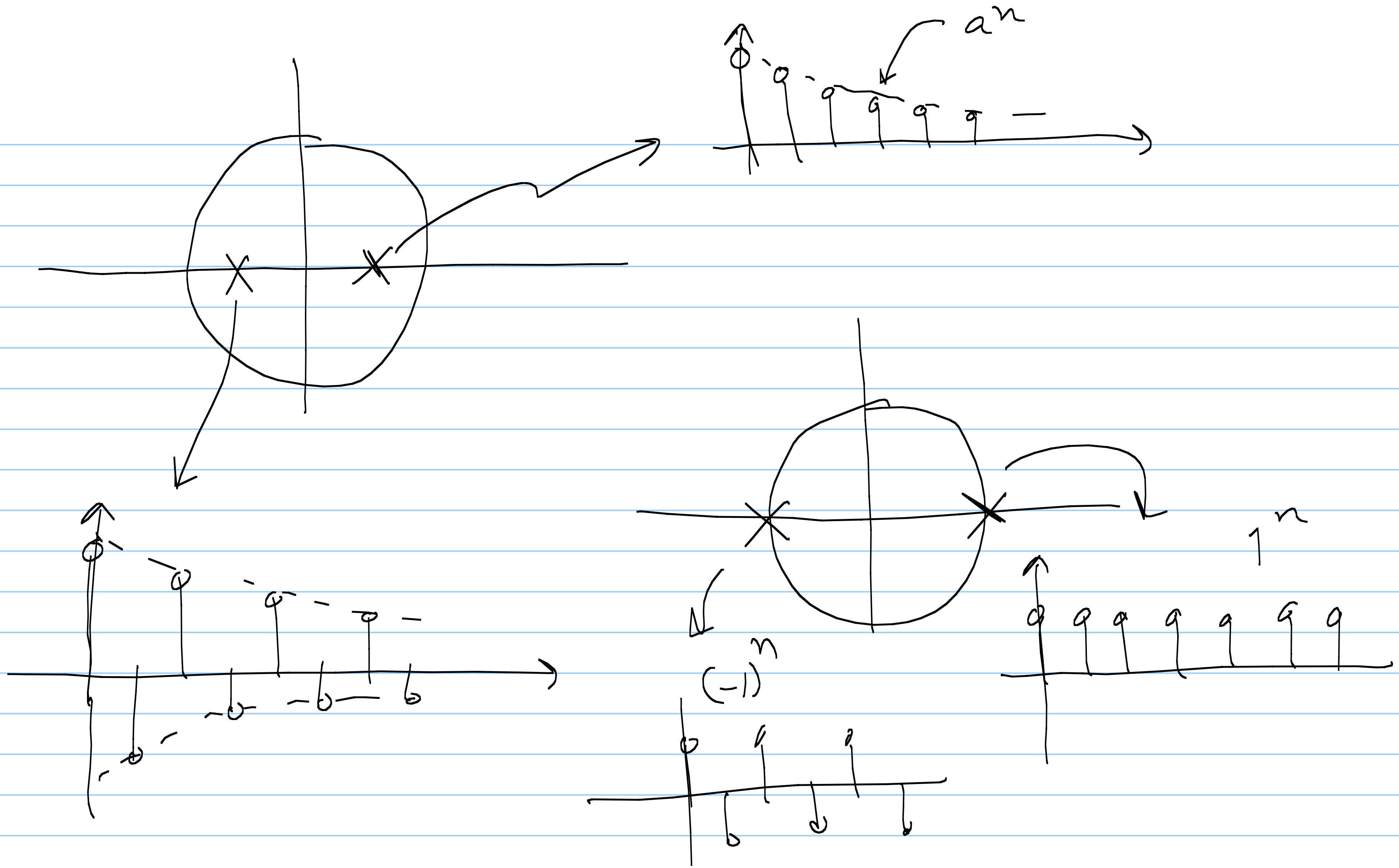
For rational TF, ROC is outside the circle corresponding to largest radius pole

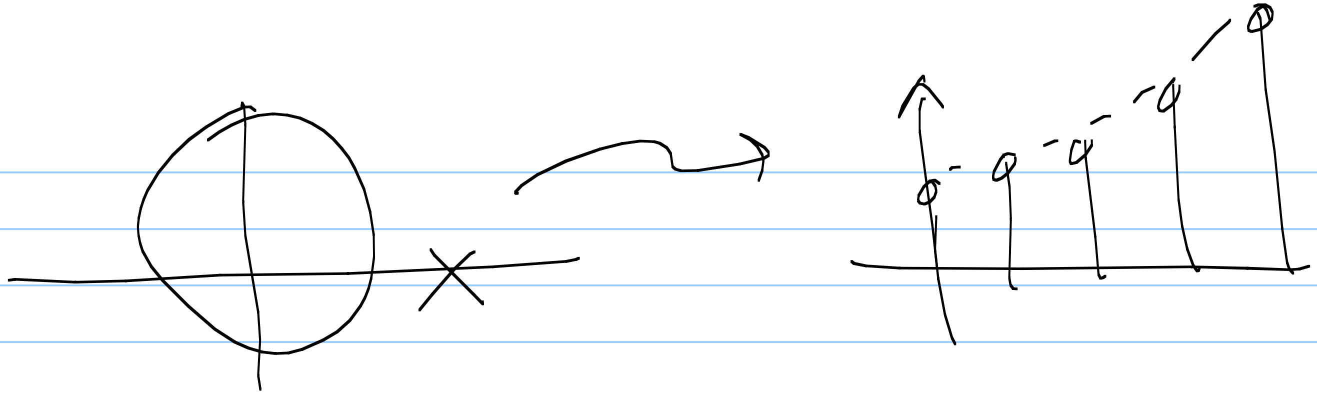
For stability, $e^{j\omega} \in$ RoC.

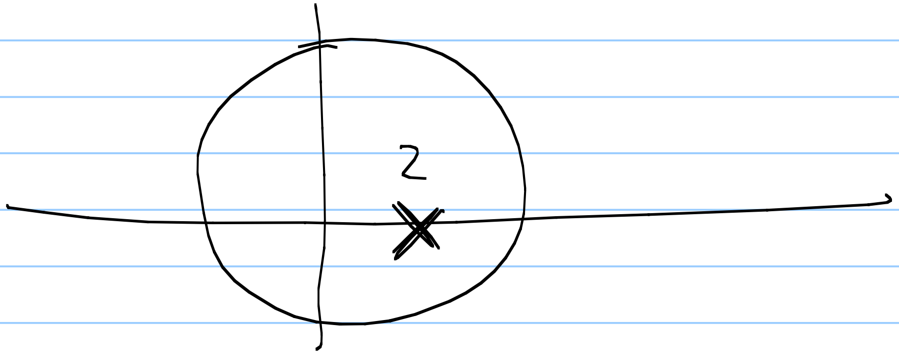
$\Rightarrow$ all poles must be <u>strictly</u>

<u>inside</u> the <u>unit circle</u>

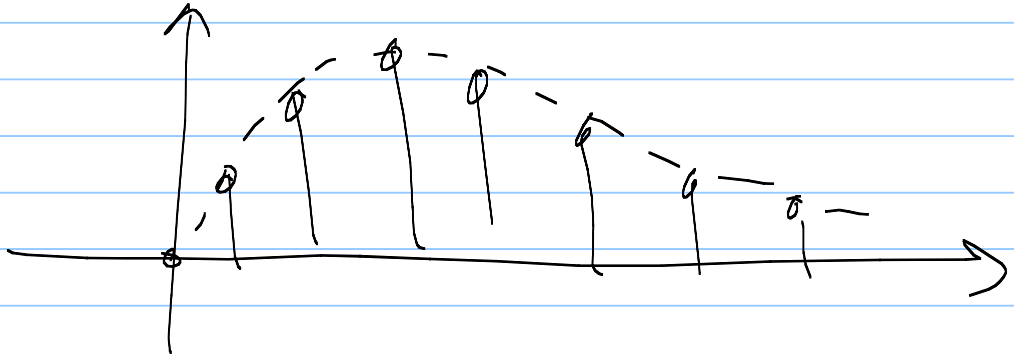
$h[n]$ is finite $\Longleftrightarrow$ $H(z)$ except for 0 and/or ∞

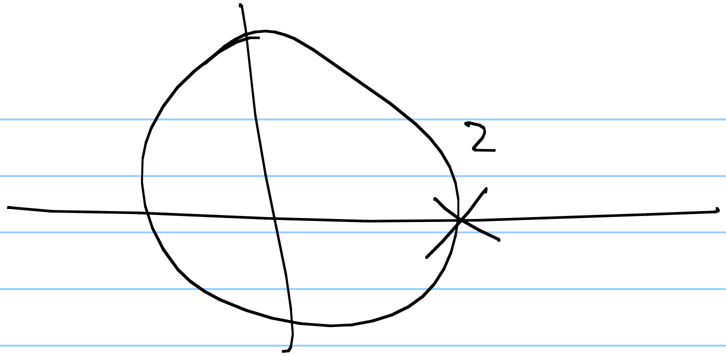

a^n
$(-1)^n$
1^n





$n a^n u[n]$

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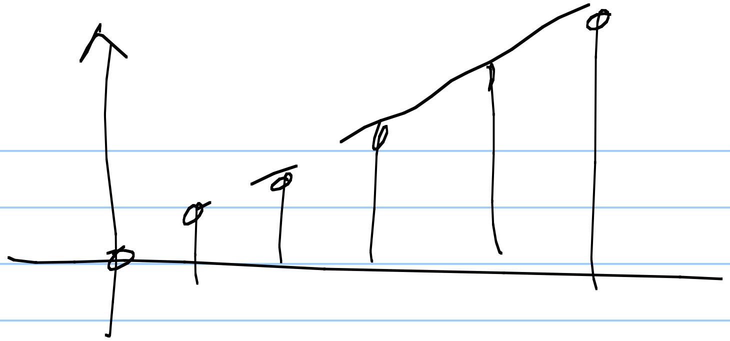

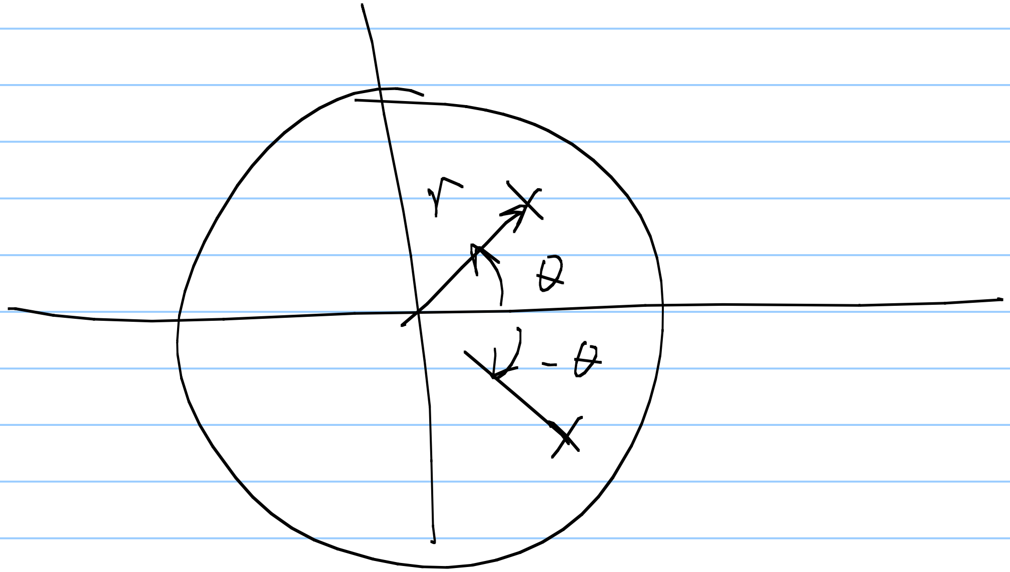
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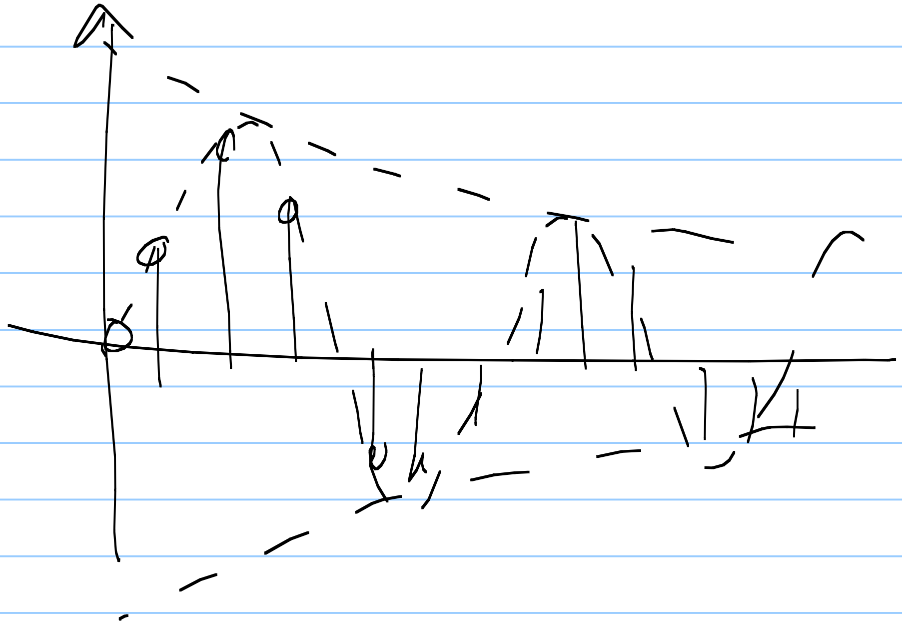


If uncancelled nontrivial poles exist,

then $h(n)$ will contain terms like $a^n u(n)$ or $n a^n u(n)$, etc., which last (theoretically) for ∞ duration $\Rightarrow$ IIR systems
(infinite impulse response)

Otherwise, $h(n)$ is finite: FIR

finite impulse response

$$H(z)\Big|_{z=e^{j\omega}} = H(e^{j\omega})$$

$$H(z) = \frac{B(z)}{A(z)} = K \, \frac{\prod_{l=1}^{M} (1 - z_l z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

$$= K \, z^{N-M} \, \frac{\prod_{l=1}^{M} (z - z_l)}{\prod_{k=1}^{N} (z - p_k)}$$

trivial poles/zeros

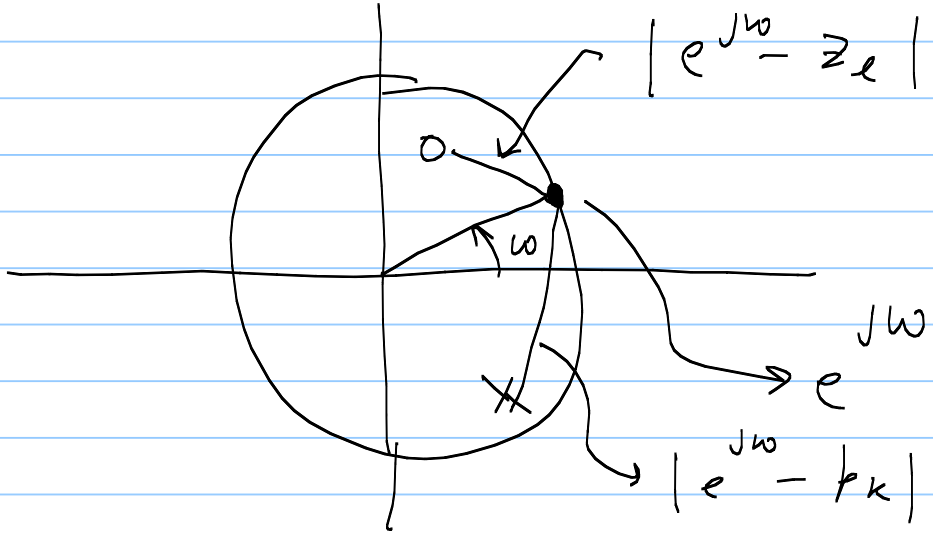
$$H(e^{jw}) = K \; e^{jw(N-M)} \frac{\prod_{l=1}^{M} (e^{jw} - z_l)}{\prod_{k=1}^{N} (e^{jw} - p_k)}$$

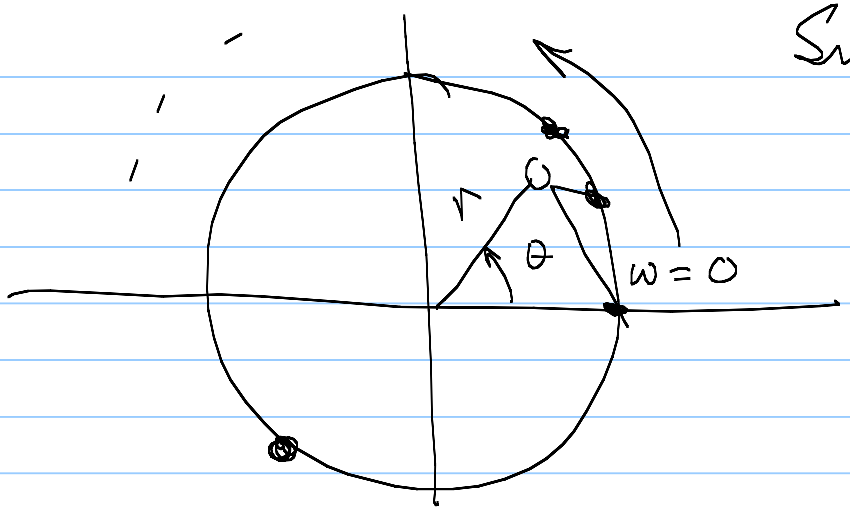
$$|H(e^{jw})| = |K| \frac{\prod_{l=1}^{M} |e^{jw} - z_l|}{\prod_{k=1}^{N} |e^{jw} - p_k|}$$

Note Title 20-10-2011

EC 5142 20 Oct 2011

$$|H(e^{j\omega})| = |k| \underbrace{|e^{j\omega(N-M)}|}_{1} \frac{\prod_{l=1}^{M} |e^{j\omega} - z_l|}{\prod_{k=1}^{N} |e^{j\omega} - p_k|}$$





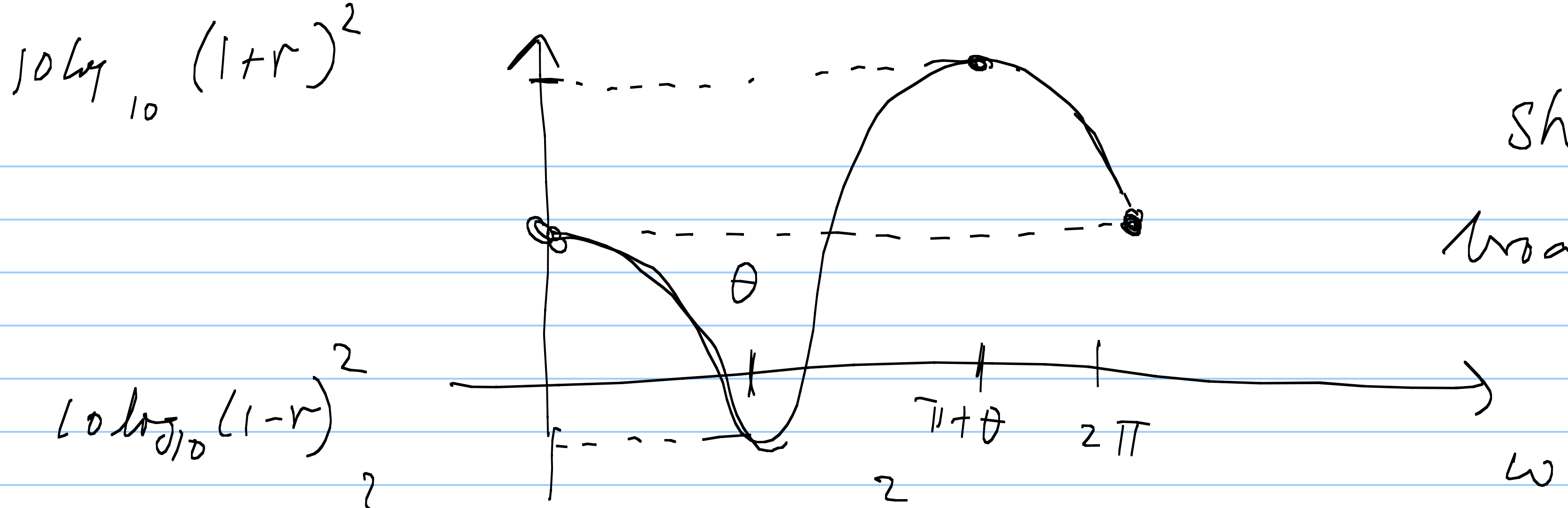
Single complex zero

$re^{j\theta}$

$$H(z) = 1 - re^{j\theta} z^{-1}$$

$$|H(e^{j\omega})|^2 = |1 - re^{j\theta} e^{-j\omega}|^2$$

$$= 1 + r^2 - 2r\cos(\omega - \theta)$$



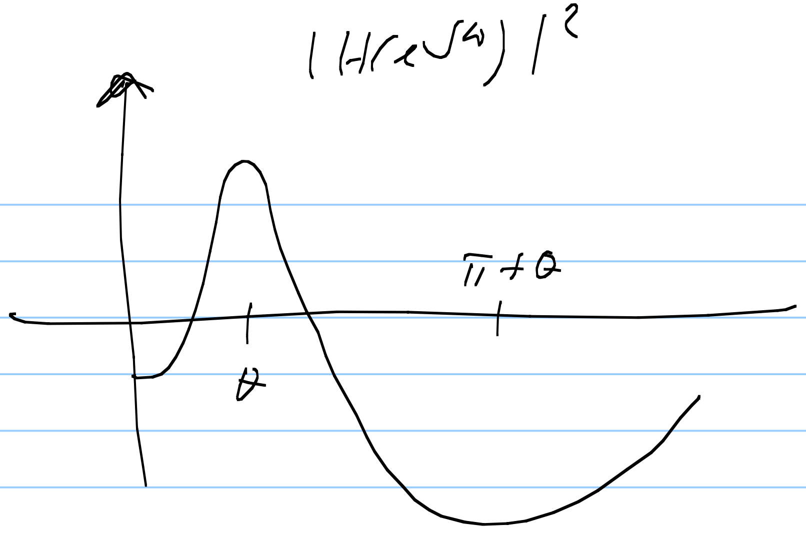
Sharper valley

broader peak

$$|H(e^{j\omega})|^2_{min} = (1-r)^2 \quad \text{for } \omega = \theta$$

$$|H(e^{j\omega})|^2_{Max} = (1+r)^2 \quad \text{for } \omega = \pi+\theta$$

$$H(z) = \frac{1}{1 - re^{j\theta} z^{-1}}$$

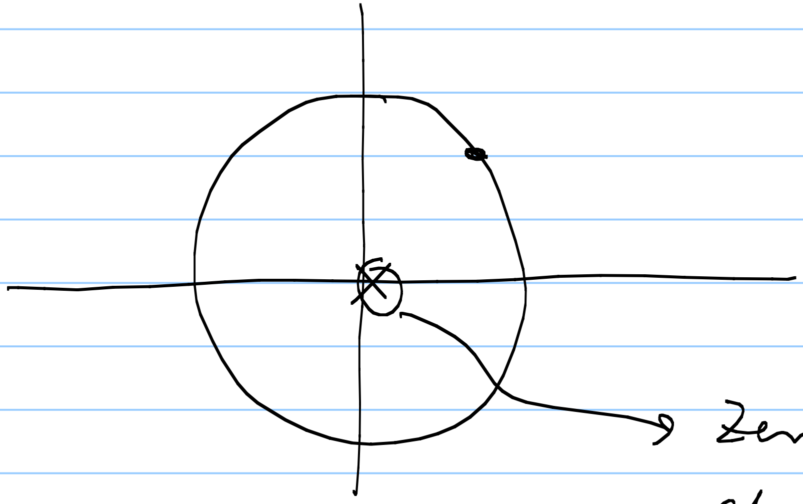


sharper peak

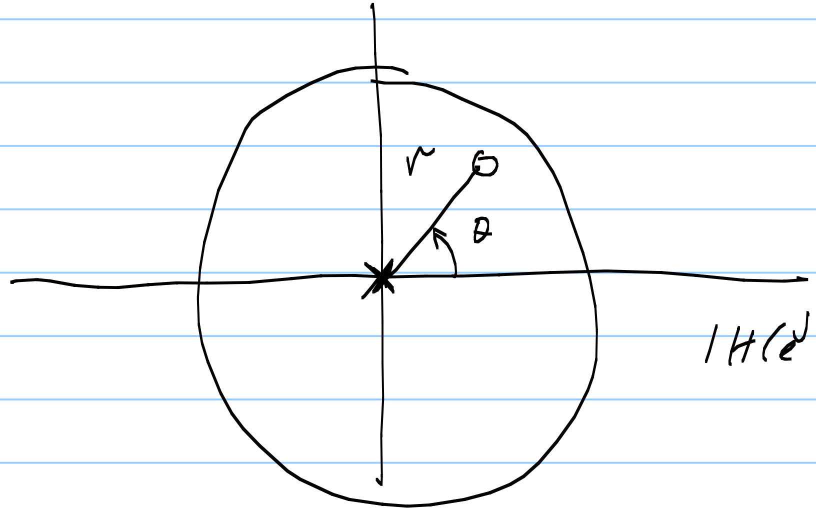
broader valley

EC 5142 Oct. 24 2011

$$|H(e^{j\omega})| = |K| \underbrace{|e^{j\omega(N-M)}|}_{\text{trivial poles \& zeros}} \frac{\prod_{l=1}^{M} |e^{j\omega} - z_l|}{\prod_{k=1}^{N} |e^{j\omega} - p_k|}$$

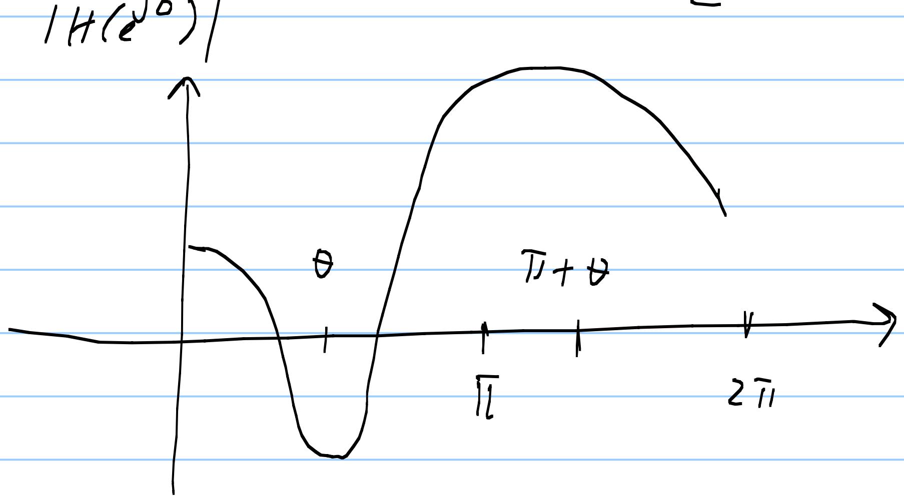


Zero or pole at the origin



$$H(z) = 1 - r e^{j\theta} z^{-1}$$

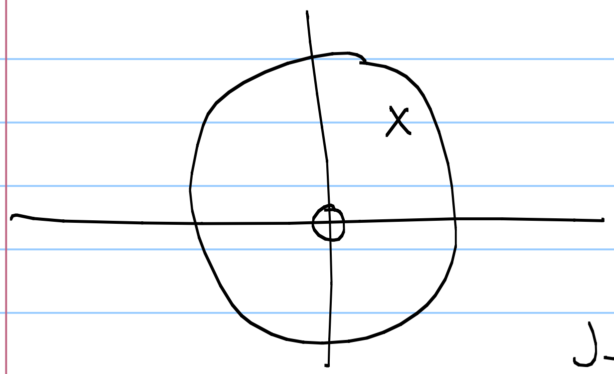
$$= \frac{z - r e^{j\theta}}{z}$$

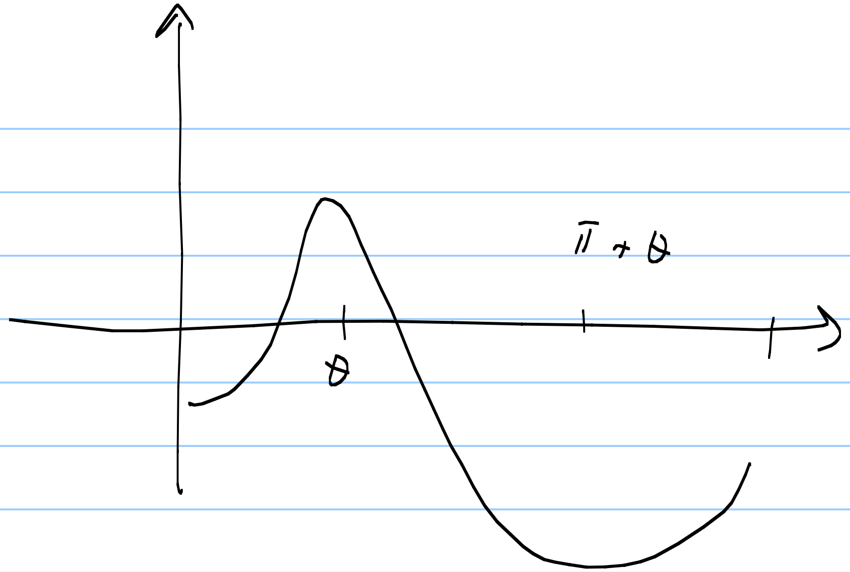


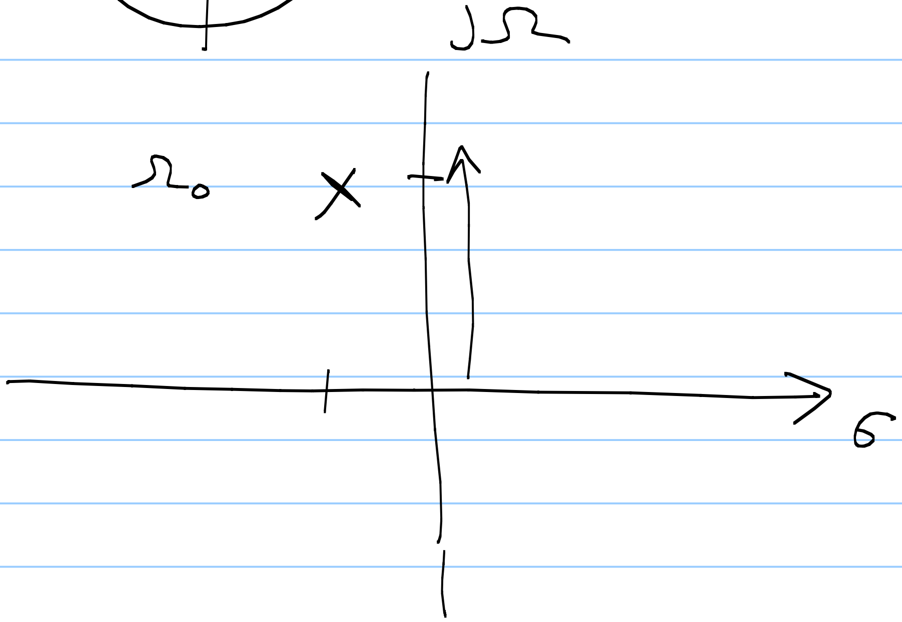
$$1 + r^2 - 2r\cos(\omega - \theta)$$

Plot $20 \log_{10} |H(e^{j\omega})|$ or $10 \log_{10} |H(e^{j\omega})|^2$

$$H(z) = \frac{1}{1 - r e^{j\theta} z^{-1}}$$



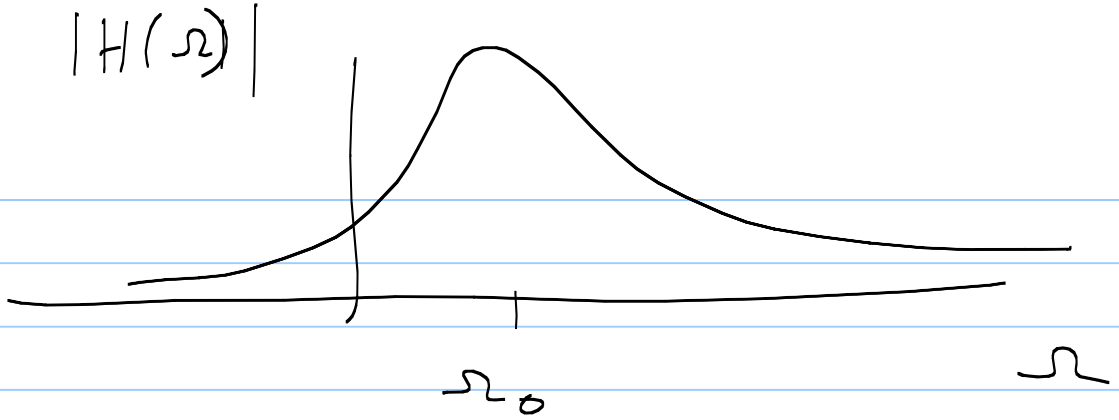


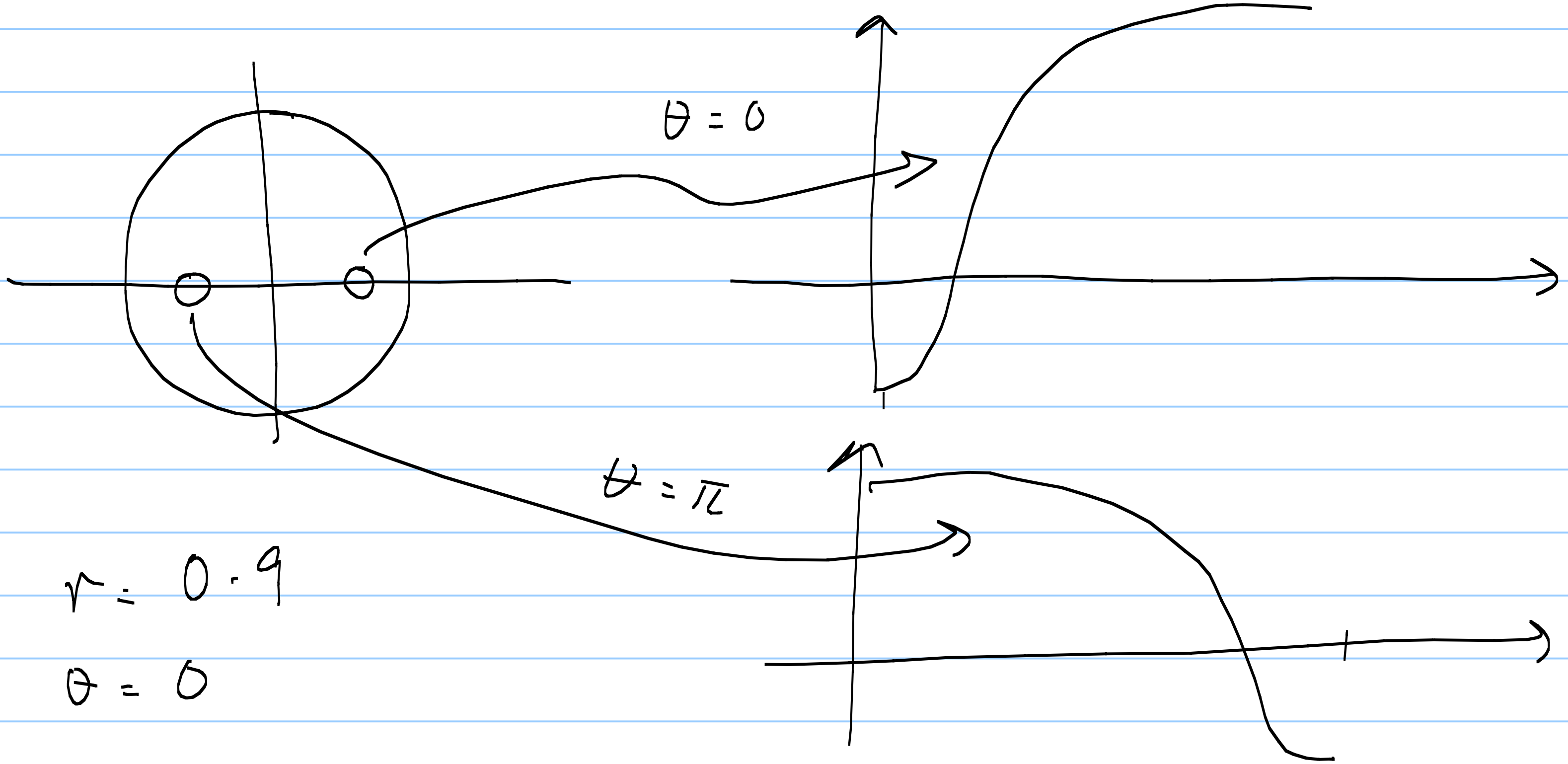


jΩ

$$H(s)\Big|_{s=j\Omega} = \frac{|K| \prod_{l=1}^{M} |(s - z_l)|}{\prod_{k=1}^{N} |s - p_k|}$$

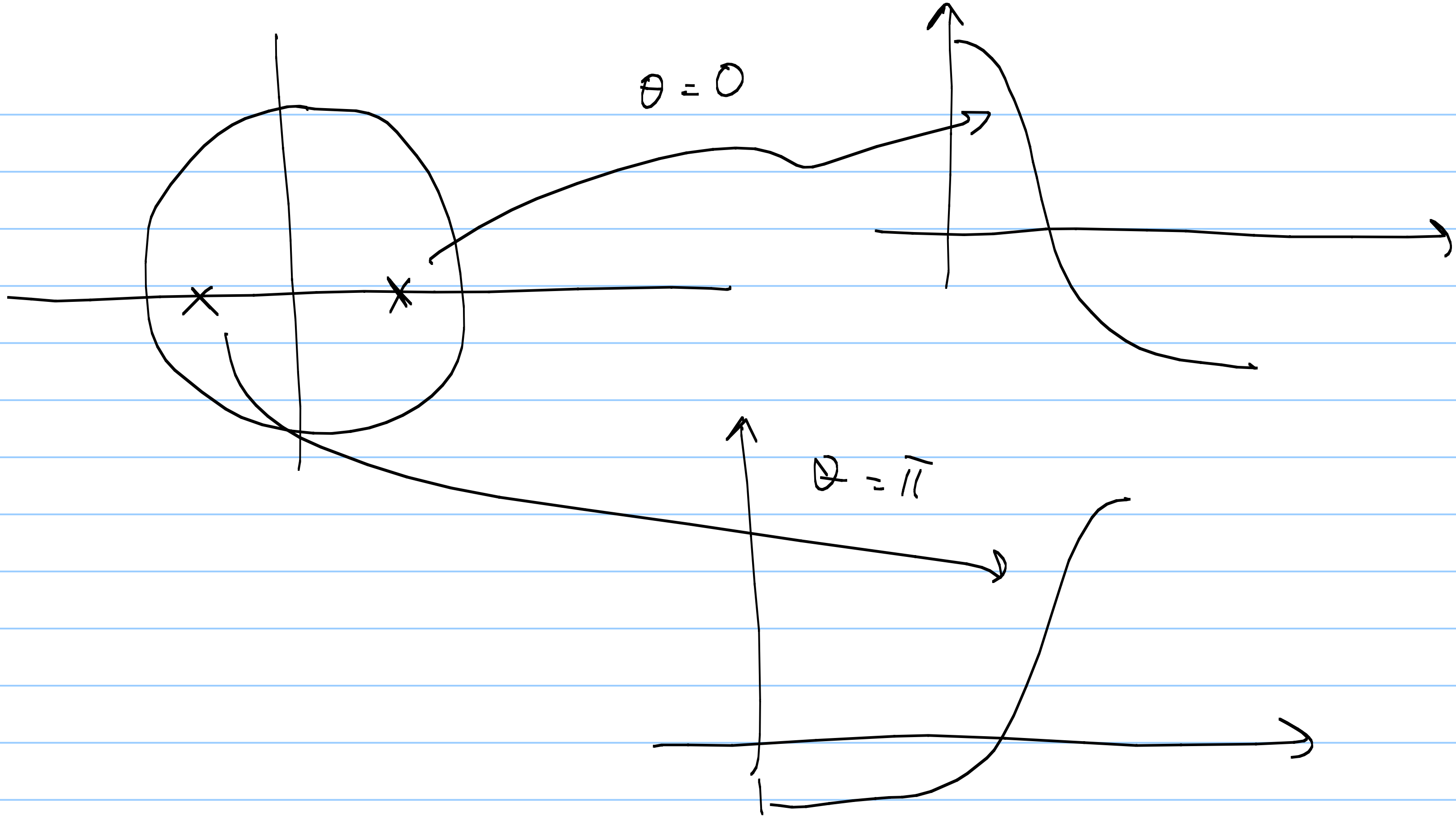
jΩ





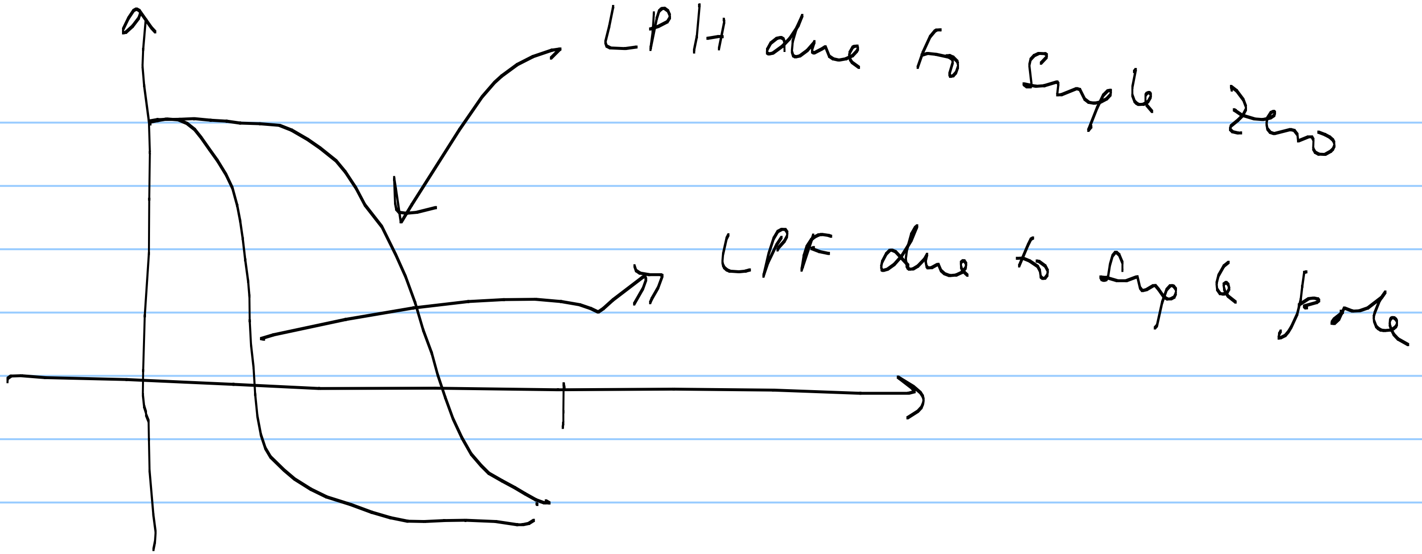
$r = 0.9$

$\theta = 0$


θ = 0
θ = π

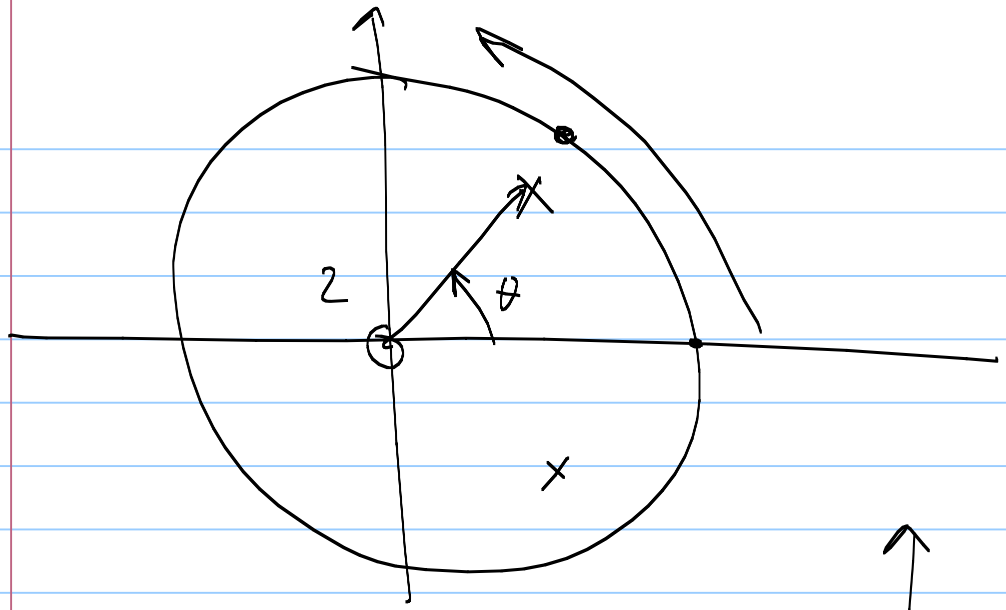
$r = 0.9$

Find the 3dB BW



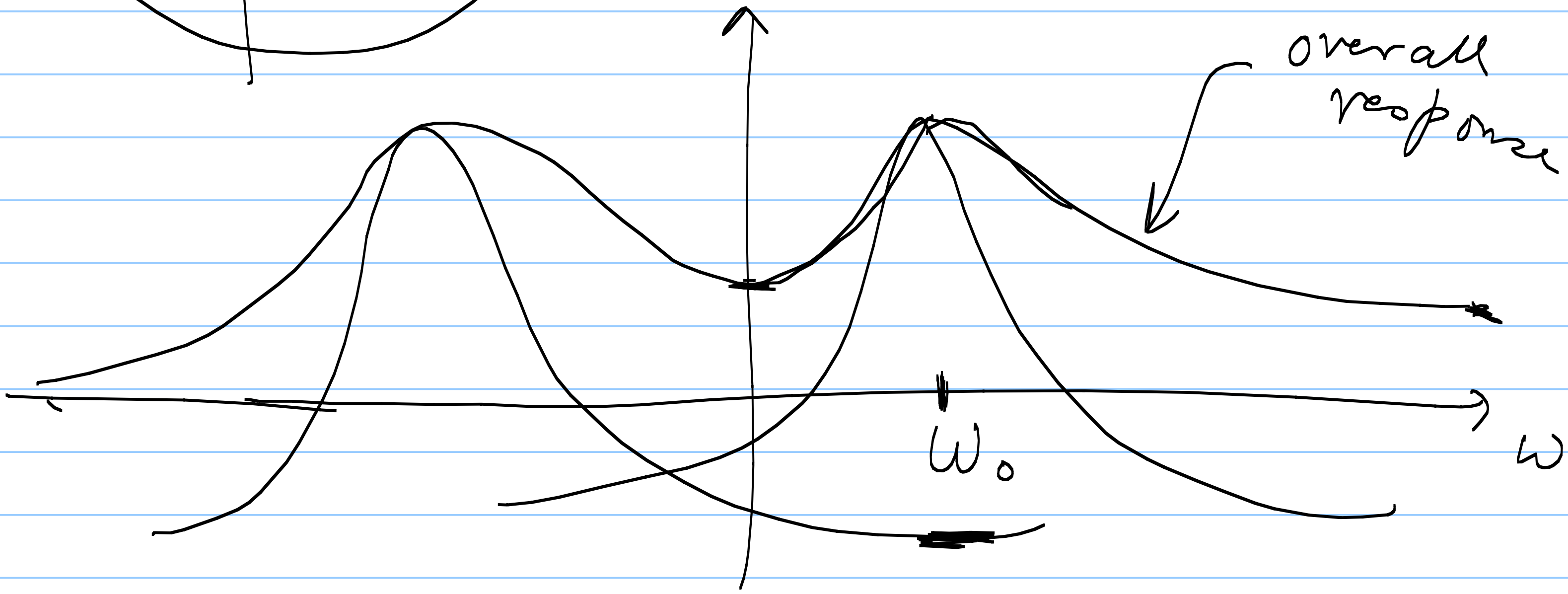
zero : $\pi/2$

pole : $\pi/30$



$$H(z) = \frac{1}{1 - 2r\cos\theta\, z^{-1} + r^{2} z^{-2}}$$

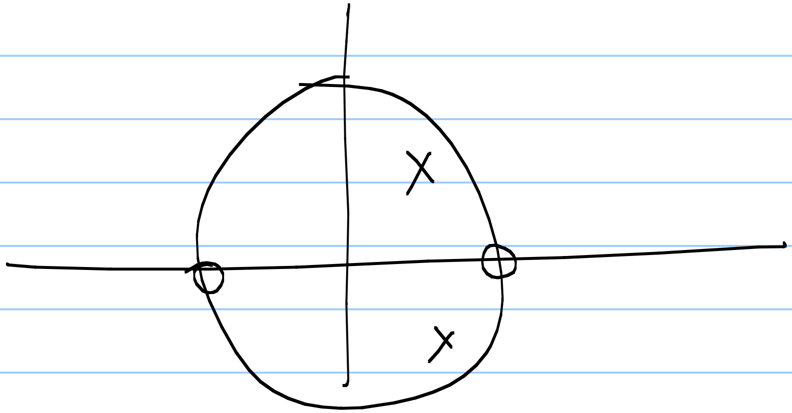
2nd order resonator



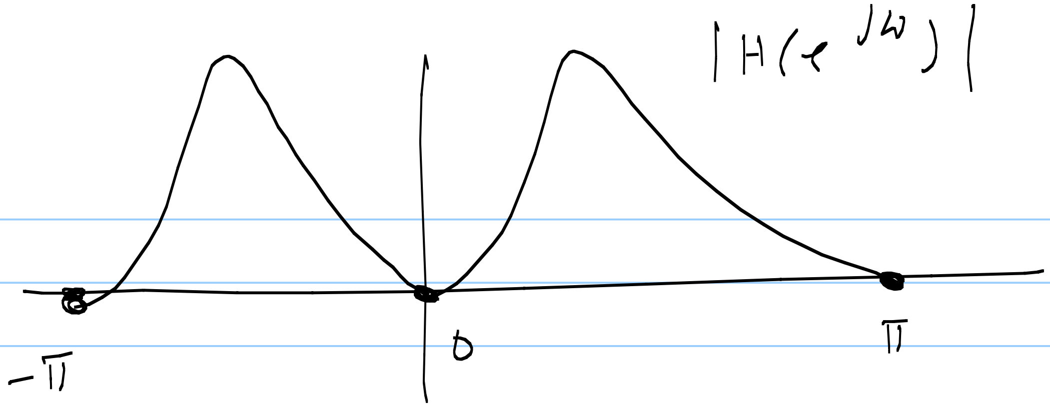
$$\omega_0 = \cos^{-1}\left[\frac{1+r^2}{2r}\cos\theta\right]$$

If $\theta = \pi/2$, $\omega_0 = \pi/2$

$$-1 \le \frac{1+r^2}{2r}\cos\theta \le 1$$

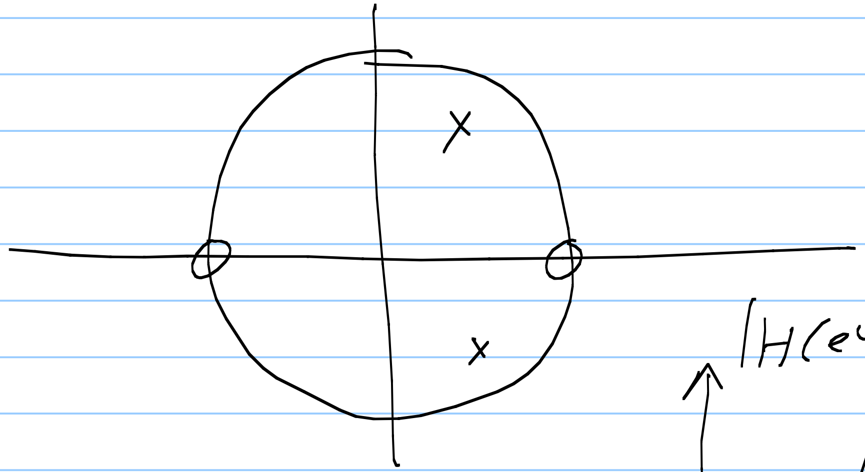


$$\frac{1-z^{-2}}{1-2r\cos\theta z^{-1}+r^2z^{-2}}$$

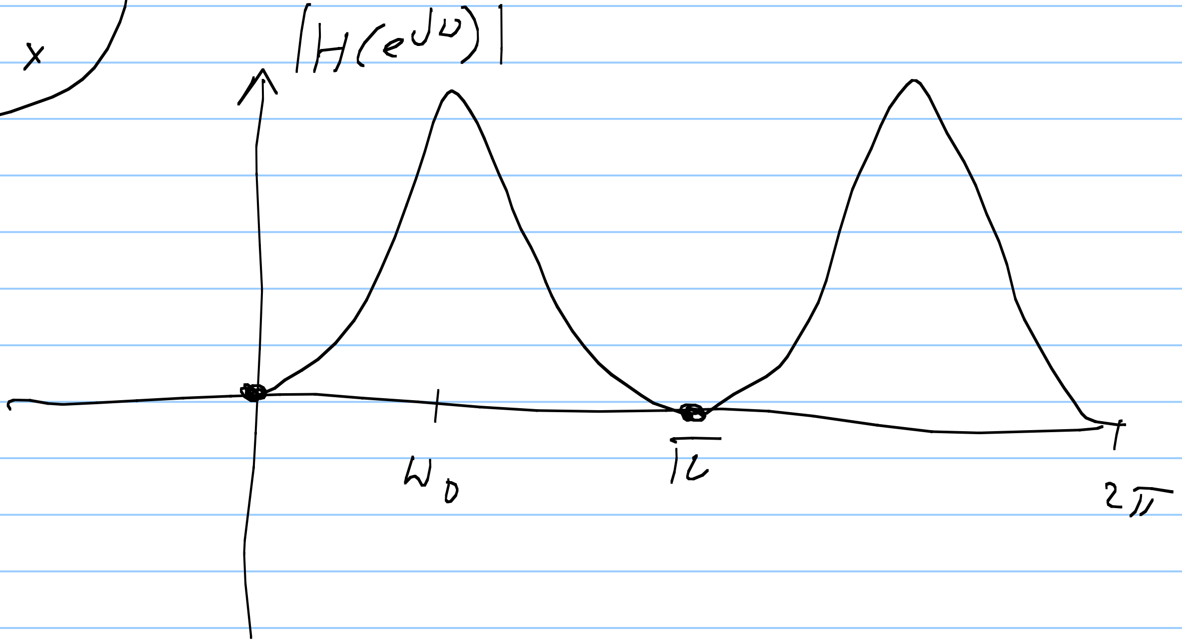

$|H(e^{j\omega})|$
$-\pi$
0
π

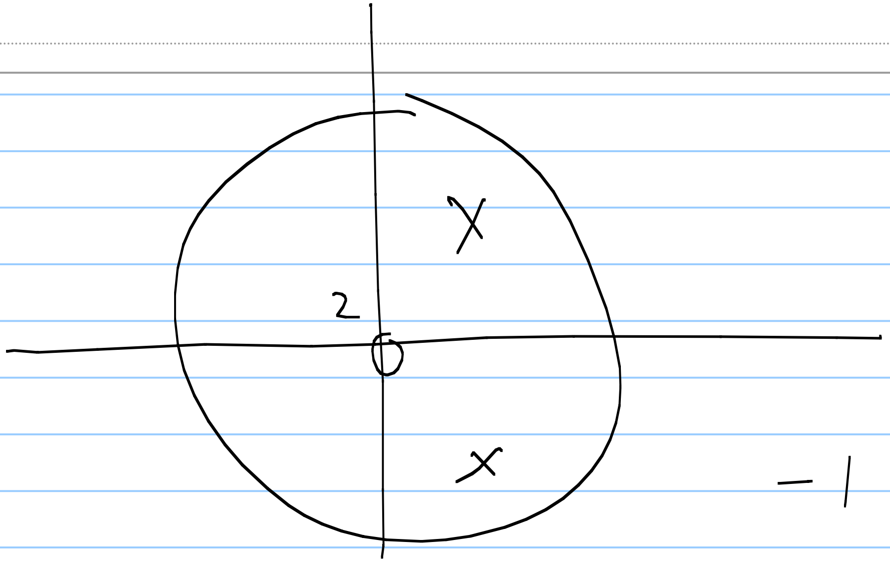
EC 5142 Oct 27 2011

Note Title 27-10-2011



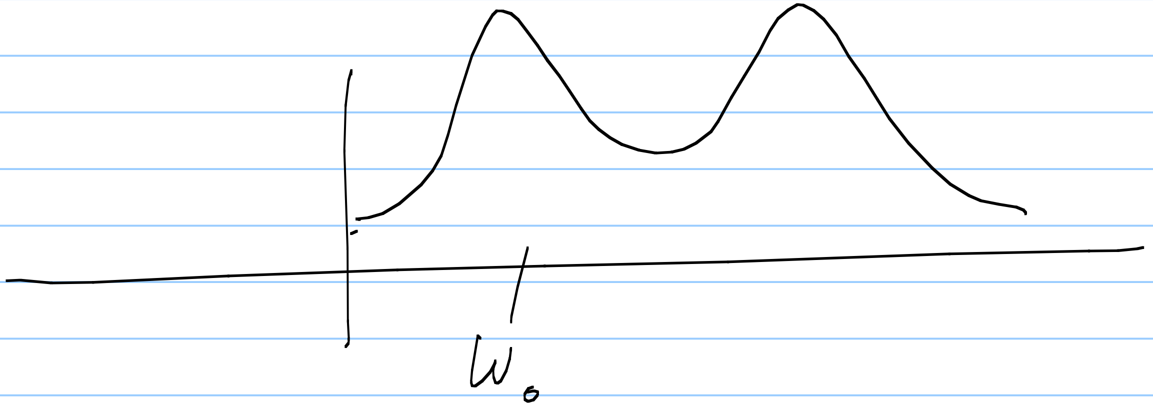
$$\frac{1 - z^{-2}}{1 - 2r\cos\theta\, z^{-1} + r^{2} z^{-2}}$$





$$W_0 = \cos^{-1}\left[\frac{1+r^2}{2r}\cos\theta\right]$$

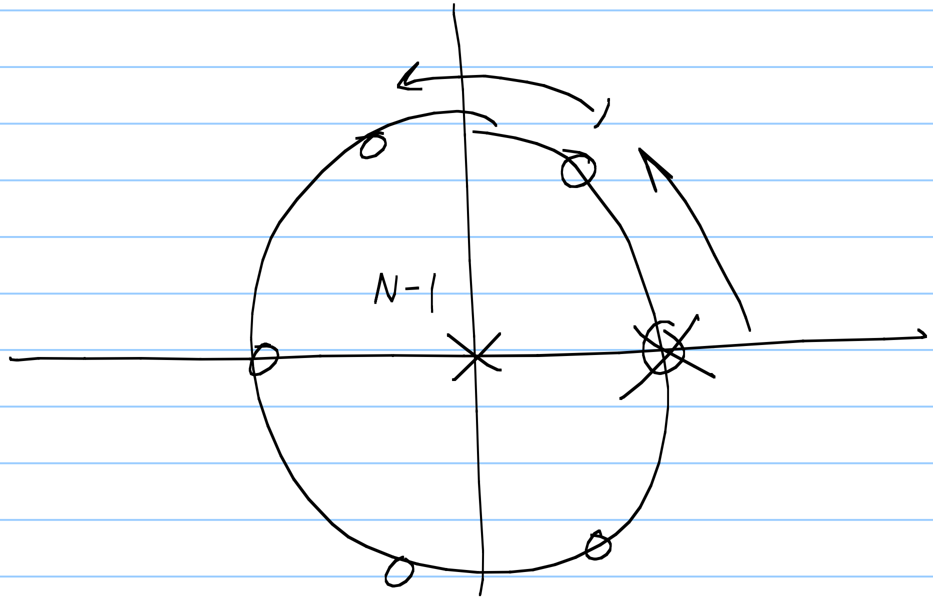
$$-1 \le \frac{1+r^2}{2r}\cos\theta \le 1$$

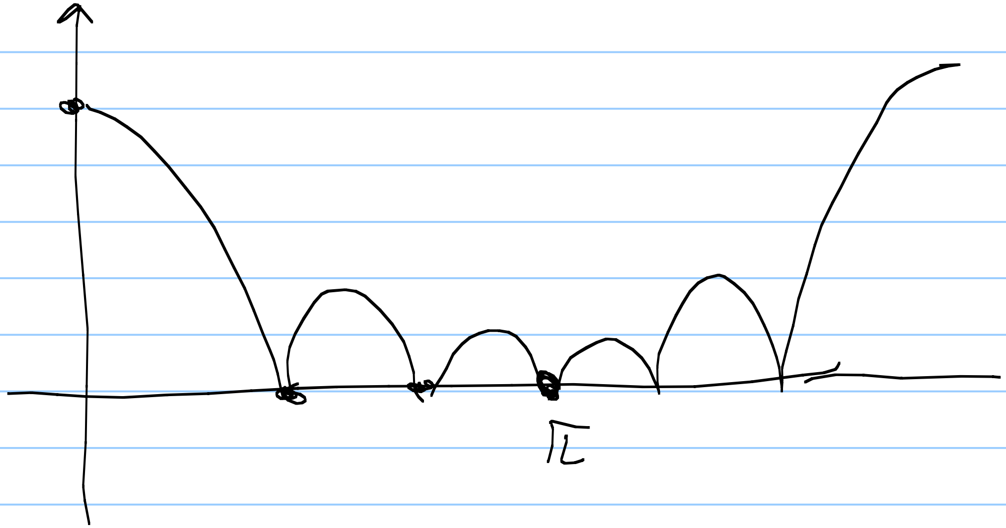


"freqz"

Matlab

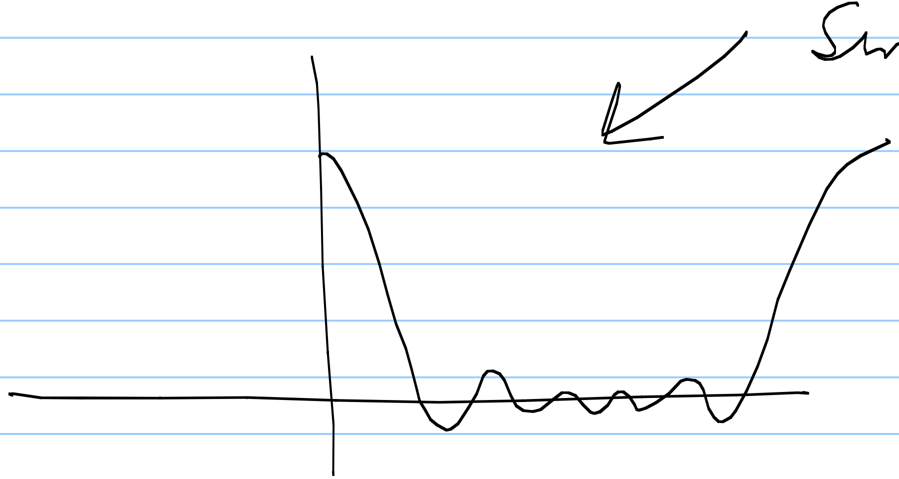
$h[n] = 1 \quad 0 \leq n \leq N-1 \longleftrightarrow \frac{1 - z^{-N}}{1 - z^{-1}}$

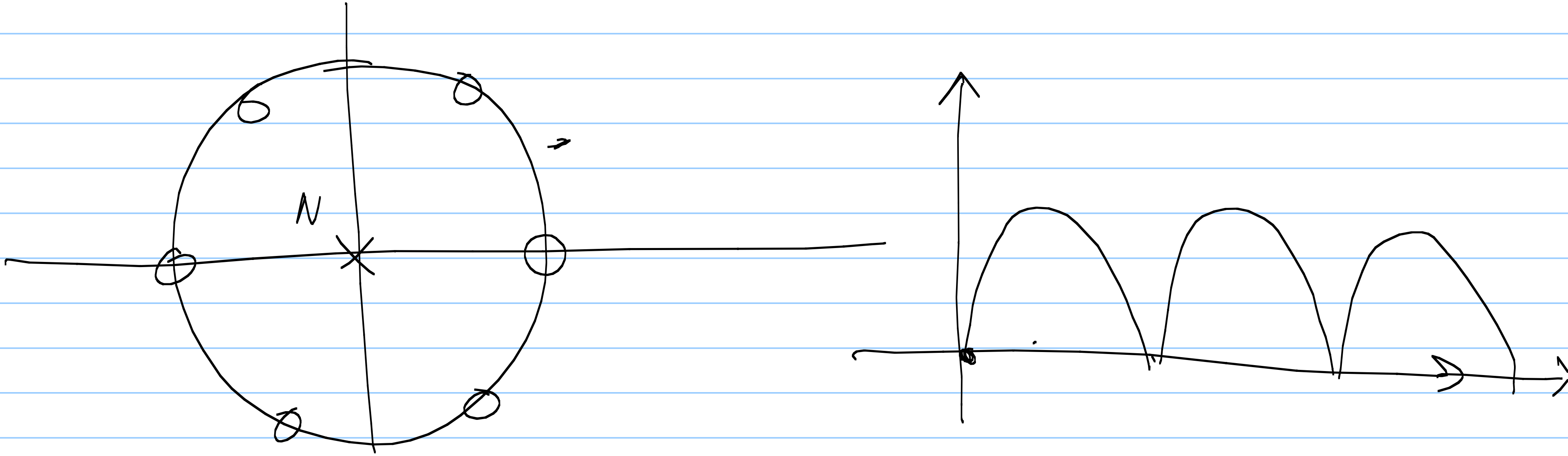




$$\left| \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \right| = \left| \frac{\cancelto{1}{e^{j\omega N/2}}}{\cancelto{1}{e^{j\omega/2}}} \; \frac{\sin N\omega/2}{\sin \omega/2} \right|$$

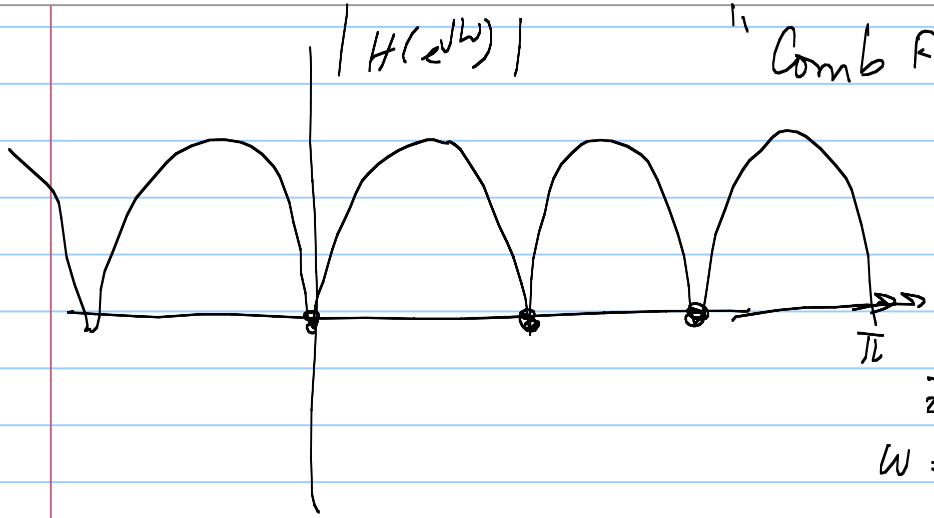
$$H(e^{j\omega}) = e^{j\omega(N-1)/2} \; \frac{\sin N\omega/2}{\sin \omega/2}$$

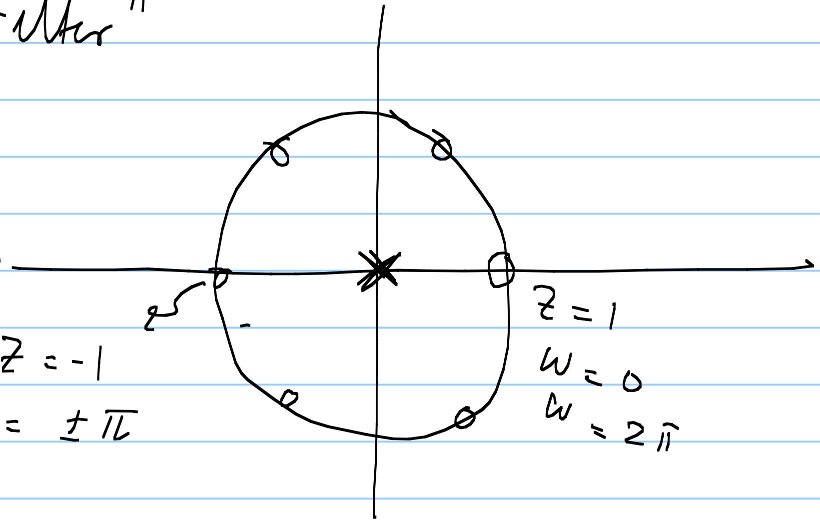



N

EC 5142 Oct. 31 2011

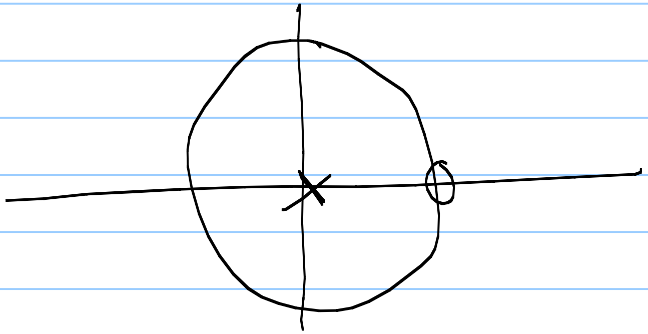
"Comb Filter"

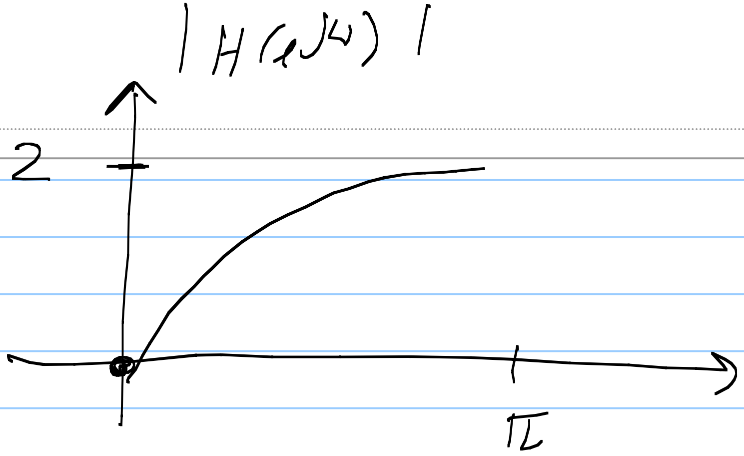




$H(z) = 1 - z^{-1}$

$= \frac{z-1}{z}$



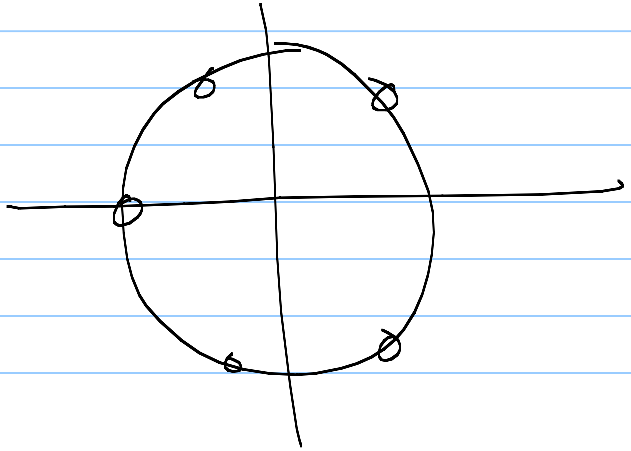


$$|1-e^{-j\omega}|$$

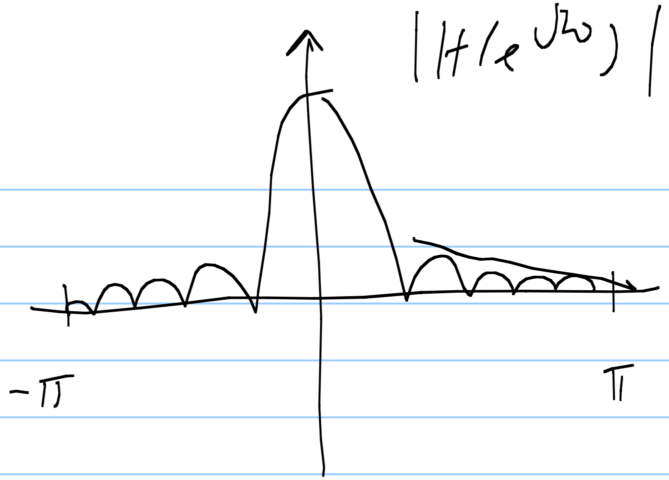
$$H(z^L) = 1 - z^{-L}$$

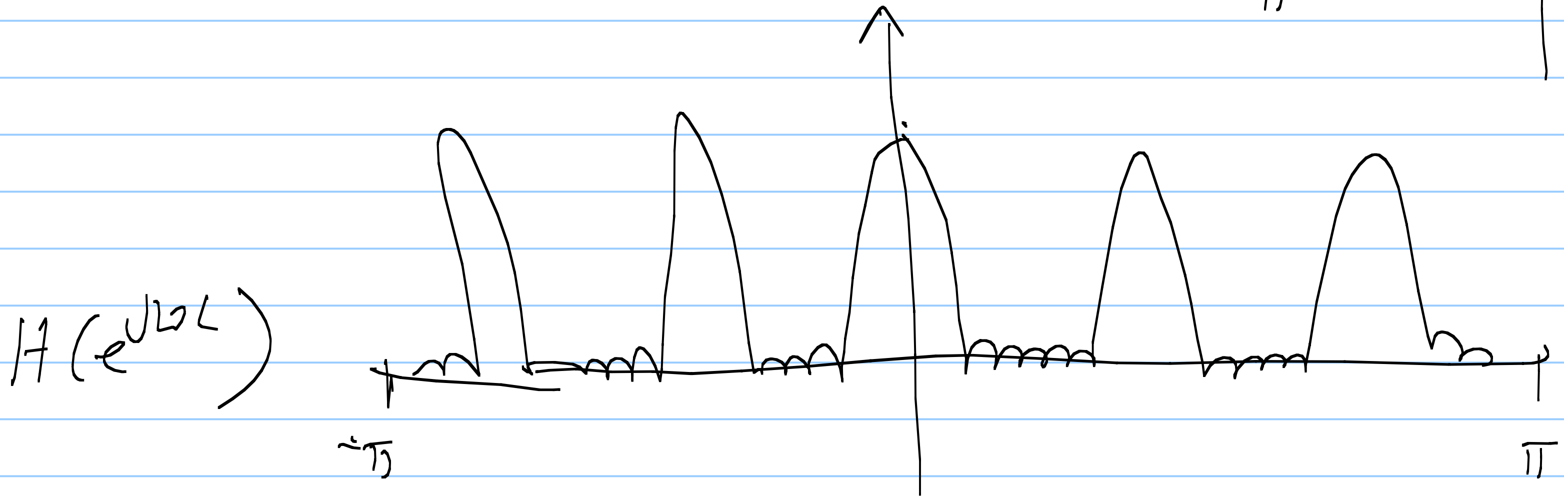
$$|1-e^{-j\omega L}|$$

$$H(z) = \frac{1-z^{-N}}{1-z^{-1}}$$



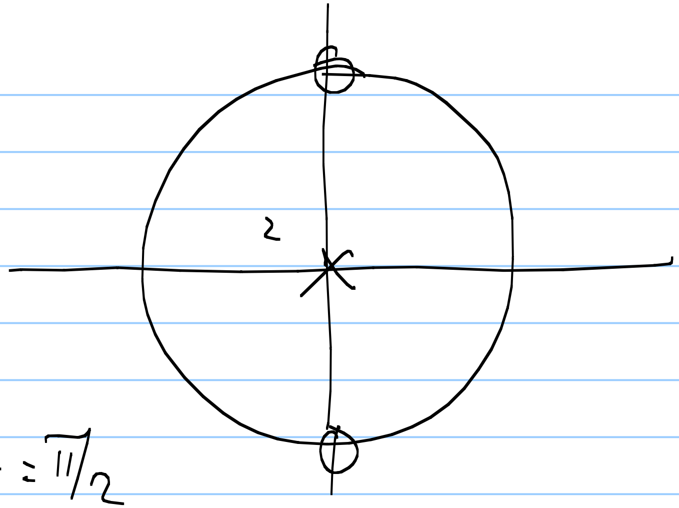
$$H(e^{j\omega}) = e^{-j(N-1)\omega/2} \frac{\sin N\omega/2}{\sin \omega/2}$$





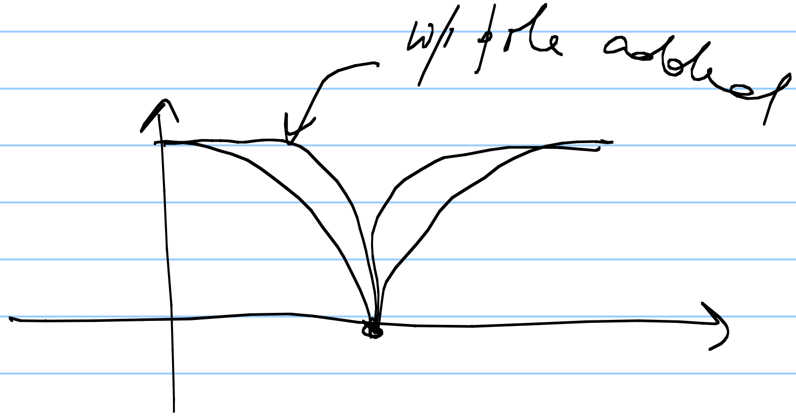
"Comb filter"

Notch Filter:

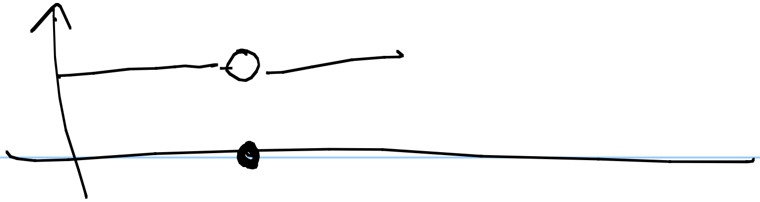


$H(z) = \quad 1 + z^{-2} \qquad r = 1, \; \theta = \pi/2$

$H(z) = \quad 1 - 2r\cos\theta \, z^{-1} + r^{2} z^{-2}$

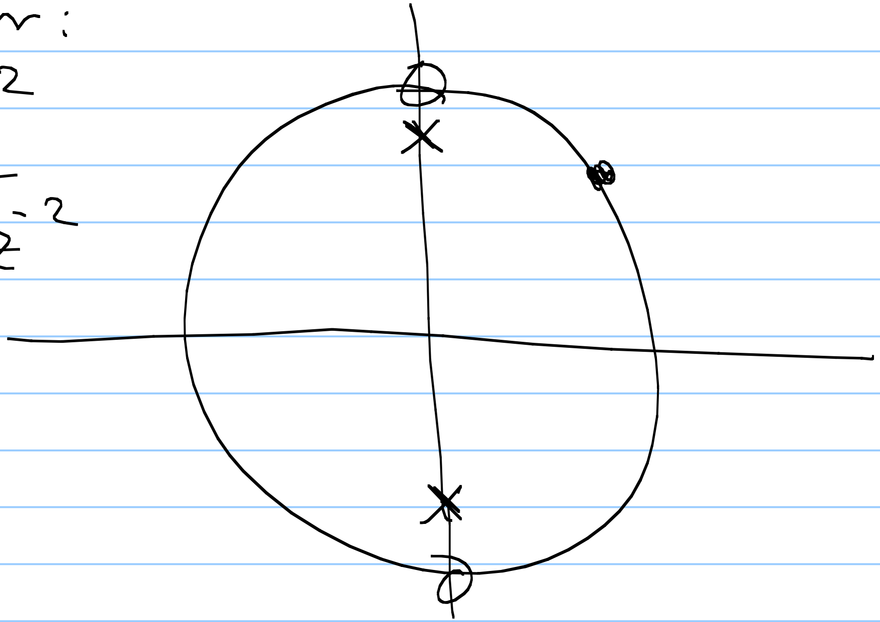


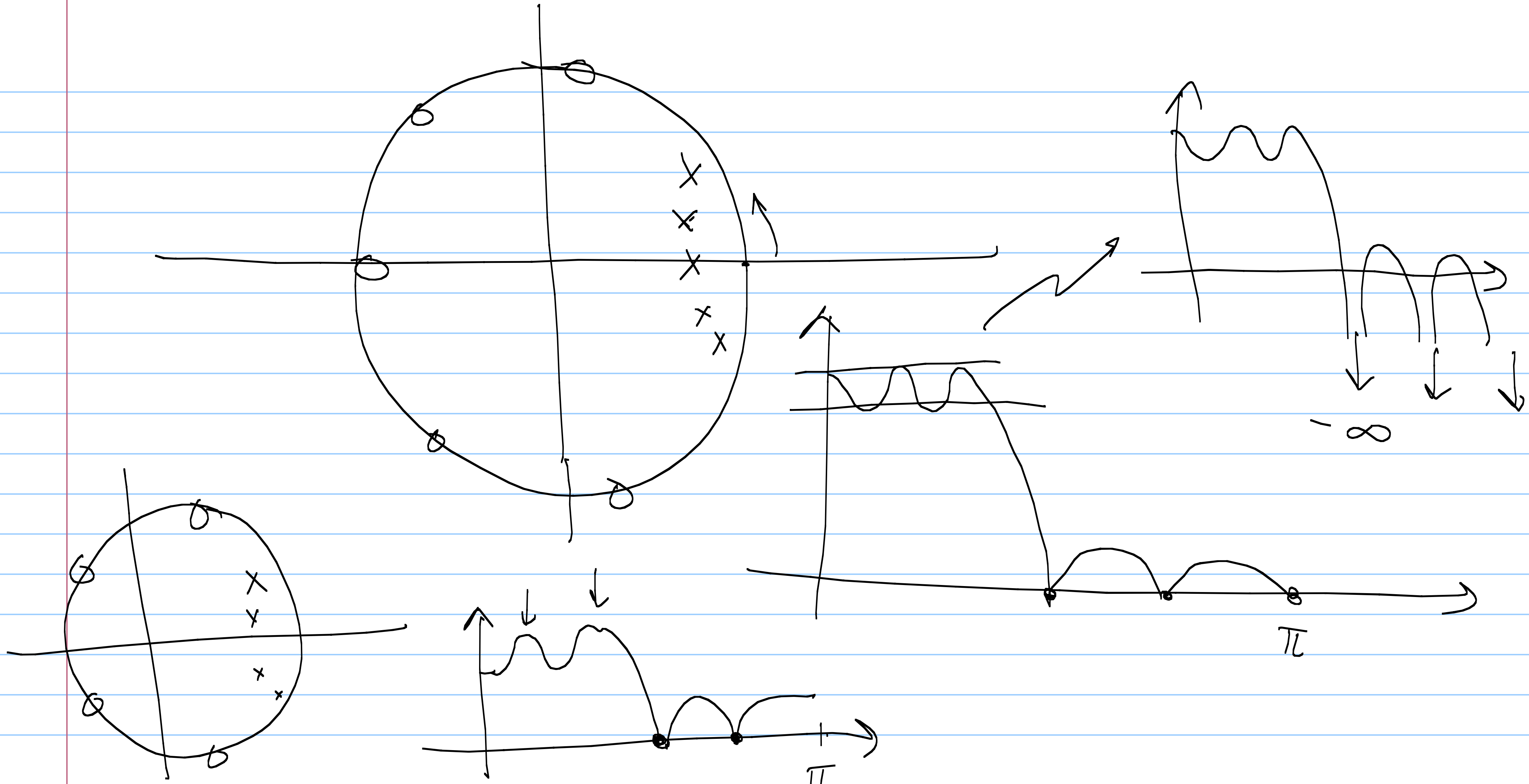
Ideal:

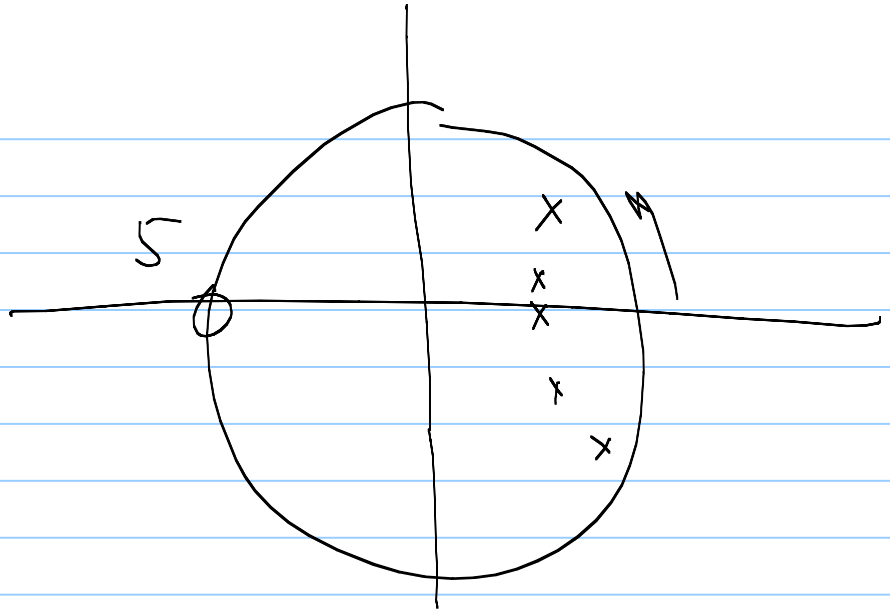


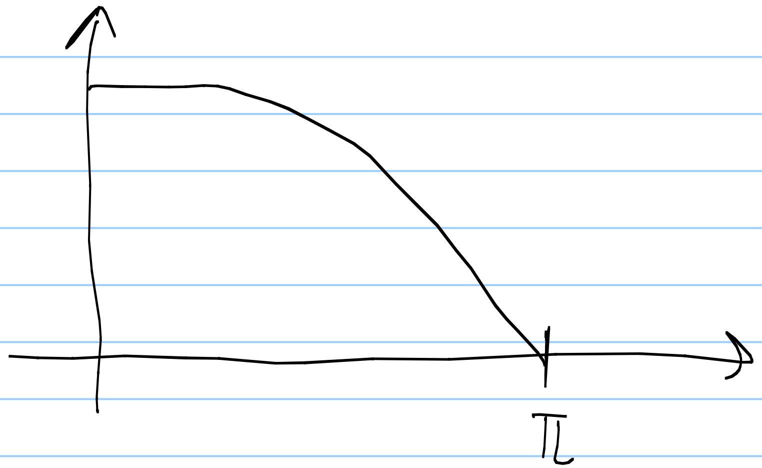
Improved Notch filter:

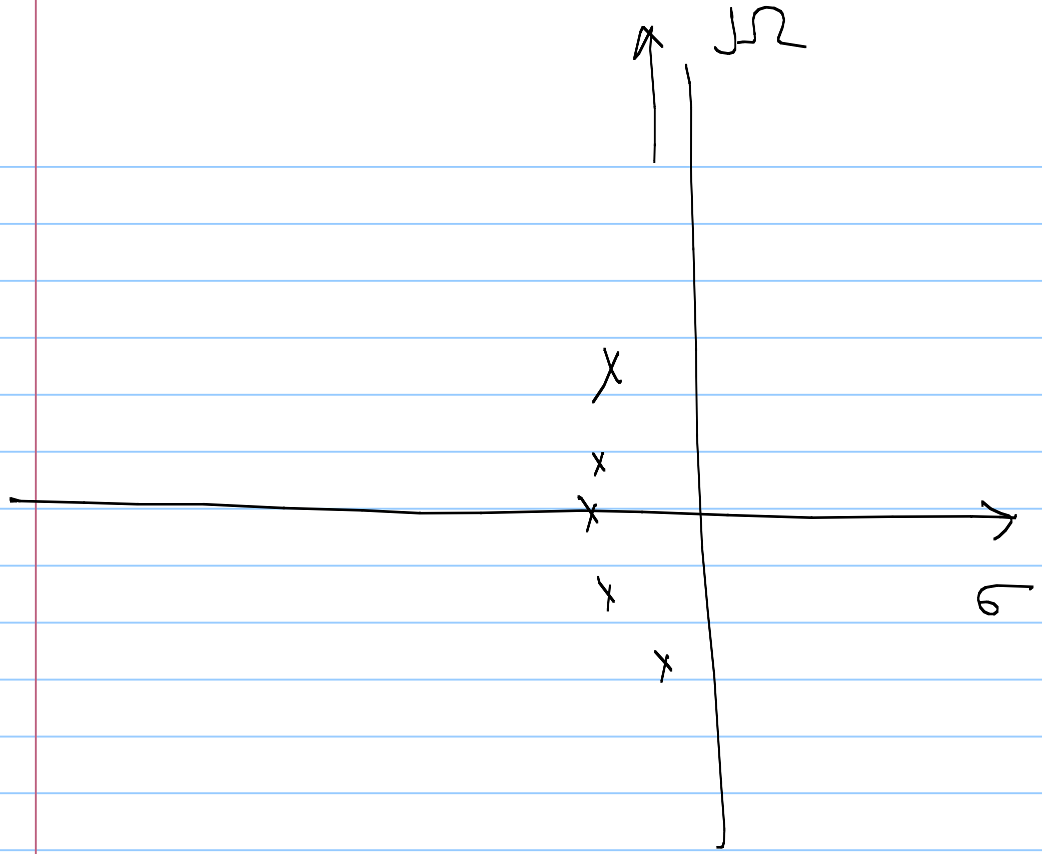
$$H(z) = \frac{1 - 2\cos\theta z^{-1} + z^{-2}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

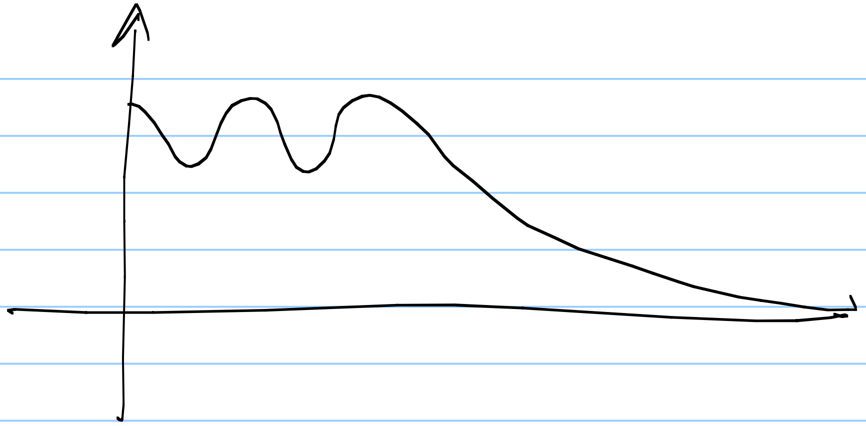


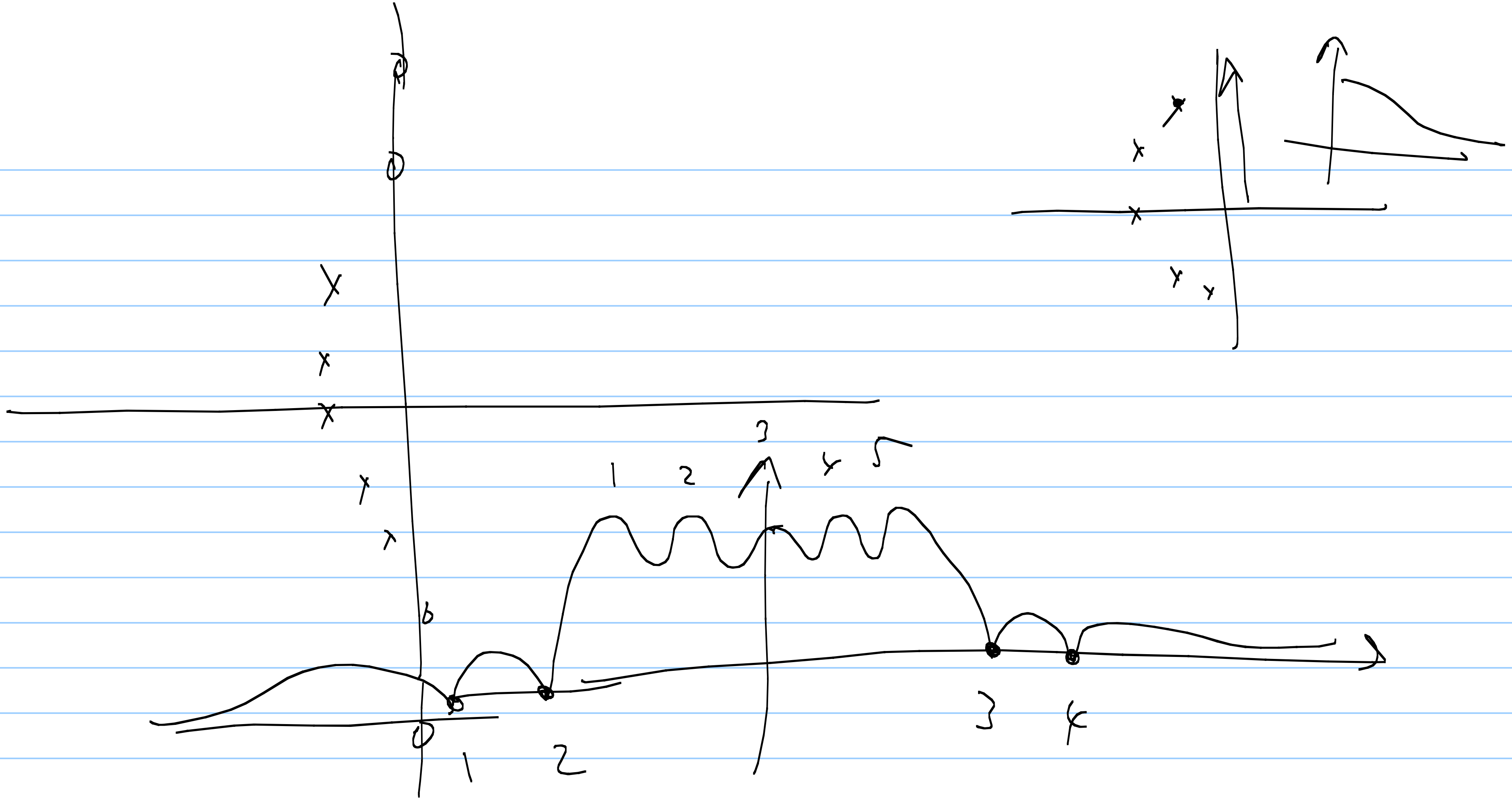

π
π
-∞


5


π


jΩ
σ




1
2
3
4
5
6
1
2
3
4

$$\measuredangle H(e^{j\omega}) = \measuredangle b_0 + \sum_{k=1}^{M} \measuredangle (1 - b_k e^{-j\omega})$$

$$- \sum_{k=1}^{N} \measuredangle (1 - a_k e^{-j\omega})$$

$$\tan^{-1}(3/4) \neq \tan^{-1}\left(-3/-4\right)$$

Single Complex Zero

$$H(z) = 1 - re^{j\theta} z^{-1}$$

$$H(e^{j\omega}) = 1 - re^{j\theta}e^{-j\omega}$$

$$= 1 - re^{-j(\omega-\theta)}$$

$$= 1 - r\cos(\omega-\theta) + jr\sin(\omega-\theta)$$

$$\measuredangle H(e^{j\omega}) = \tan^{-1}\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}$$

EC 5142 Nov. 1 2011

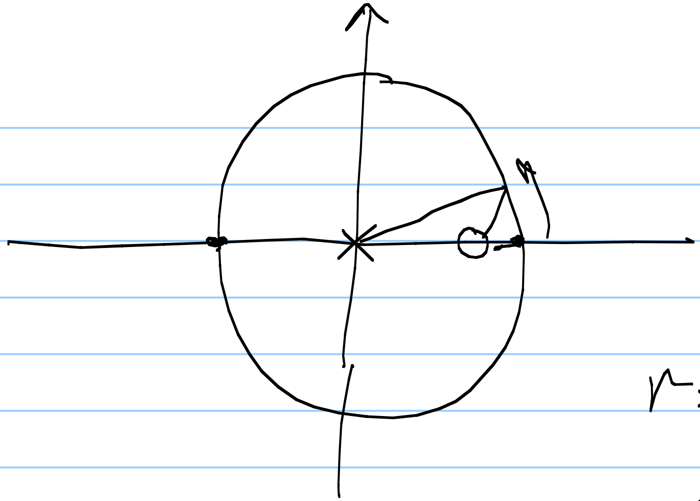
$$H(e^{j\omega}) = b_0 \, e^{j\omega(N-M)} \frac{\prod_{l=1}^{M}(e^{j\omega}-z_l)}{\prod_{k=1}^{N}(e^{j\omega}-p_k)}$$

$$\measuredangle H(e^{j\omega}) = \measuredangle b_0 + \underbrace{\omega(N-M)} + \sum_{l=1}^{M} \measuredangle(e^{j\omega}-z_l) - \sum_{k=1}^{N} \measuredangle(e^{j\omega}-p_k)$$

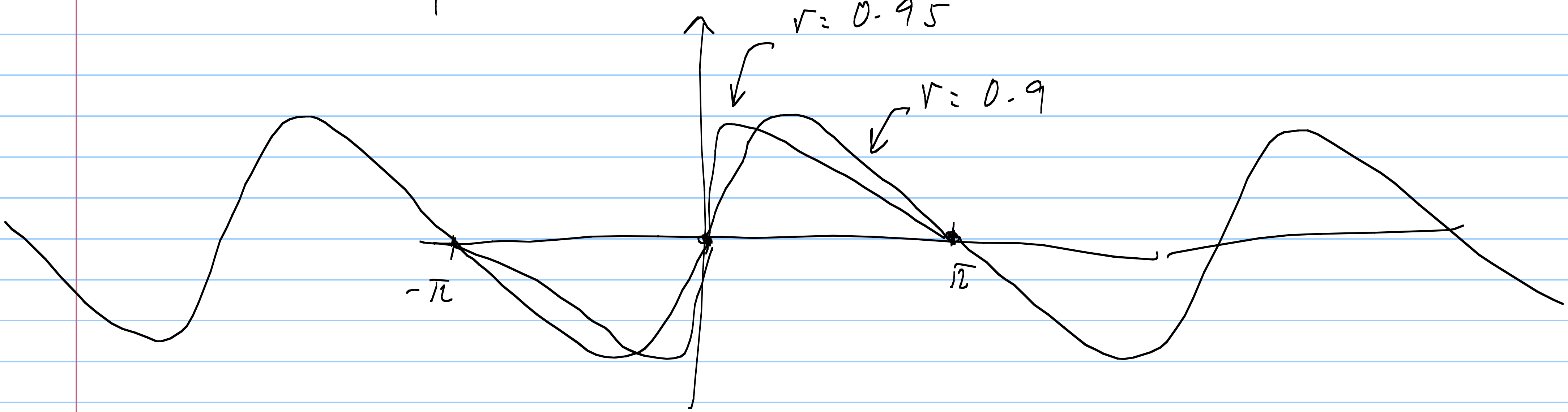
phase contribution due to trivial poles/zeros

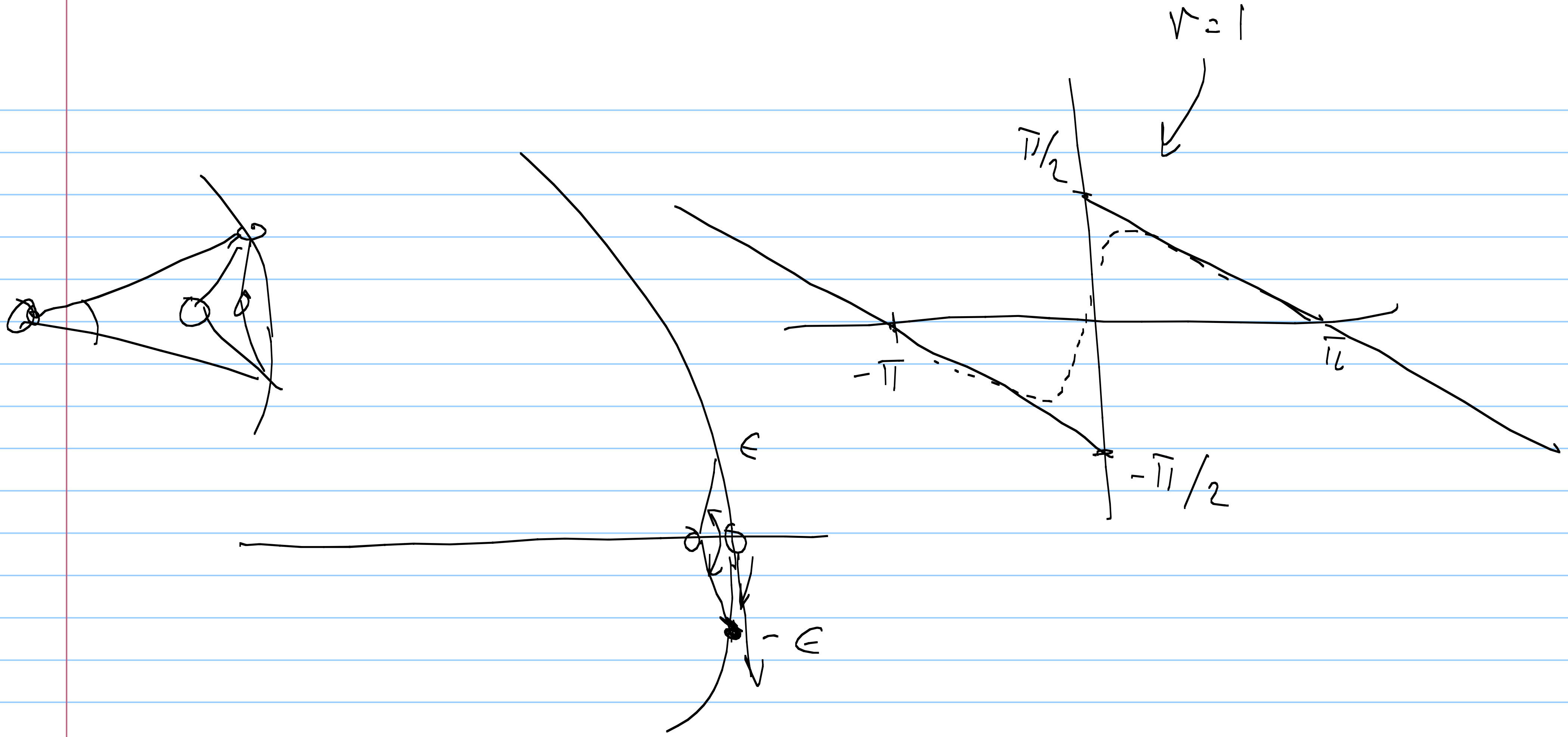
Simple complx zero

$$\angle H(e^{j\omega}) = \tan^{-1} \frac{r \sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}$$



$r = 0.9, \quad \theta = 0$

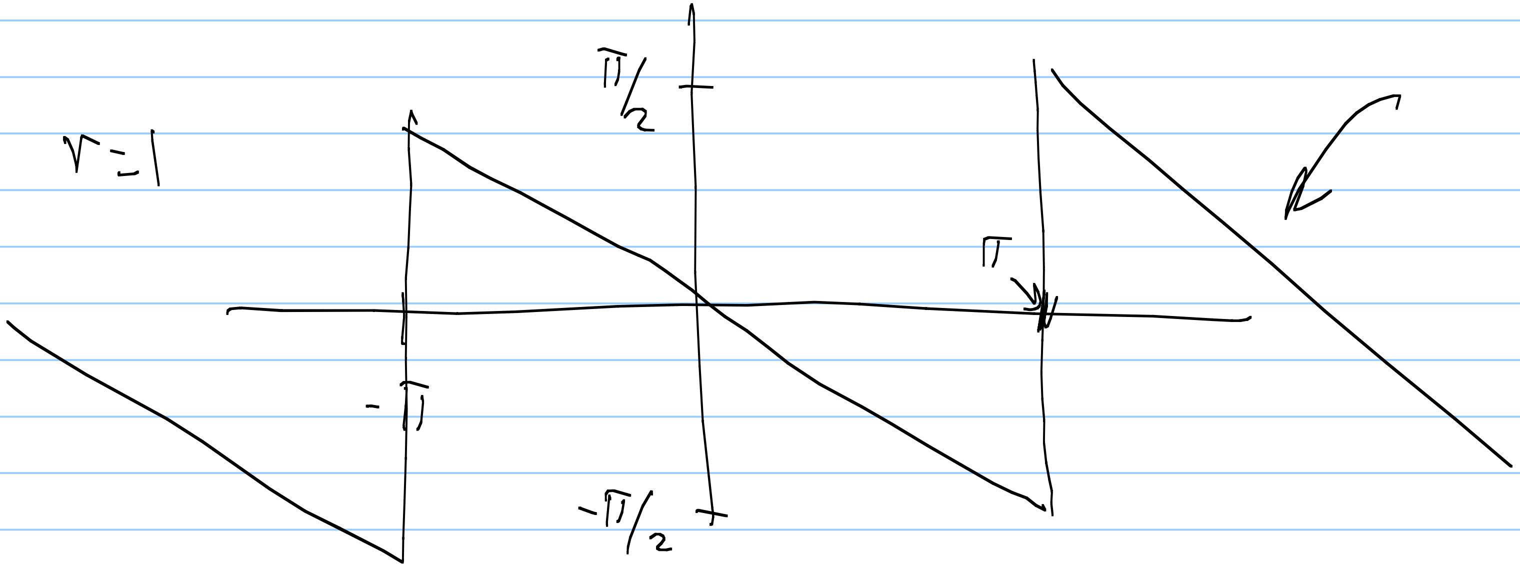


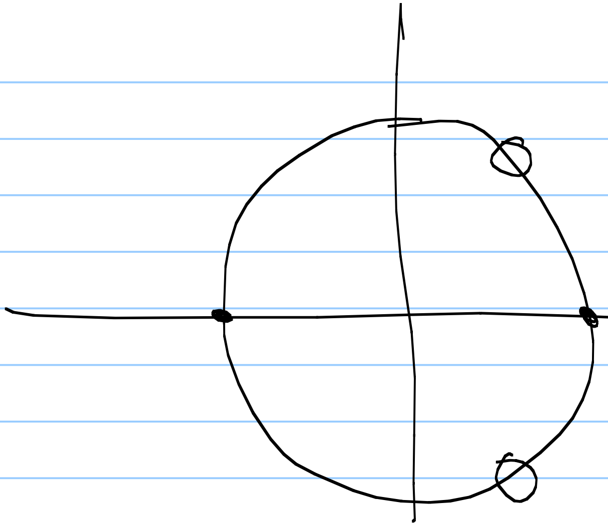

r = 1
π/2
-π
π
-π/2
ε
-ε

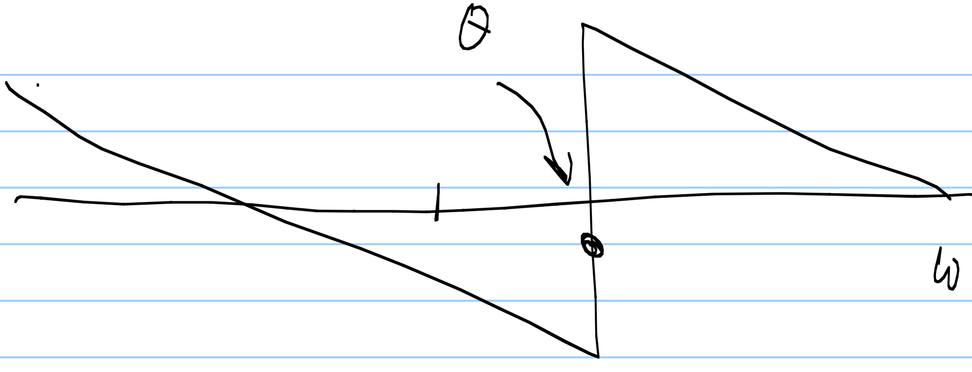
$$\tan^{-1}\frac{\sin w}{1-\cos w} = \begin{cases} \pi/2 - \frac{w}{2} & w > 0 \\ -\frac{\pi}{2} - \frac{w}{2} & w < 0 \end{cases}$$

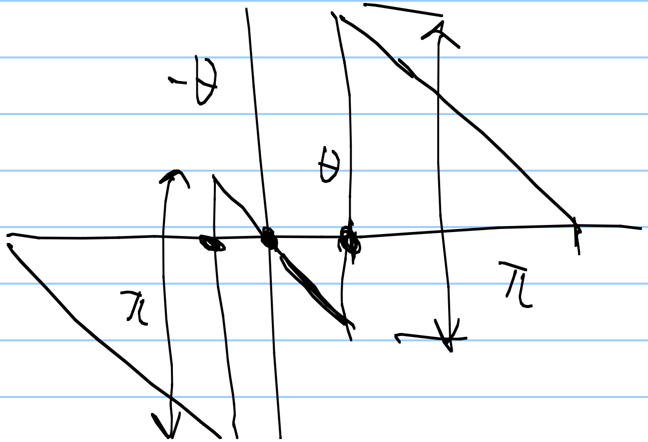
$\theta = 0, \quad r = 1$

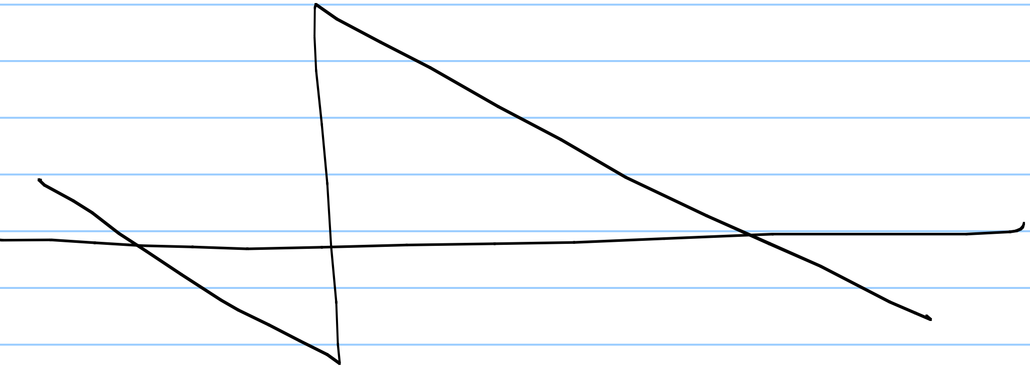
For $\theta = \pi, \quad r = 1$



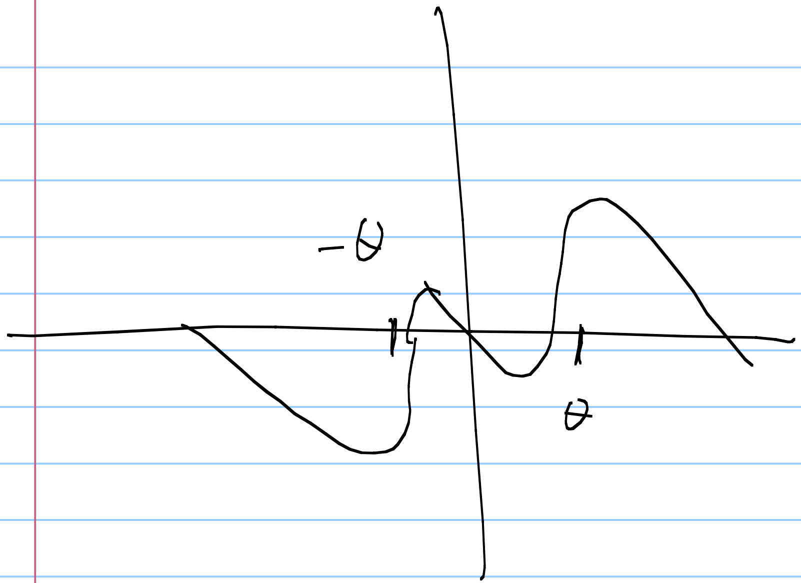


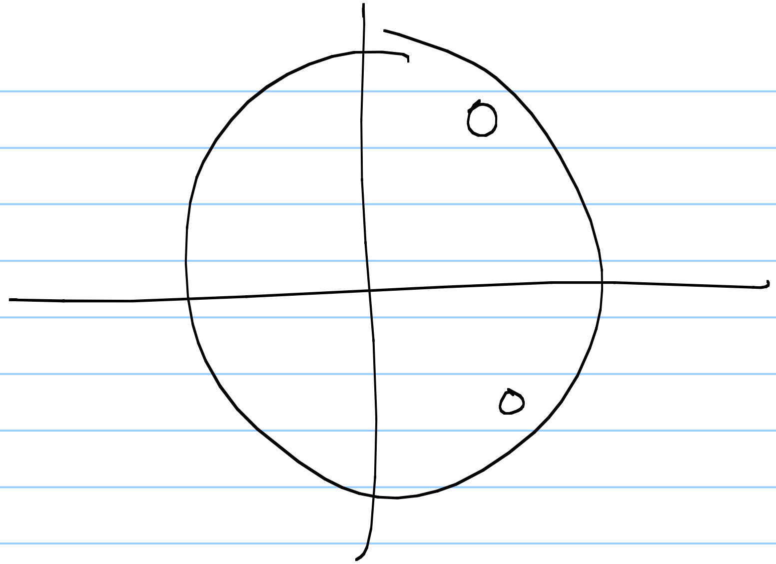


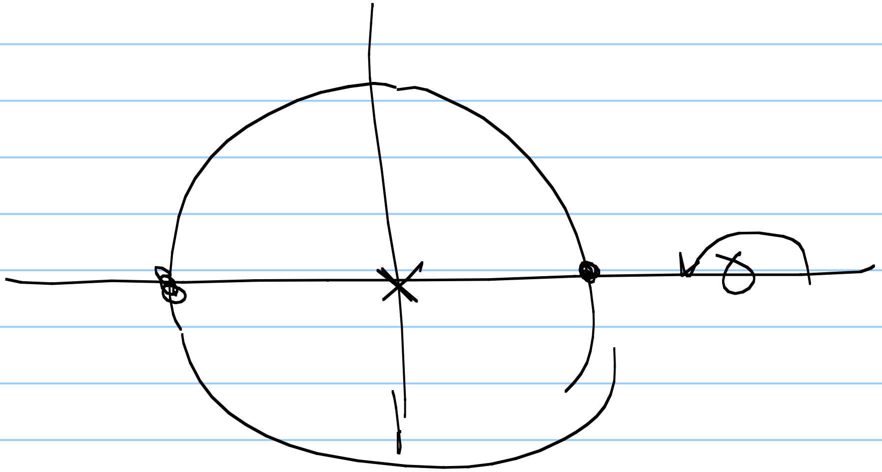


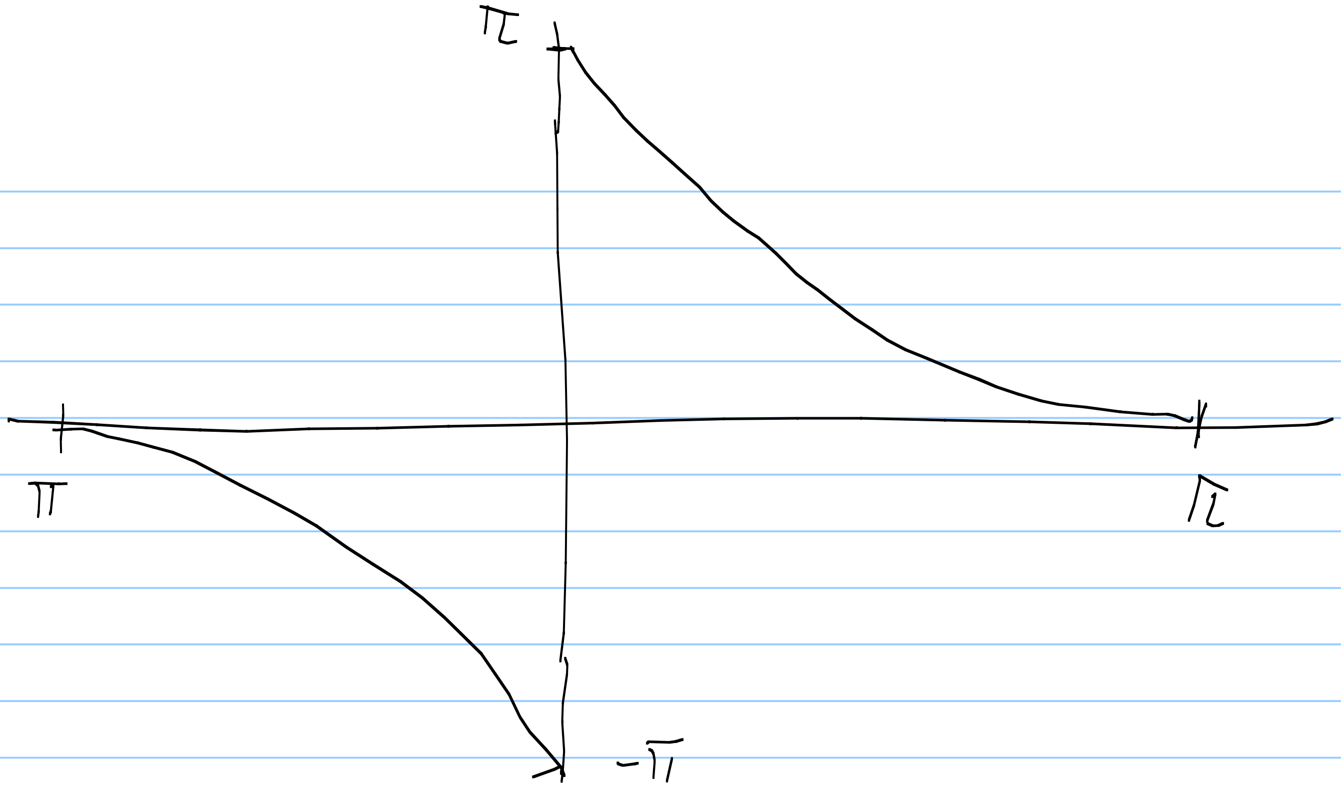


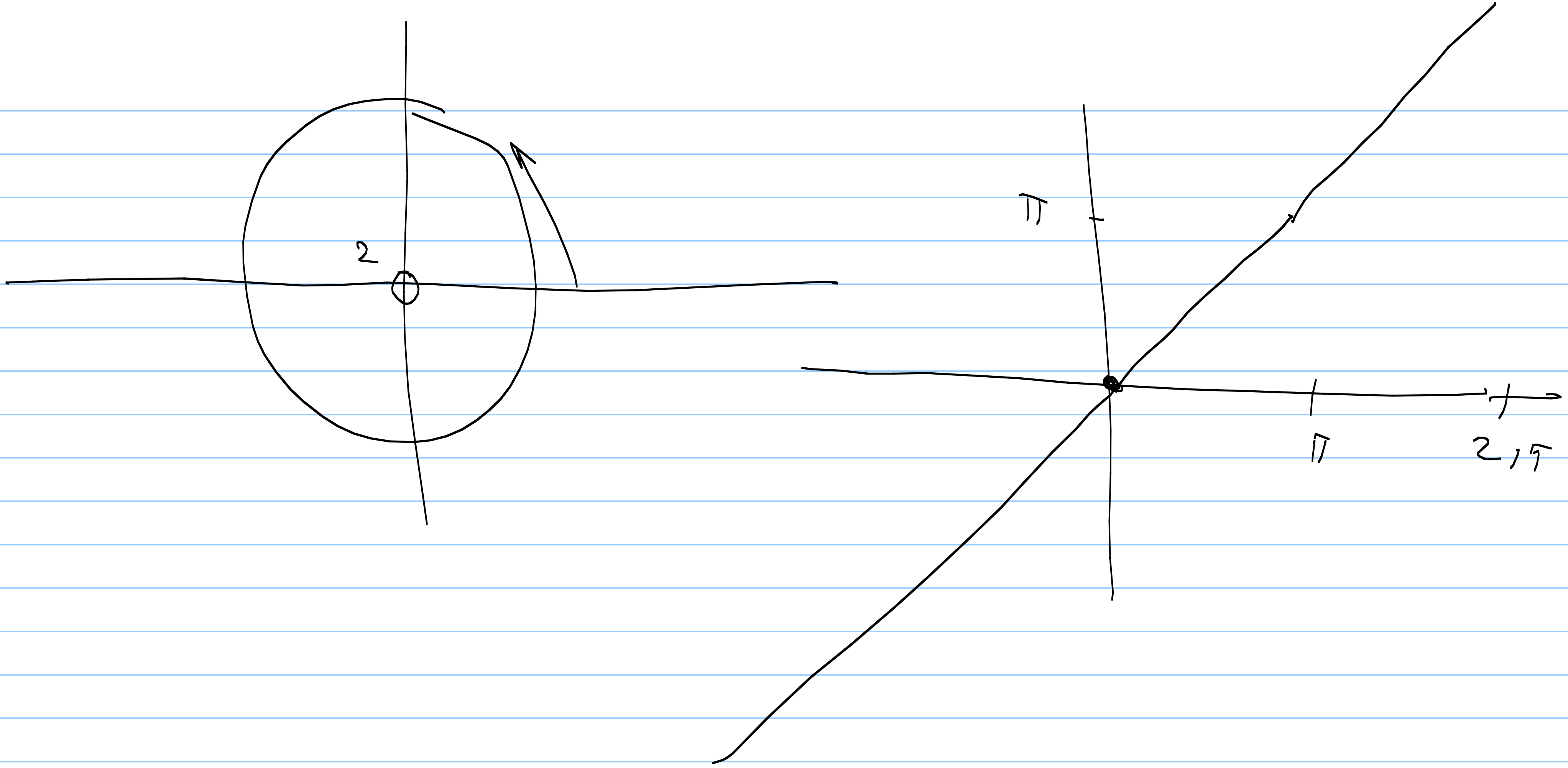
$\omega = -\theta$


-θ
θ

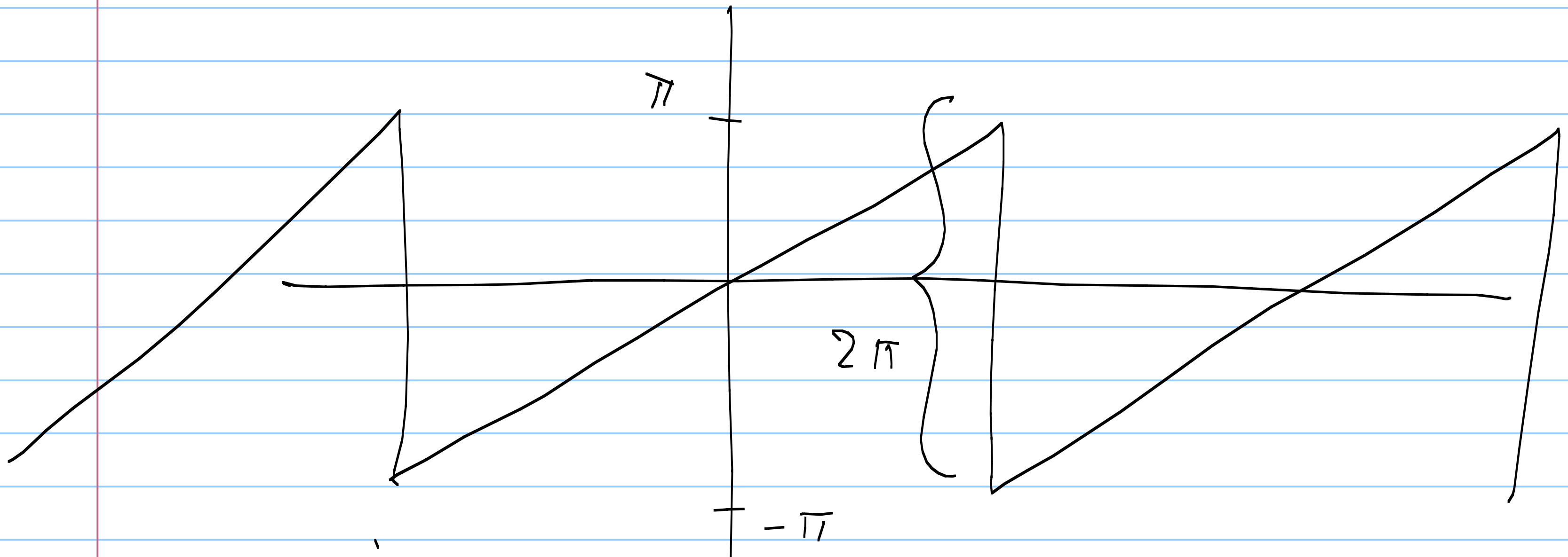




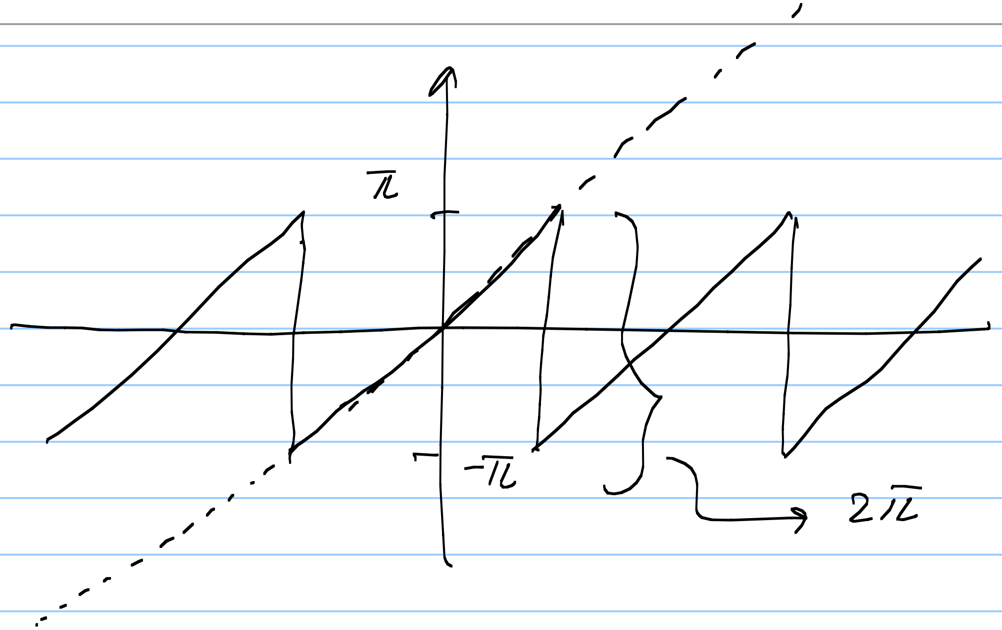

π
π
π
−π


2
π
π
2π

Principal phase. $-\pi \le \theta < \pi$



EC 5142 Nov 02 2011



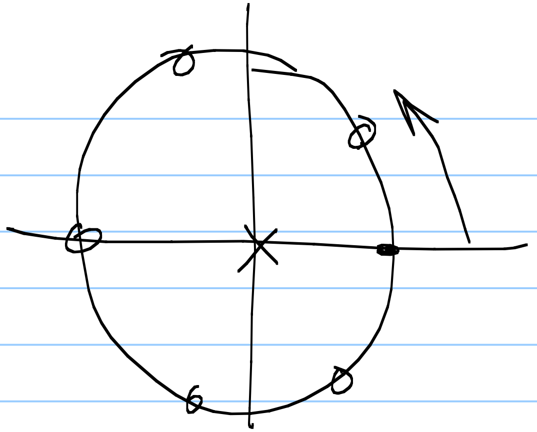
Principal Phase: $-\pi \leq \Phi(\omega) < \pi$ "wrapped phase"

$\hookrightarrow$ principal $\measuredangle H(e^{j\omega})$

From $\underline{\Phi}(\omega)$ we can get back $\phi(\omega)$ by adding or subtracting required multiples of 2π

For rational transfer function, the $\phi(\omega)$ is continuous except for possible jumps of π

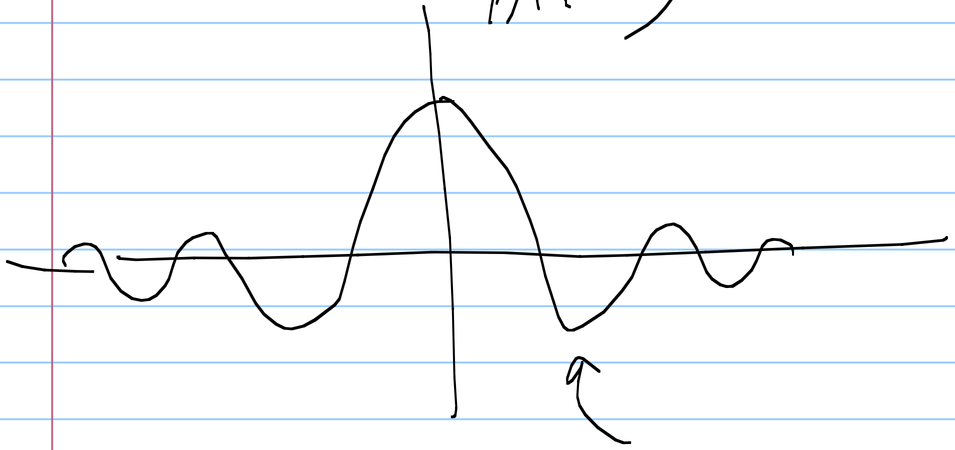
$$H(e^{j\omega}) = |H(e^{j\omega})| \, e^{j \measuredangle H(e^{j\omega})}$$



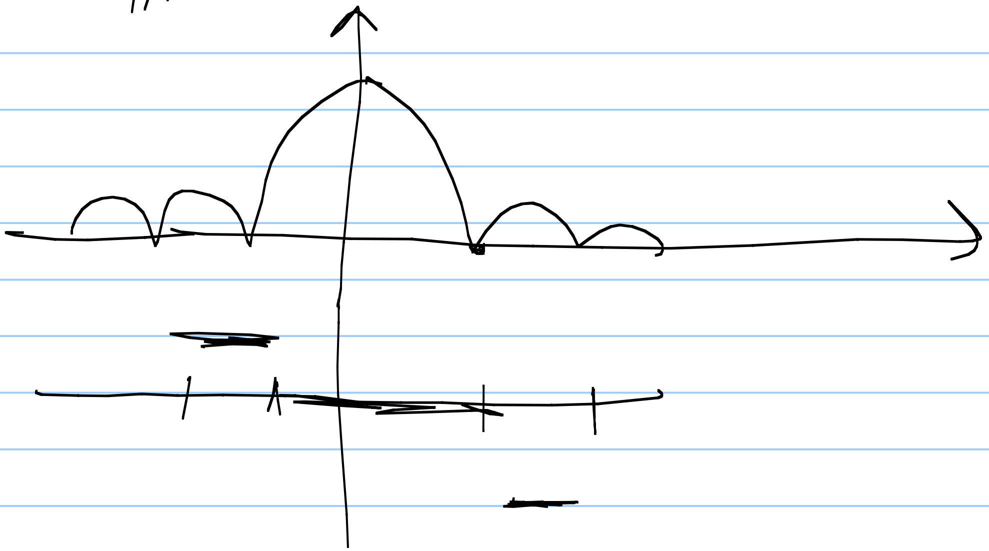
$h[n] = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$

$$\Longleftrightarrow \quad \frac{\sin 5\omega/2}{\sin \omega/2}$$

$H(e^{j\omega})$

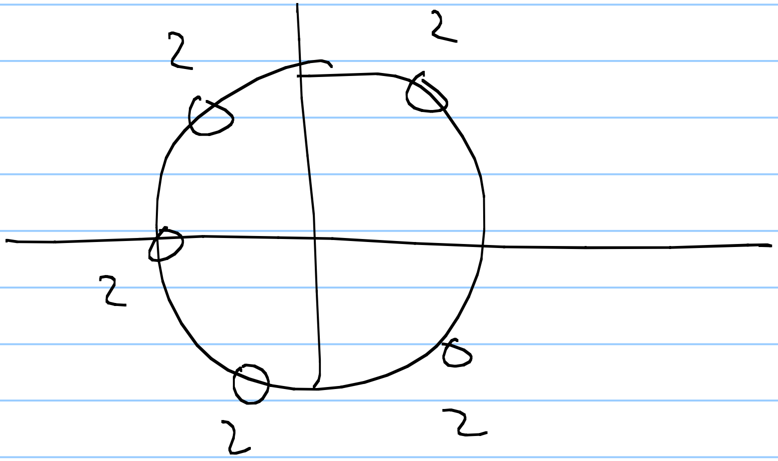


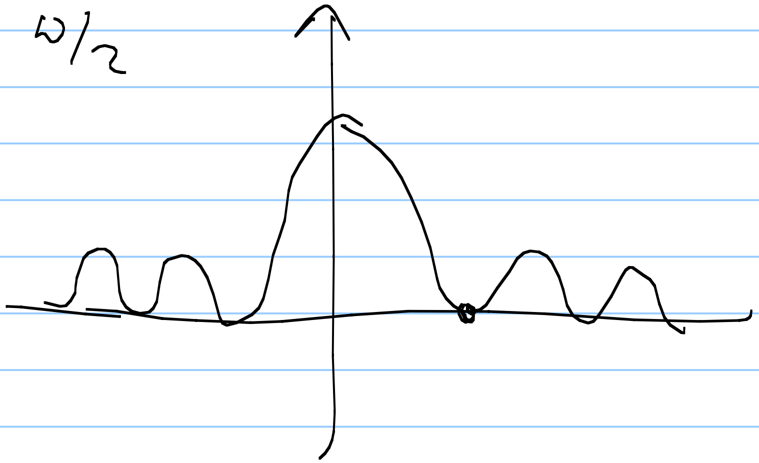
$|H(e^{j\omega})|$



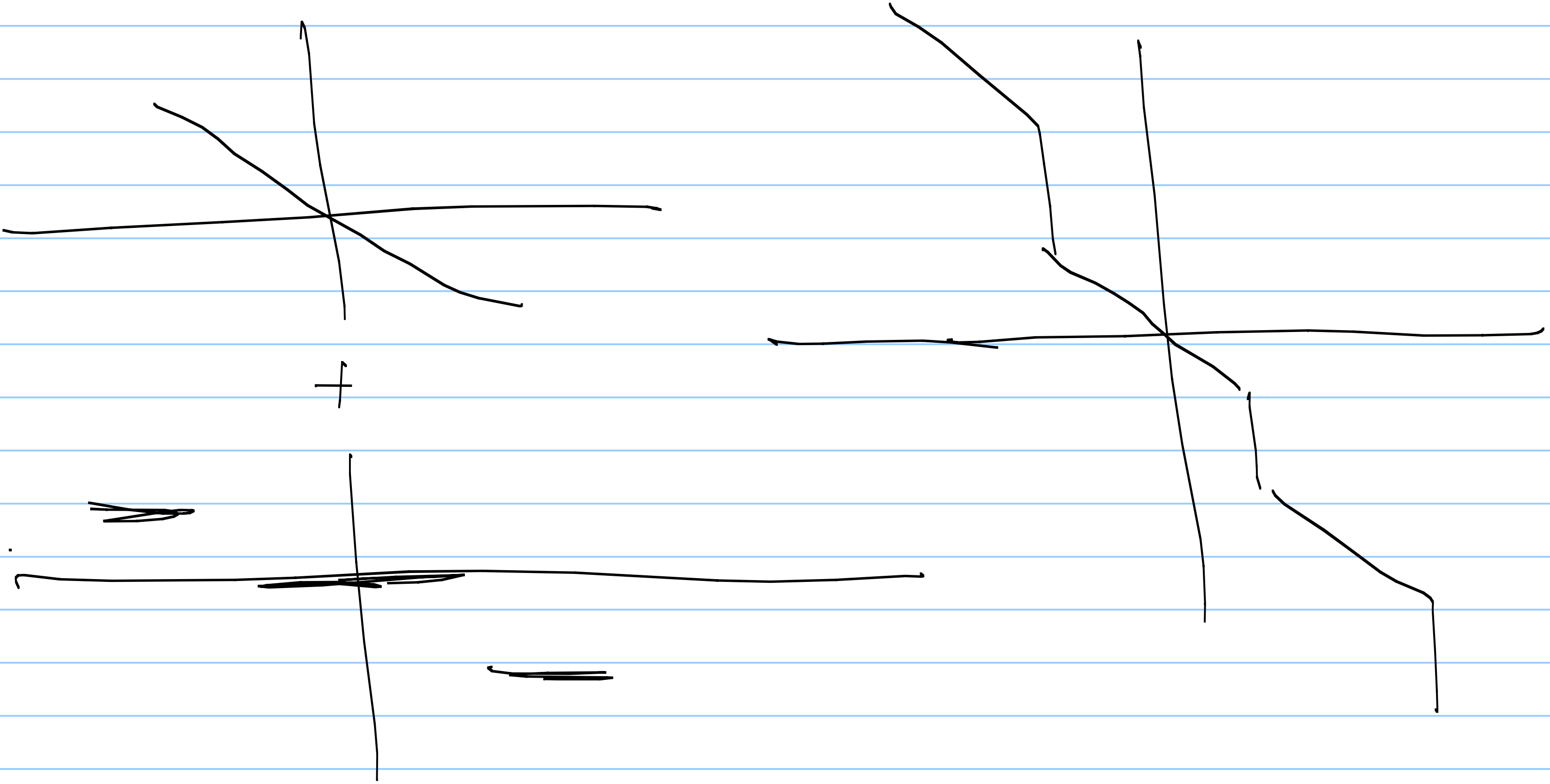
$$h[n] * h[n] \Longleftarrow \{1 \quad 2 \quad 3 \quad 4 \quad \underset{\uparrow}{5} \quad 4 \quad 3 \quad 2 \quad 1\}$$

$$H(e^{jw}) \cdot H(e^{jw}) = \frac{\sin^2 5w/2}{\sin^2 w/2}$$





$$h(n) = \{\overset{\downarrow}{1}, 1, 1, 1, 1\} \longleftrightarrow e^{-j2w} \overbrace{\frac{\sin 5w/2}{\sin w/2}}^{\longrightarrow A(w)}$$



H = freqz(b, a)

$$\phi(\omega) \longrightarrow \measuredangle H(e^{j\omega})$$

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\phi(\omega)}$$

$$= \underbrace{A(\omega)}_{\text{real-valued}} e^{j\phi_1(\omega)}$$

$\phi_1(\omega) \longrightarrow$ continuous

true for rational TF

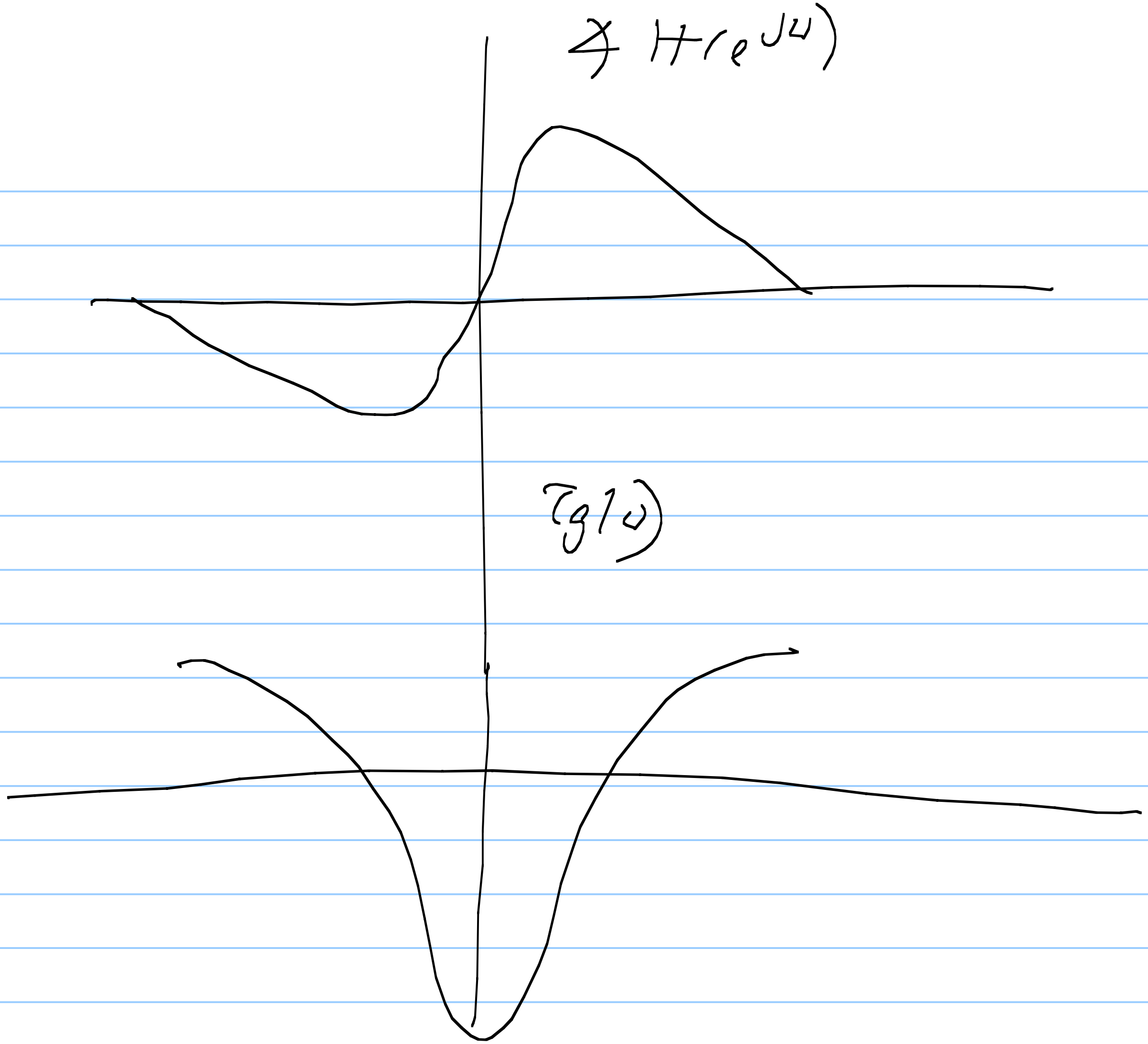
Group Delay: $$-\frac{d}{d\omega}\phi(\omega) = \tau_g(\omega)$$

Simple complex zero

$$\phi(\omega) = \tan^{-1} \frac{r \sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}$$

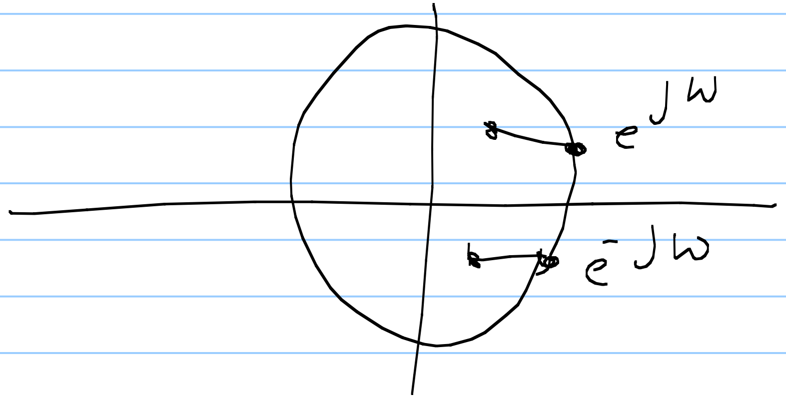
$$-\frac{d}{d\omega}\phi(\omega) = \frac{r(r - \cos(\omega - \theta))}{|1 - re^{j\theta}e^{j\omega}|^2}$$

Single complex zero



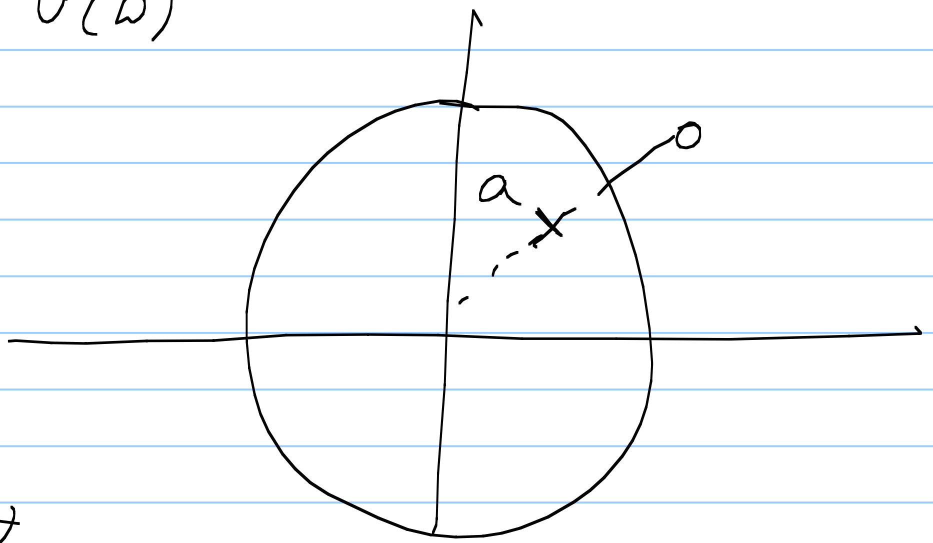
$$H(z) = \frac{-a^{*} + z^{-1}}{1 - a z^{-1}}$$

$$H(e^{jw}) = \frac{-a^{*} + e^{-jw}}{1 - a e^{-jw}} = \frac{e^{-jw} - a^{*}}{e^{-jw}(e^{jw} - a)}$$



$$|H(e^{j\omega})| = \left| \frac{e^{-j\omega} - a^*}{e^{j\omega} - a} \right| = 1$$

$$H(e^{j\omega}) = e^{j\theta(\omega)}$$



pole: $a = re^{j\theta}$

zero: $1/a^* = \frac{1}{r}e^{j\theta}$ reflected across the unit circle

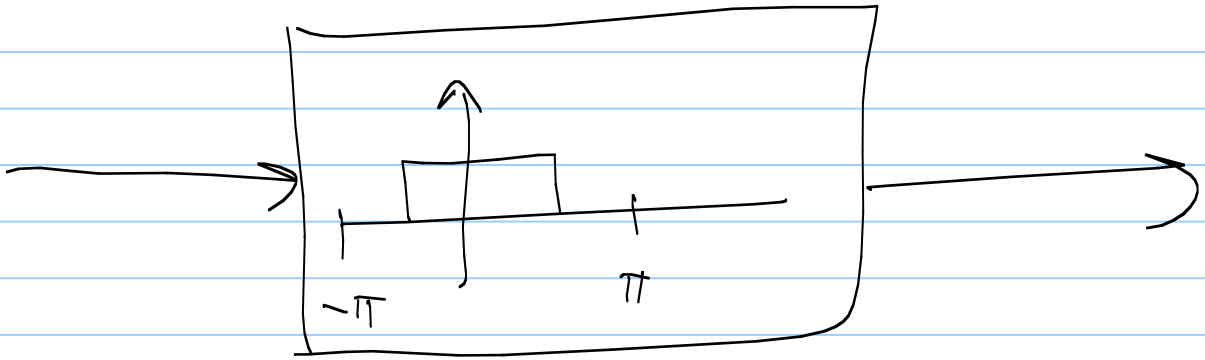
$$H(z) = \frac{-a^{*}\left(1 - \frac{1}{a^{*}} z^{-1}\right)}{1 - a z^{-1}}$$

$|H(e^{j\Omega})| = 1$ "All pass" Section

$$H_{Ap}(z) = \frac{a_N^{*} + a_{N-1}^{*} z^{-1} + \cdots + a_1^{*} z^{-(N-1)} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{z^{-N} A^*(z^*)}{A(z)}$$

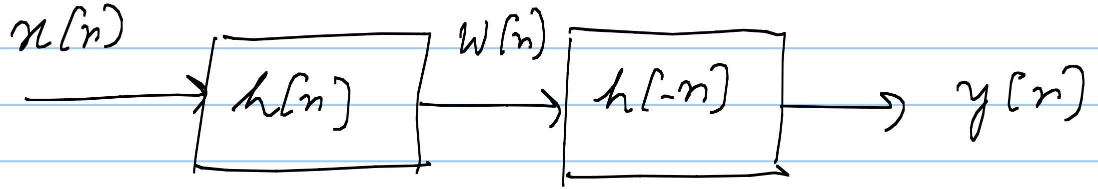
$$\begin{cases} 1 & |w| < w_c \\ 0 & \text{otherwise} \end{cases}$$



EC 5142 Nov 03 2011

Note Title

03-11-2011



$$W(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})\, H(e^{-j\omega})$$

For real-valued $h[n]$, $H(e^{-j\omega}) = H^*(e^{j\omega})$

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})\, H^*(e^{j\omega})$$

$$= X(e^{j\omega})\, |H(e^{j\omega})|^2$$

$|H(e^{j\omega})|^2$ = "zero phase filter"

Not practical because $h[-n]$ is non-causal if $h[n]$ is Causal

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$$

$$x[n] = A_1 \cos \omega_1 n + A_2 \cos \omega_2 n$$

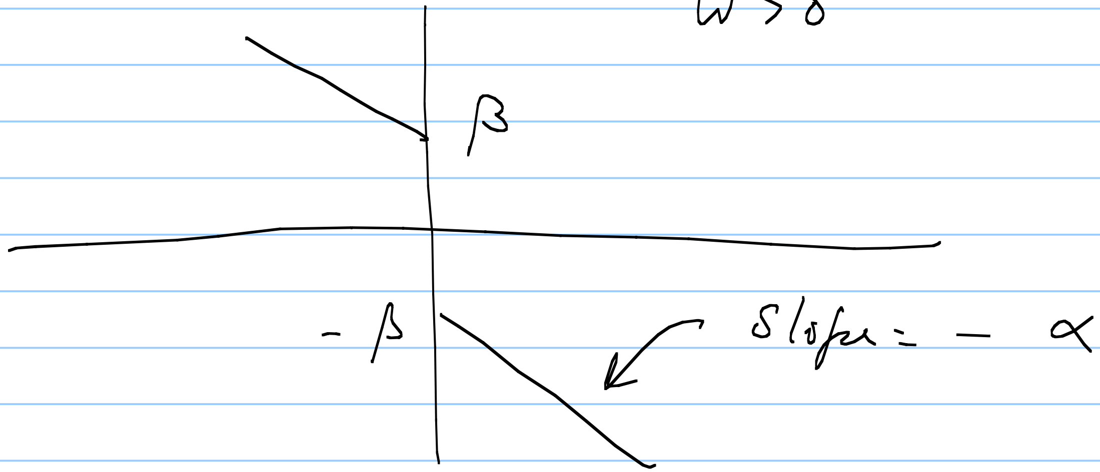
$$x[n-n_0] = A_1 \cos \omega_1 (n-n_0) + A_2 \cos \omega_2 (n-n_0)$$

$$= A_1 \cos(\omega_1 n - \overbrace{\omega_1 n_0}^{\phi_1}) + A_2 \cos(\omega_2 n - \overbrace{\omega_2 n_0}^{\phi_2})$$

$$\phi(\omega) = -\alpha\omega + \beta$$

generalized linear phase

$\omega > 0$

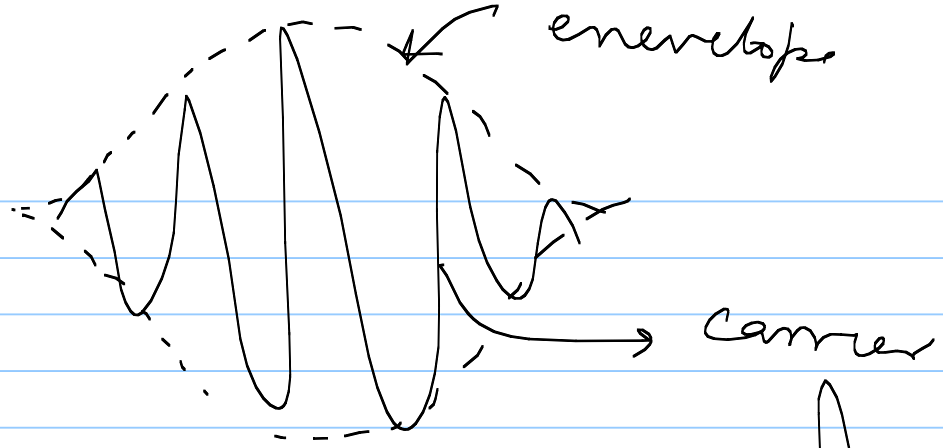


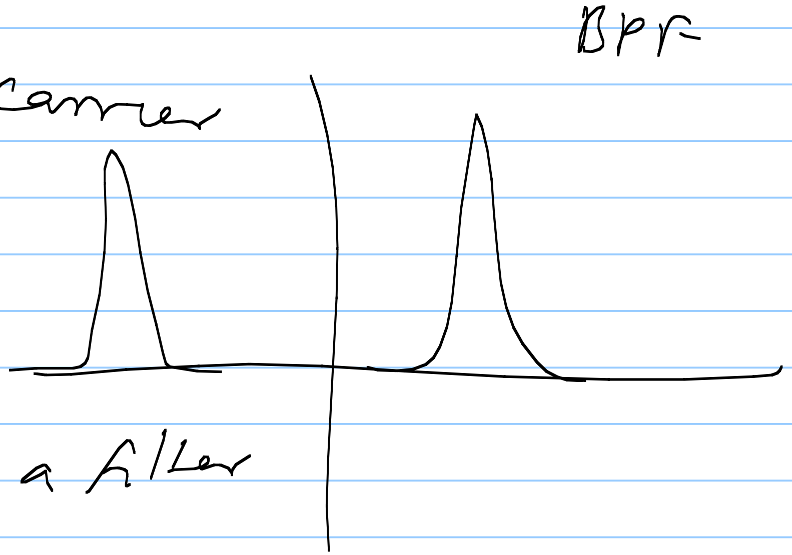
$$\tau_g(\omega) = -\frac{d\phi}{d\omega} = \alpha$$

Phase delay : $\tau_{ph}(\omega) = -\dfrac{\phi(\omega)}{\omega}$

For $\phi(\omega) = -\alpha\omega$ $\quad \tau_g(\omega) = \alpha$

$$\tau_{ph}(\omega) = \alpha$$





Under what conditions is a filter linear phase ?

$$\phi(\omega) = -\alpha\omega + \beta$$

$$H(e^{j\omega}) = \sum_n h[n] e^{-j\omega n}$$

$$= |H(e^{j\omega})| e^{j(-\alpha\omega+\beta)}$$

$$\sum_n h[n] \cos\omega n - j \sum_n h[n] \sin\omega n = |H(e^{j\omega})| e^{j(-\alpha\omega+\beta)}$$

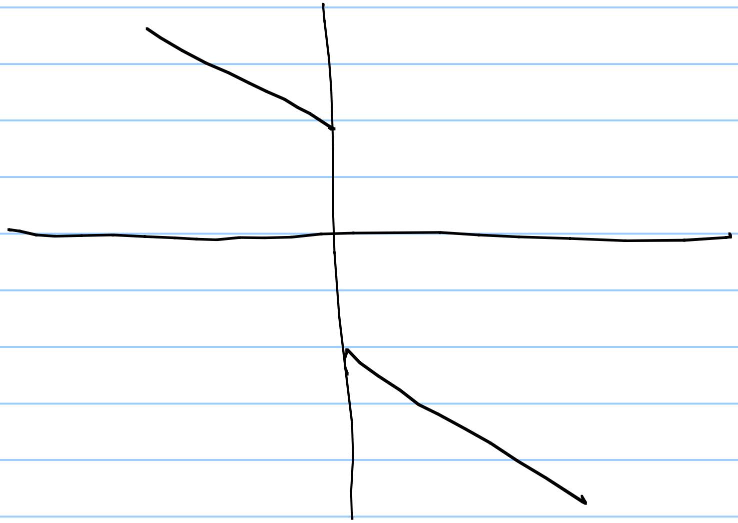
$$\frac{-\sum_n h[n] \sin\omega n}{\sum_n h[n] \cos\omega n} = \frac{\sin(-\alpha\omega+\beta)}{\cos(-\alpha\omega+\beta)}$$

$$\sum_n{}' h(n) \sin \omega n \cos(-\alpha\omega + \beta)$$

$$+ \sum{}' \sin(-\alpha\omega + \beta)\, h(n) \cos \omega n = 0$$

$$\sum_n{}' h(n) \sin((-\alpha + n)\omega + \beta) = 0$$

EC5142 Nov 10 2011



Generalized linear phase.

$\phi(\omega) = -\alpha\omega + \beta \qquad \omega > 0$

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j(-\alpha\omega+\beta)} = \sum_n h[n] e^{-j\omega n}$$

$$\sum_n h[n] \sin\left(-\overline{\alpha + n}\ \omega + \beta\right) = 0$$

$h[n] = 0 \ \forall n$ is the trivial solution

First, assume $\beta = 0$

$$\sum_n h[n] \sin\left(-\overline{\alpha + n}\ \omega\right) = 0$$

Assume $h[n] = 0$ for n outside $[0, N-1]$

i.e. $h[n]$ is an FIR filter

$$\sum_{n=0}^{N-1} h[n] \sin\left(\overline{-\alpha+n}\;\omega\right) = 0$$

$$h[0] \sin(-\alpha\omega)$$

$$h[N-1] \sin\left(\overline{-\alpha+N-1}\;\omega\right)$$

$$\alpha = -\alpha + N-1 \Rightarrow \alpha = \frac{N-1}{2}$$

$$h[0] = h[N-1]$$

$$h[1] \sin\left(\overline{-\alpha+1}\,\omega\right)$$

$$h[N-2] \sin\left(\overline{-\alpha+N-2}\,\omega\right)$$

$$\alpha - 1 = -\alpha + N - 2 \Rightarrow \alpha = \frac{N-1}{2}$$

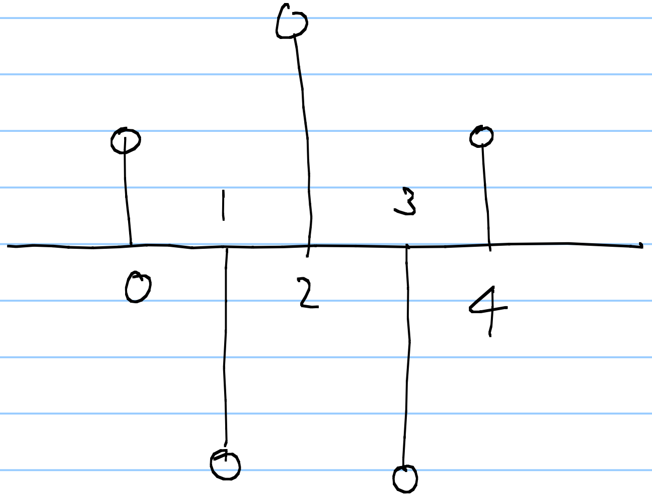
$$h[1] = h[N-2]$$

In general, if $h[N-1-n] = h[n]$

& $\alpha = \frac{N-1}{2}$ = Centre of symmetry pairwise cancellation will occur.

Eg:

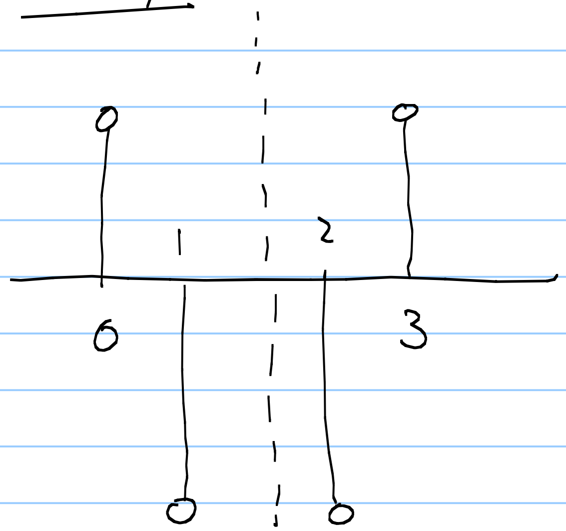
N = 5



$$h[2]\,\sin\left(-\frac{5-1}{2}+\overset{2}{n}\quad\omega\right)=0$$

Since $n=\frac{N-1}{2}$ & $\alpha=\frac{N-1}{2}$

For <u>N = 4</u>



$$\phi(\omega) = -\alpha\omega$$

$$= -\frac{N-1}{2}\omega$$

$$\tau_g(\omega) = \frac{N-1}{2}$$

Now consider $\beta \neq 0$. Let $\beta = \frac{\pi}{2}$

$$\sum_{n=0}^{N-1} h[n] \sin\left(\overline{-\alpha+n}\ \omega + \pi/2\right) = 0$$

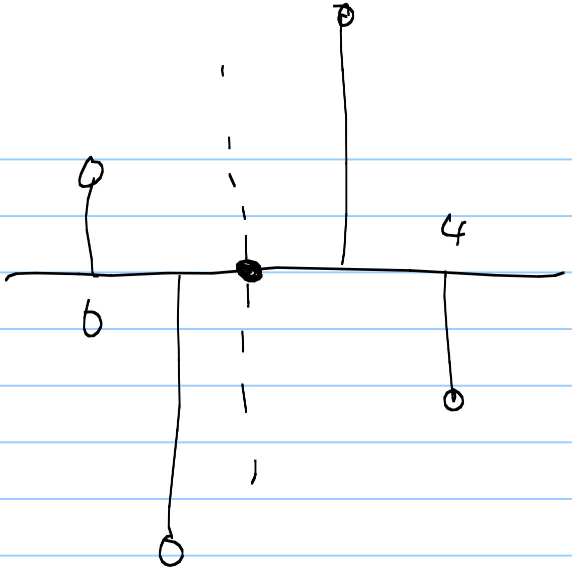
$$\sum_{n=0}^{N-1} h(n) \cos\left(\overline{-\alpha+n}\ \omega\right) = 0$$

$$h[0] \cos(-\alpha\omega)$$

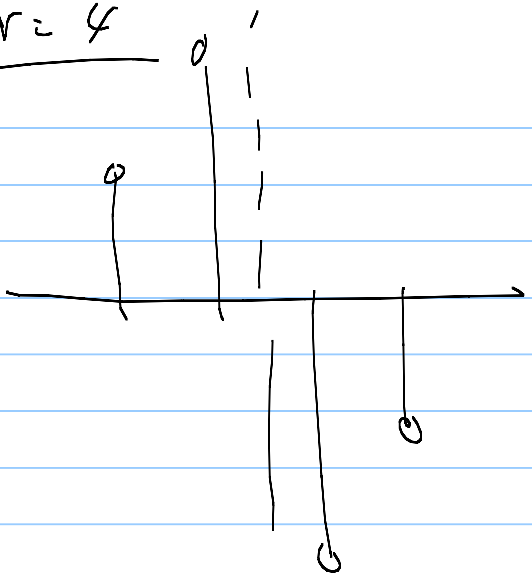
$$h[N-1] \cos\left(\overline{-\alpha+N-1}\ \omega\right) \qquad \alpha = \frac{N-1}{2}$$

$$h[0] = -h[N-1]$$

$N = 5$

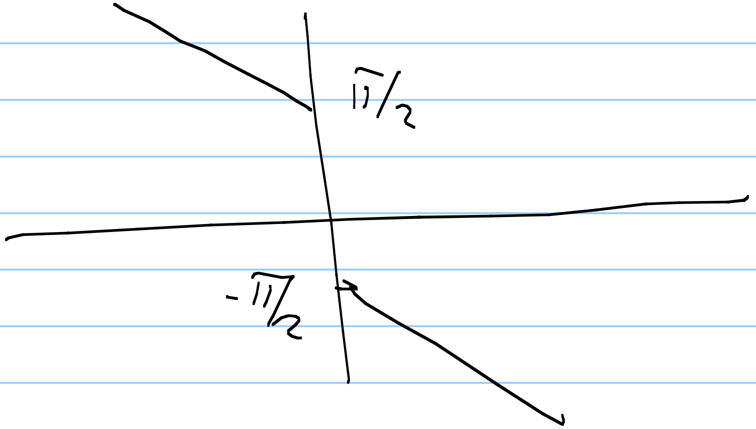


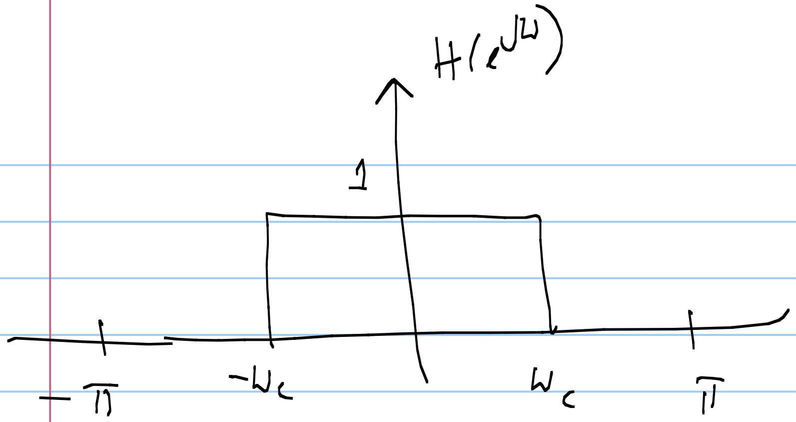
$N = 4$



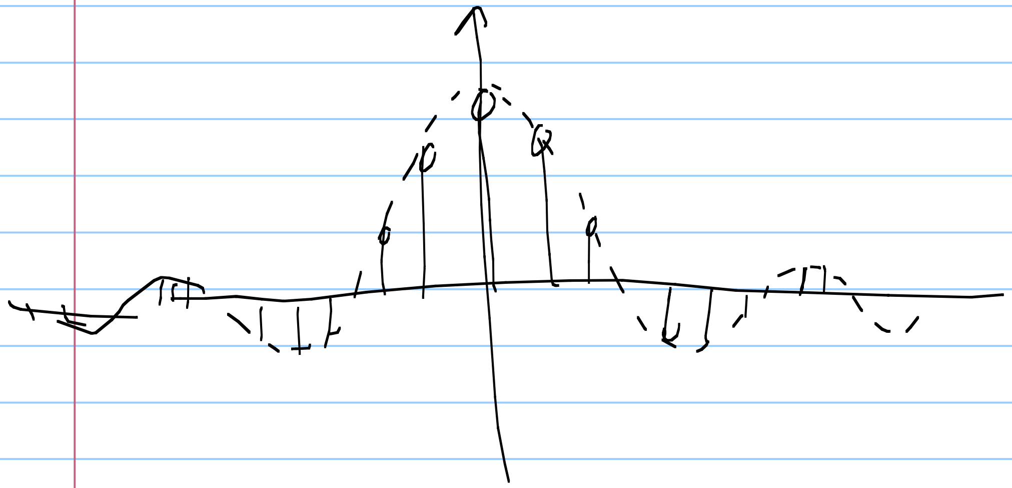
$$\phi(\omega) = -\frac{N-1}{2}\omega + \frac{\pi}{2}$$

$$\tau_g(\omega) = \frac{N-1}{2}$$

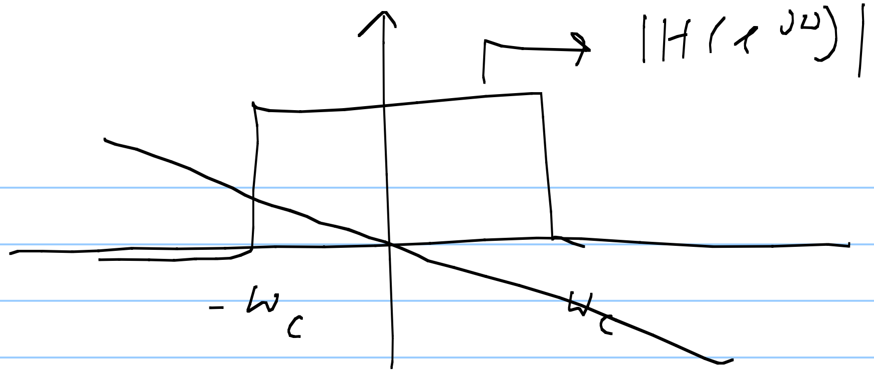




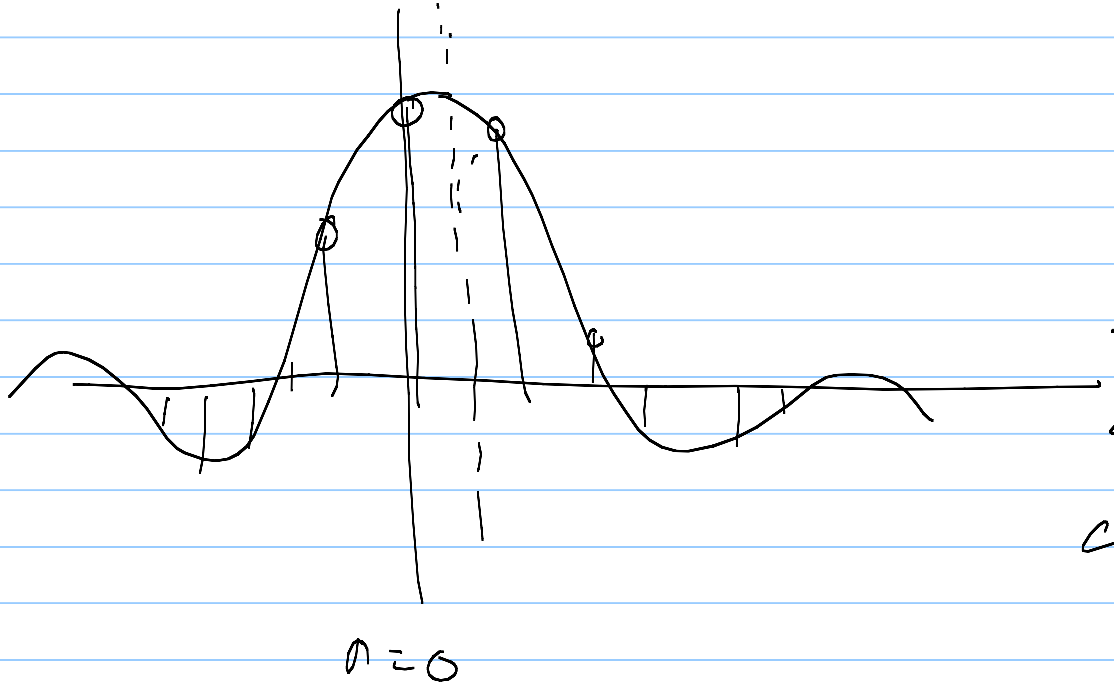
$$h(n) = \frac{\sin w_c n}{\pi n}$$



IIR



$$H(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & |\omega| < w_c \\ 0 & w_c < |\omega| < \pi \end{cases}$$

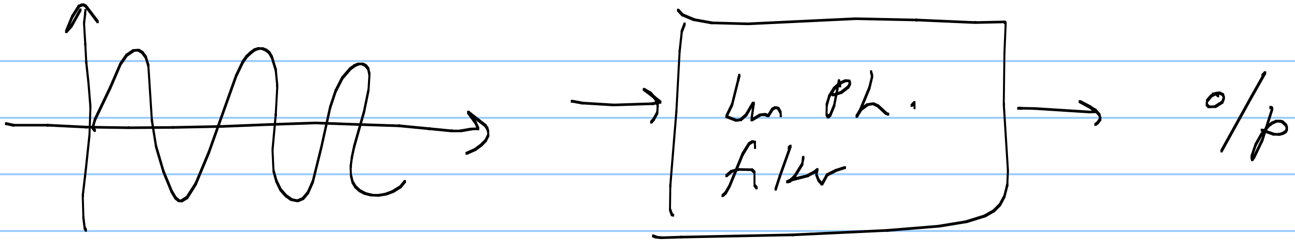


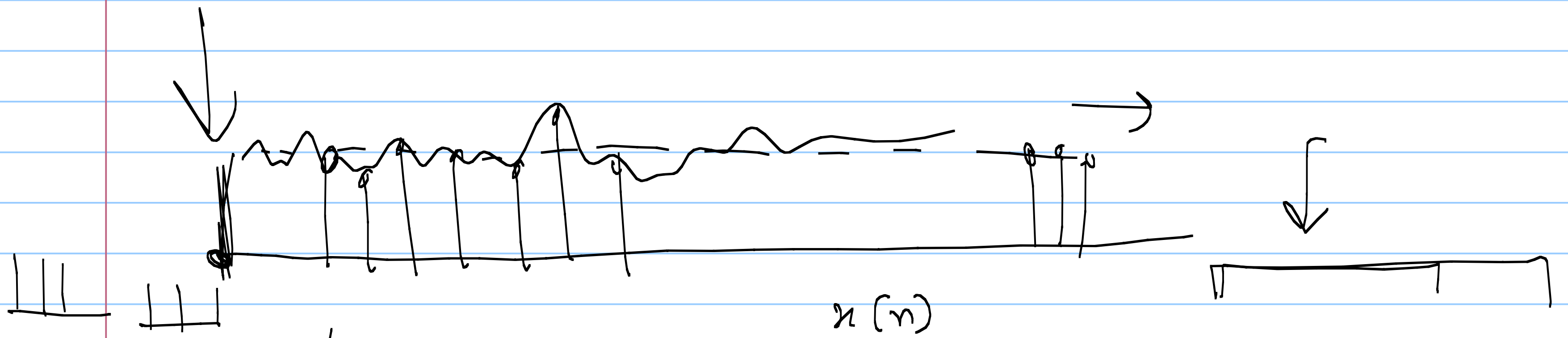
Symmetry is sufficient but not necessary for linear phase in the case for IIR filters

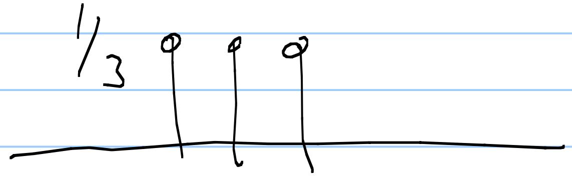
However, for FIR filters, symmetry is both necessary & sufficient for generalized linear phase

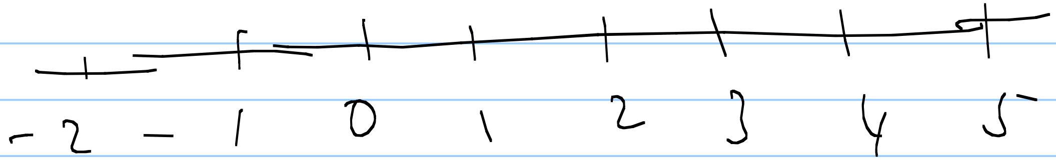
Length: N

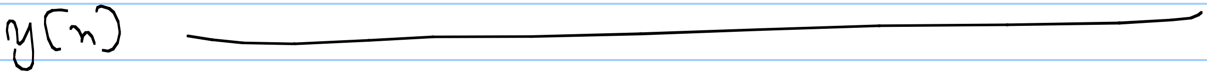
Symmetry	Length	Name	$\tau_g(\omega)$	
Even	Odd	Type I	$\frac{N-1}{2}$	integer
Even	Even	Type II	"	integer + 1/2
Odd	Odd	Type III	"	integer
Odd	Even	Type IV	"	integer + 1/2


Lin Ph.
filter
o/p

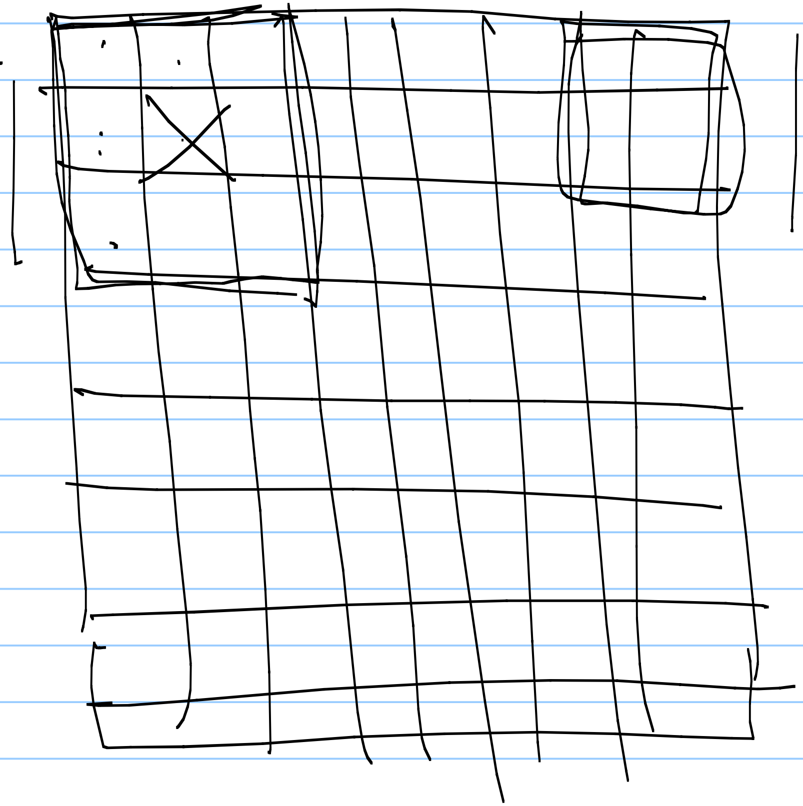

x(n)


1/3


-2
-1
0
1
2
3
4
5


y(n)

1/9



1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$$H(e^{j\omega}) = |H(e^{j\omega})| \, e^{j\theta(\omega)}$$ $\theta(\omega)$: linear phase

Consider $G(z) = \dfrac{1}{H(z)}$

$$G(e^{j\omega}) = \frac{1}{|H(e^{j\omega})|} \, e^{-j\theta(\omega)}$$

$$h[N-1-n] = h[n]$$

$$h[n] \longleftrightarrow H(z)$$

$$h[n+N] \longleftrightarrow z^{N} H(z)$$

$$h[N-n] \longleftrightarrow z^{-N} H(z^{-1})$$

$$h[N-1-n] \longleftrightarrow z^{-(N-1)} H(z^{-1})$$

$$h[n] = \pm h[N-1-n]$$ (real-valued)

$$H(z) = \pm z^{-(N-1)} H(z^{-1})$$

For the general case

$$h[n] = \pm h^*[N-1-n]$$

$$H(z) = \pm z^{-(N-1)} H^*(z^{*-1})$$

If z_0 is a zero of $H(z)$,

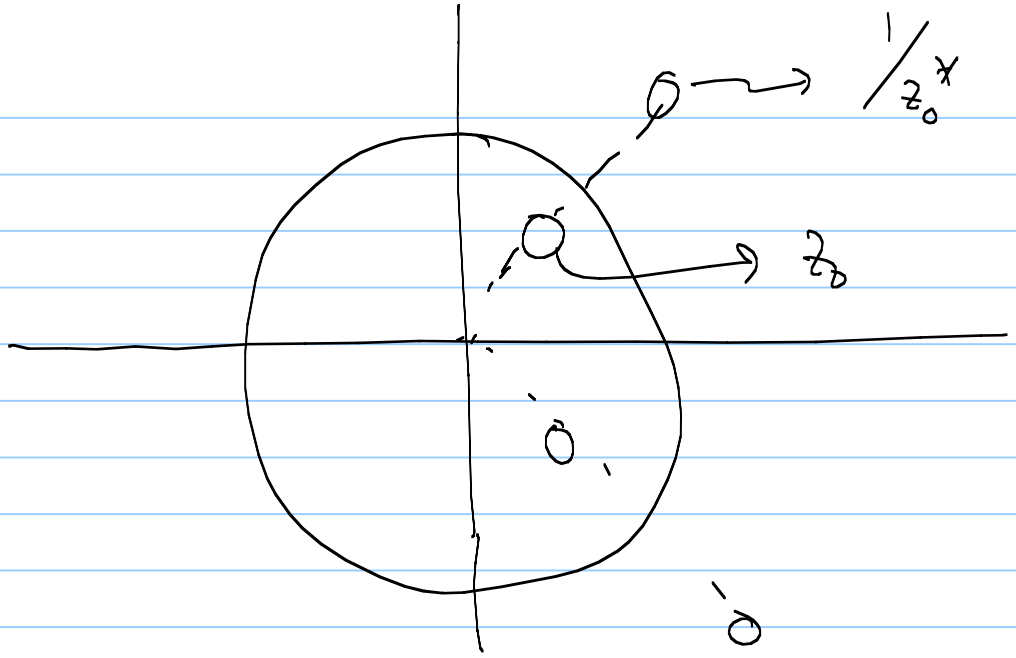
$$H(z_0) = 0$$

$$H(z_0) = \pm z_0^{-(N-1)} H^*(z_0^{*-1}) = 0$$

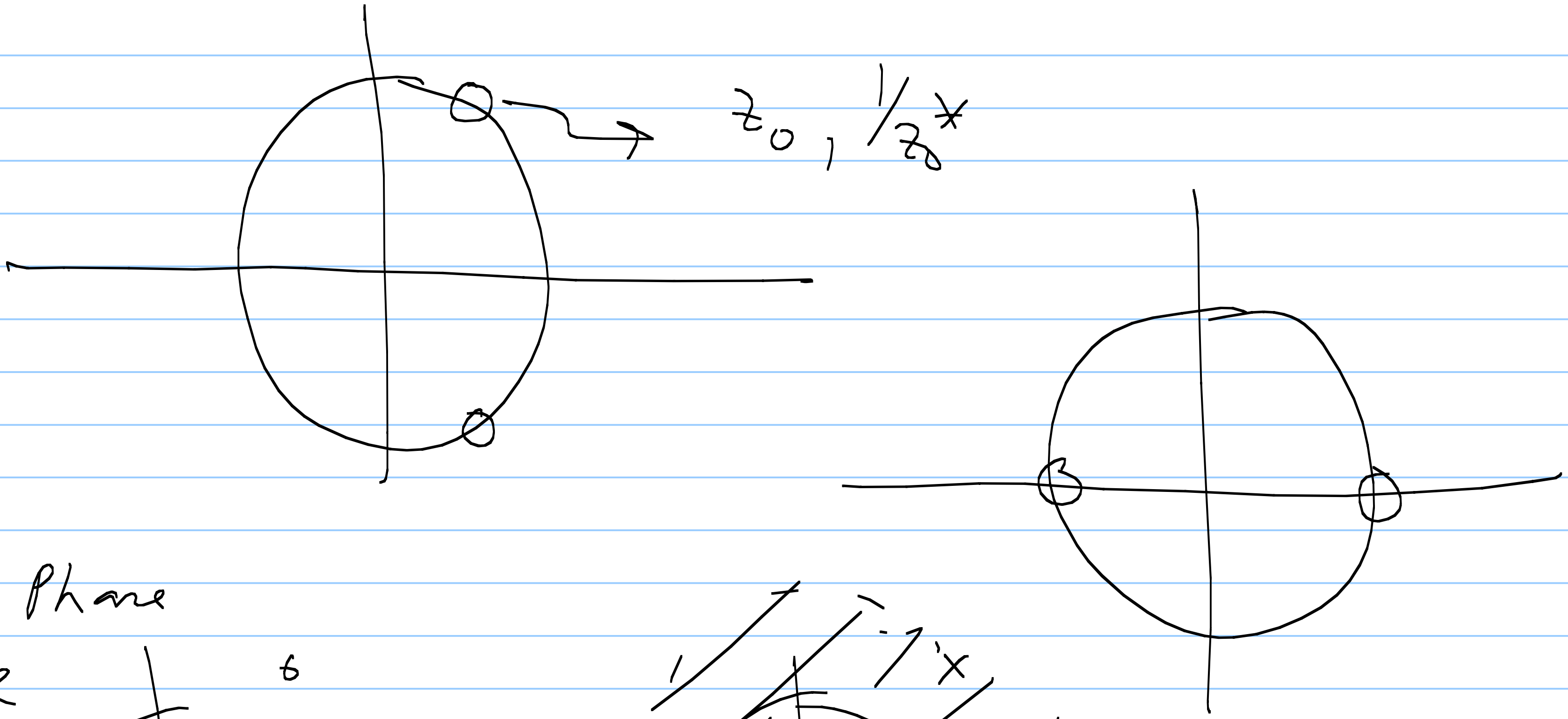
$$\Rightarrow \quad H^*(z_0^{*-1}) = 0$$

$\Rightarrow \quad \frac{1}{z_0^*}$ is also a zero

If $z_0 = re^{j\theta}$, then $\frac{1}{z_0^*} = \frac{1}{r} e^{j\theta}$

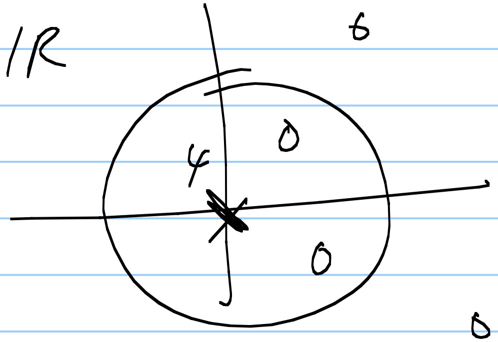


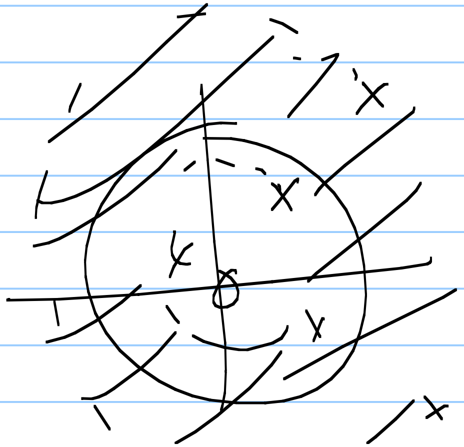
$$r e^{j\theta}, \quad r e^{-j\theta}, \quad \frac{1}{r} e^{j\theta}, \quad \frac{1}{r} e^{-j\theta}$$



Min Phase

FIR





Min. Phase IIR

EC 5142 Nov. 14 2011

For FIR filters, $h[N-1-n] = \pm h[n]$

gives rise to linear phase

For $\beta = \pi/2$, $h[N-1-n] = -h[n]$

i.e. sequence is real-valued & odd

$h[n] \longleftrightarrow H(e^{j\omega})$

↳ imaginary & odd

$H(e^{j\omega}) = j\ G(e^{j\omega})$.

Where $G(e^{j\omega})$ is real-valued

$$H(e^{j\omega}) = e^{j\pi/2} \, G(e^{j\omega})$$

Type I length odd, symmetry is even $M = N-1$

$$H(e^{j\omega}) = \sum_{n=0}^{\boxed{N-1} \rightarrow M} h[n] \, e^{-j\omega n}$$

$$= h[0] + h[M] e^{-j\omega M}$$

$$+ h[1] e^{-j\omega} + h[M-1] e^{-j\omega(M-1)}$$

$$\vdots$$

$$+ \; h\left[\frac{M}{2}-1\right] e^{-j\omega\left(\frac{M}{2}-1\right)} + h\left[\frac{M}{2}+1\right] e^{-j\omega\left(\frac{M}{2}+1\right)}$$

$$+ \; h\left[\frac{M}{2}\right] e^{-j\omega M/2} \qquad e^{-j\omega\frac{M}{2}} \cdot e^{j\omega}$$

$h[0] = h[M] \qquad h[1] = h[M-1]$, etc.

$$H(e^{j\omega}) = e^{-j\omega\frac{M}{2}} h[0]\left[e^{+j\omega M/2} + e^{-j\omega M/2}\right]$$

$$+ \; \vdots$$

$$+ \; e^{-j\omega\frac{M}{2}} \quad h\left[\frac{M}{2}-1\right]\left[e^{j\omega} + e^{-j\omega}\right]$$

$$+ \quad e^{-j\omega \frac{M}{2}} \, h\left[\frac{M}{2}\right]$$

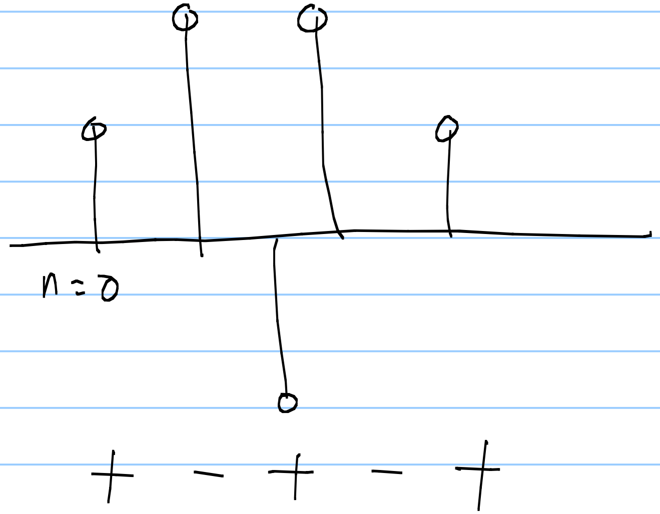
$$= e^{-j\omega\frac{M}{2}} \left[\sum_{n=0}^{\frac{M}{2}-1} h[n] \, 2 \cos\left(\omega\left(\frac{M}{2} - n\right)\right) + h\left[\frac{M}{2}\right] \right]$$

$$= e^{-j\omega\frac{M}{2}} \underbrace{A(\omega)}_{}$$

real-valued continuous function

Constrained Zeros

Type I $\quad \underline{N=5}$



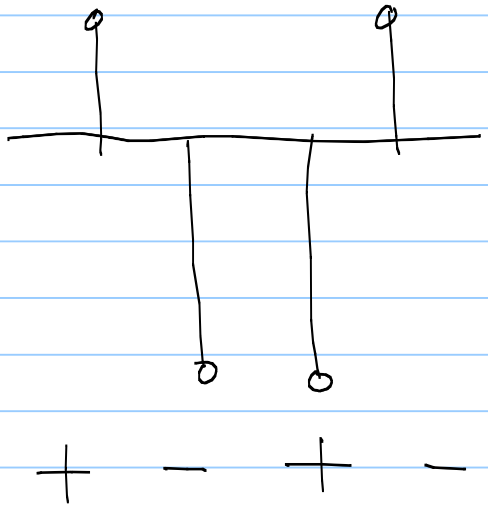
$$H(z)\Big|_{z=1} = \sum_{n=0}^{N-1} h(n)\, 1^{-n}$$

does not always add up to zero

$$H(z)\Big|_{z=-1} = \sum_{n=0}^{N-1} h(n)\, (-1)^{-n}$$

does not always add up to zero

Type II



$$\sum_{n=0}^{N-1} h(n) = 0 \ ?$$

$H(1) = 0$ not always

$H(-1) = 0$ always

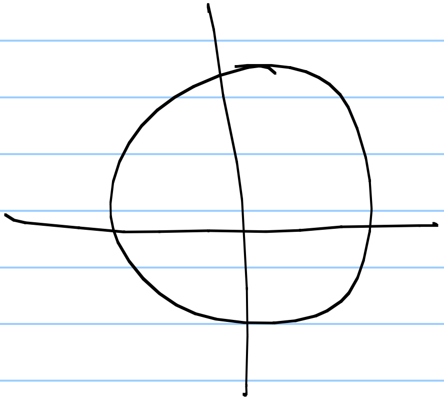
Type III $\displaystyle\sum_{n=0}^{N-1} h(n) = 0$ since it is odd symmetric

$$\sum_{n=0}^{N-1} h(n)(-1)^{n} = 0$$

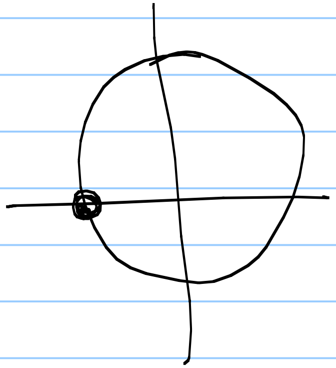
Type IV

$$\sum_{n=0}^{N-1} h(n) = 0$$ since it is odd symm

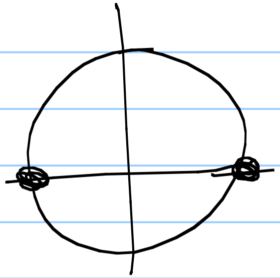
$$\sum_{n=0}^{N-1} h(n)\,(-1)^{n}$$ not always zero



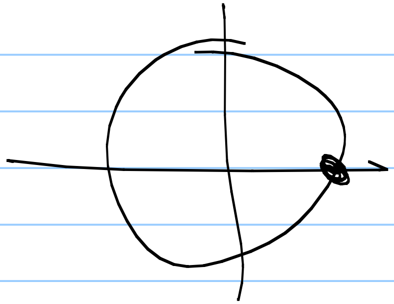
I



II



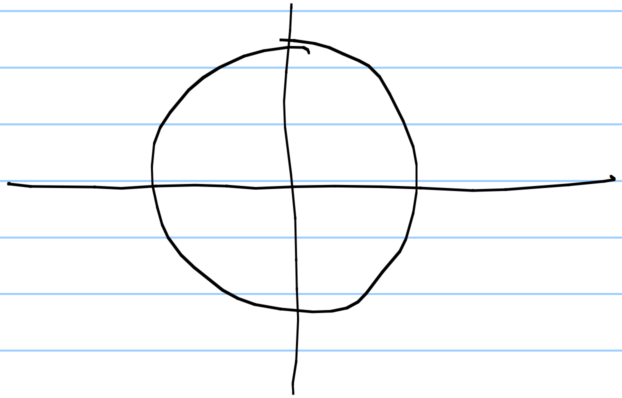
III



IV

EC 5142 15 Nov. 2011

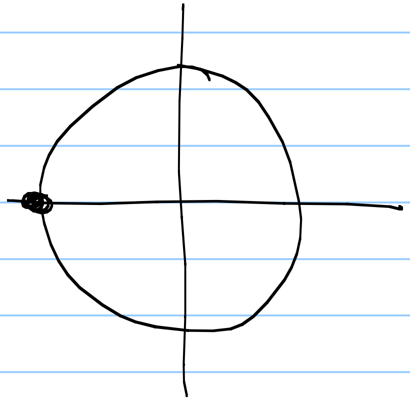
Constrained Zeros:



I

Even Symmetry

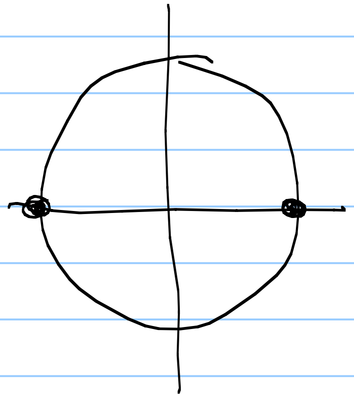
Odd length



II

Even Symm

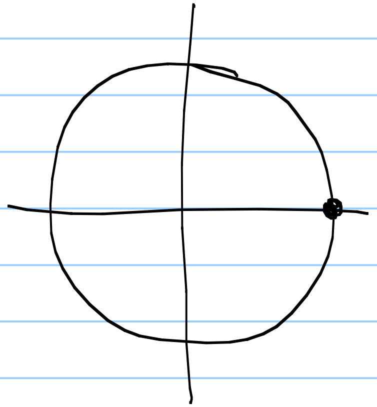
Even Length



III

Odd Symm

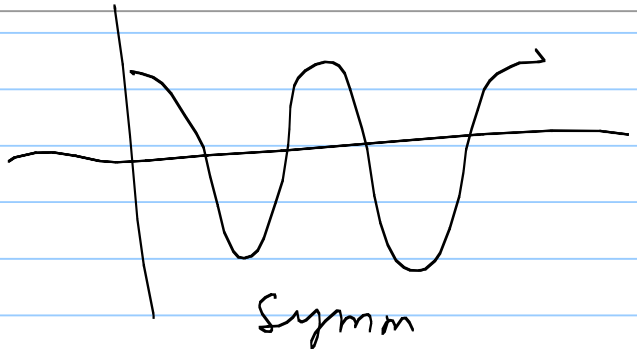
Odd length

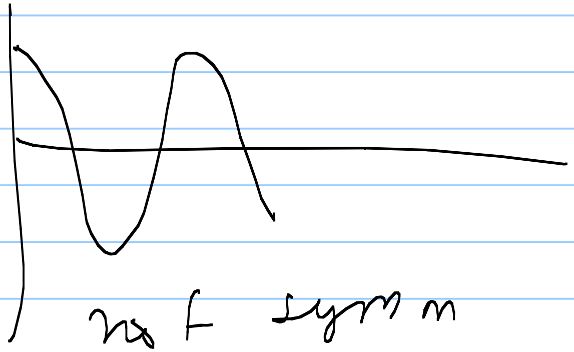


IV

Odd Symm

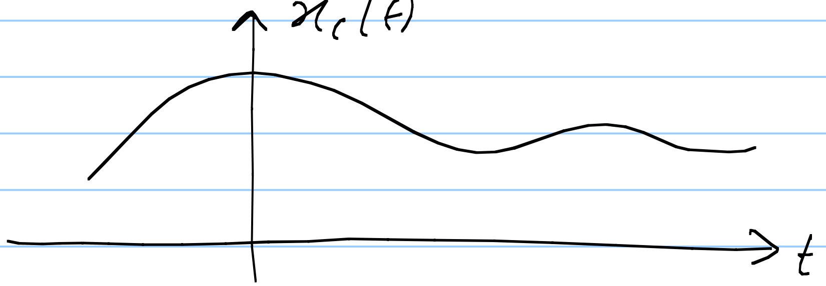
Even Length



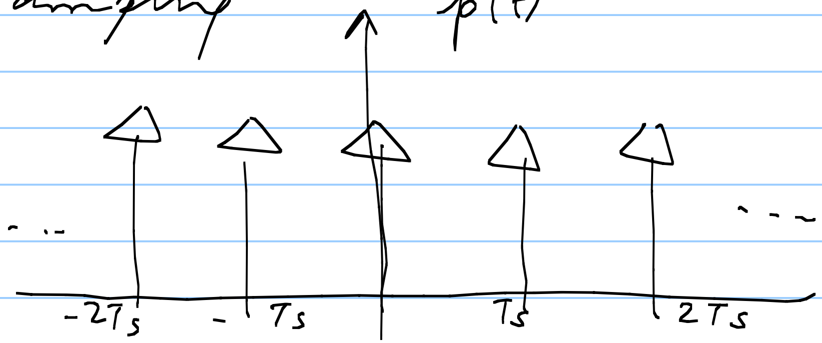


Sampling:

$x_c(t) \longrightarrow$ impulse train sampling $p(t)$



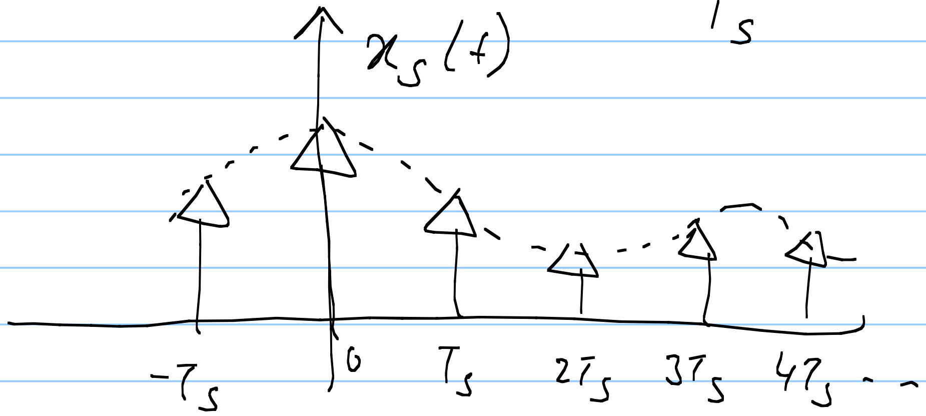
×



$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \longleftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(F - kF_s)$$

$$F_s = \frac{1}{T_s}$$

$$x_c(t) \cdot p(t) = x_s(t)$$



$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(t)\, \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT_s)\, \delta(t - nT_s)$$

$$X_s(F) = \int_{-\infty}^{\infty} x_s(t) e^{-j2\pi F t} dt$$

$$= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x_c(nT_s)\, \delta(t - nT_s) \right] e^{-j2\pi F t} dt$$

$$\overset{?}{=} \sum_{n=-\infty}^{\infty} x_c(nT_s) \int_{-\infty}^{\infty} \delta(t - nT_s)\, e^{-j2\pi F t} dt$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT_s)\, e^{-j2\pi F n T_s}$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT_s) e^{-j 2\pi \frac{F}{F_s} n}$$

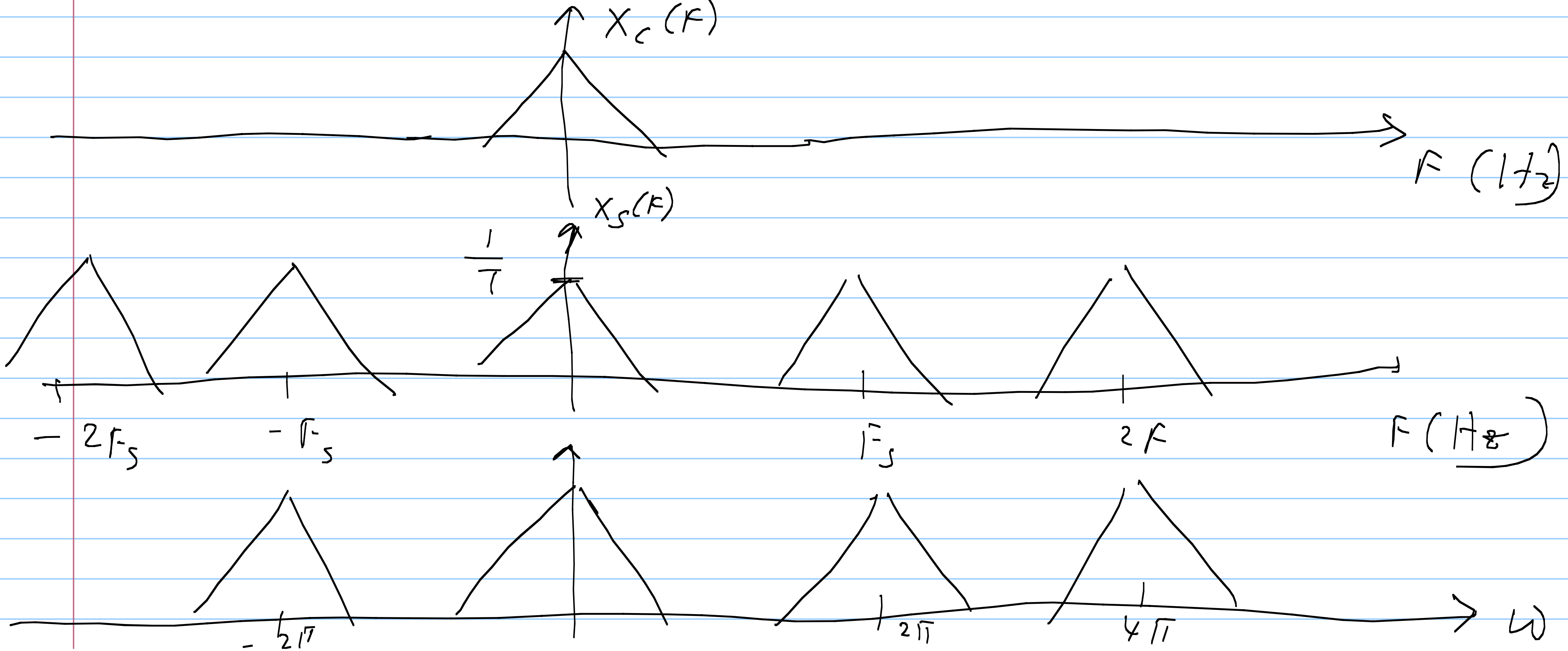
$$P(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(F - kF_s)$$

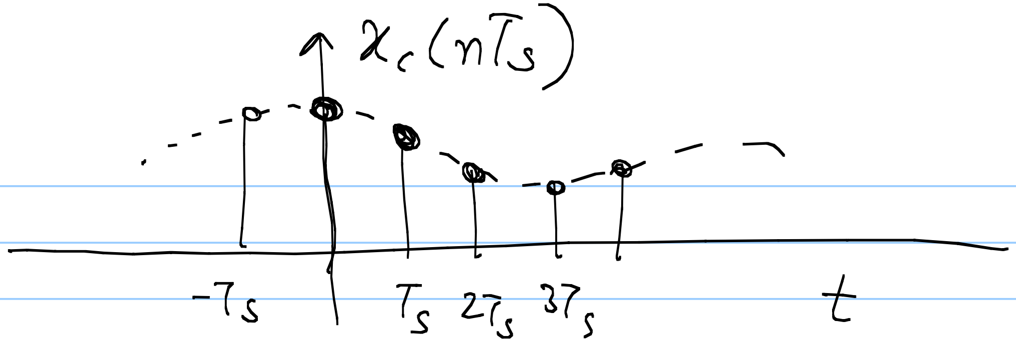
$$X_s(F) = X_c(F) * P(F)$$

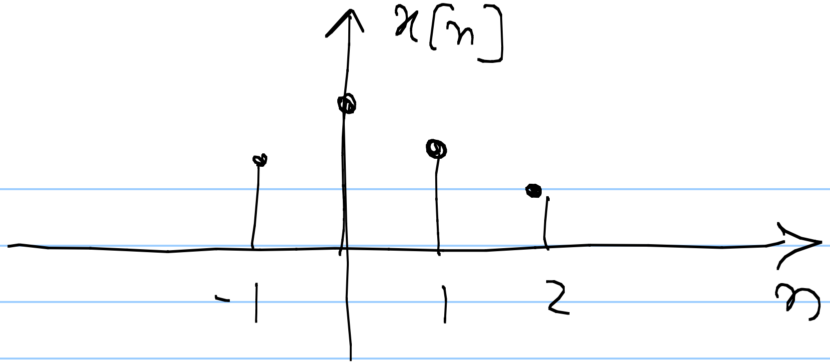
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(F - kF_s) * X_c(F)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(F - kF_s)$$

$$X_s(F) = \sum_{n=-\infty}^{\infty} x_c(nT_s)\, e^{-j 2\pi \frac{F}{F_s} n}$$







$$x[n] = x_c(nT_s)$$

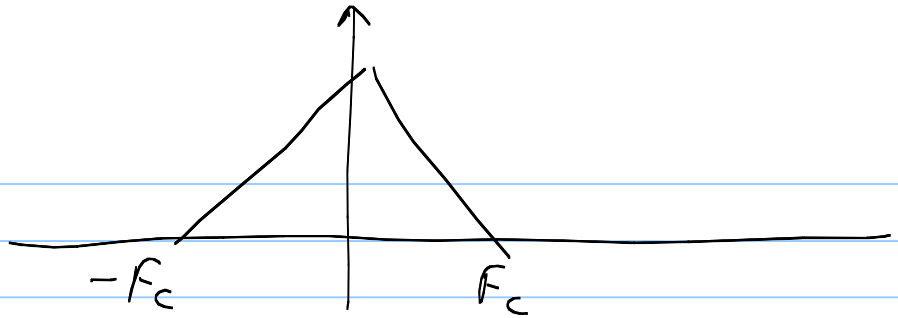
$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

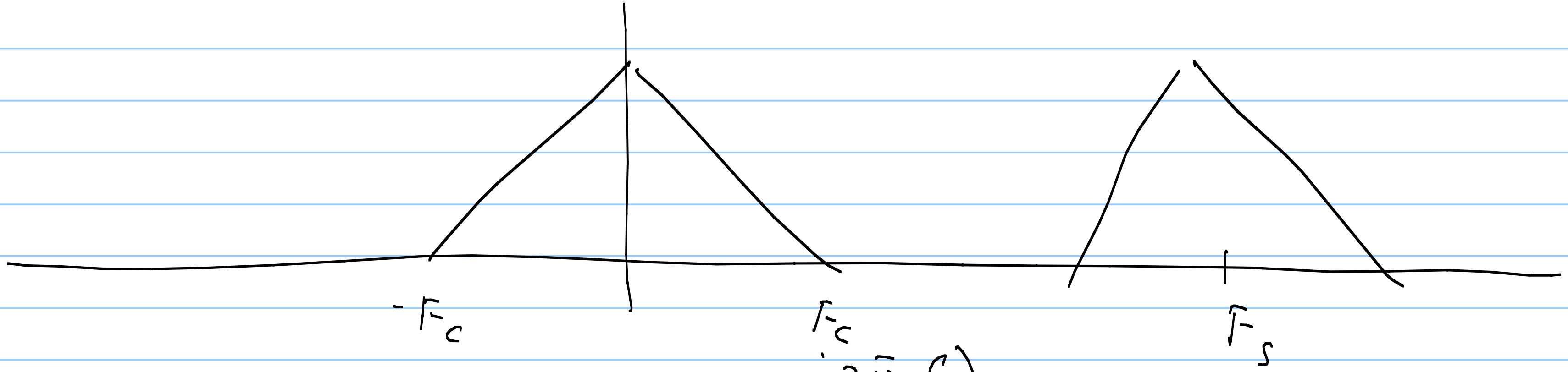
$$X_s(F) = \sum_{n=-\infty}^{\infty} \underbrace{x_c(nT_s)}_{x[n]} e^{-j 2\pi \frac{F}{F_s} n}$$

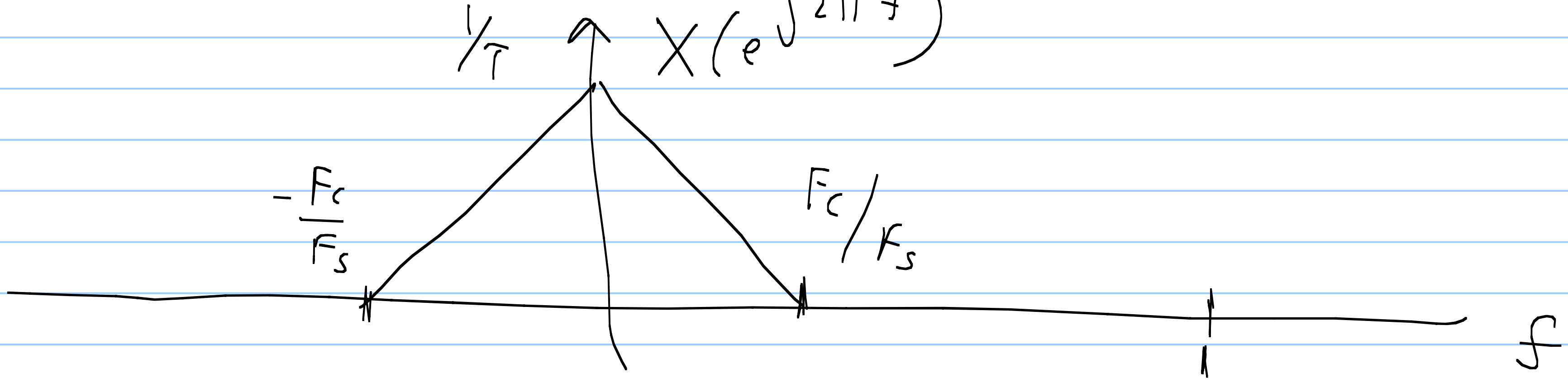
$$\frac{F}{F_s} = f \qquad\qquad \omega = 2\pi f$$

$$F \longrightarrow \frac{\omega}{2\pi T_s}$$

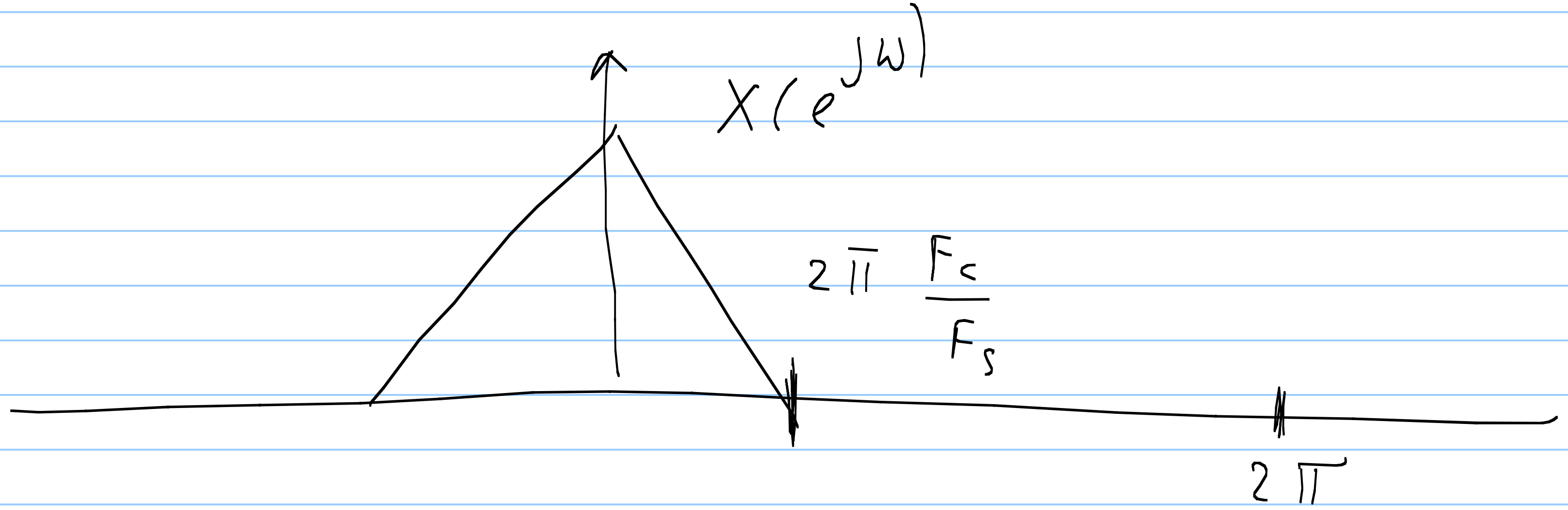
$$2\pi \, \frac{F}{F_s} \, n \longrightarrow 2\pi \, \frac{\omega}{2\pi T_s} \, \frac{1}{F_s} \, n = \omega n$$


-Fc
Fc


-Fc
Fc
Fs


1/T
X(e^{j2πf})
-Fc/Fs
Fc/Fs
f

$X(e^{j2\pi f})$ or $X(e^{jw})$



EC 5142 Nov. 16 2011

$$x_c(t) \xleftrightarrow{CTFT} X_c(F)$$

$$x_s(t) \xleftrightarrow{CTFT} X_s(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(F - kF_s)$$

$$X_s(F) = \sum_{n=-\infty}^{\infty} x_c(nT_s) e^{-j 2\pi \frac{F}{F_s} n}$$

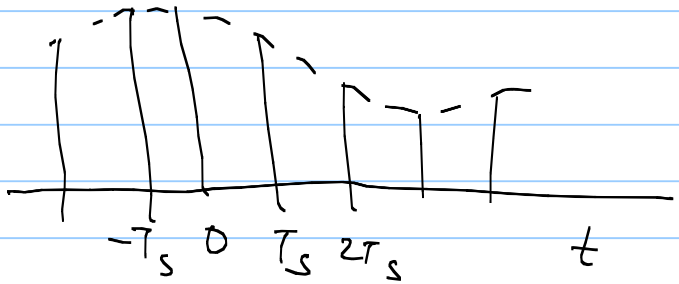
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

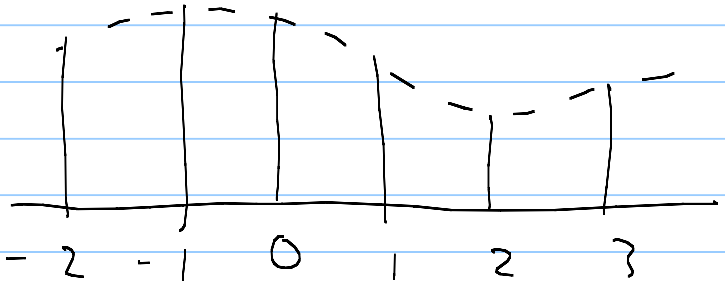
$$x[n] \equiv x_c(nT_s)$$

Note Title 16-11-2011

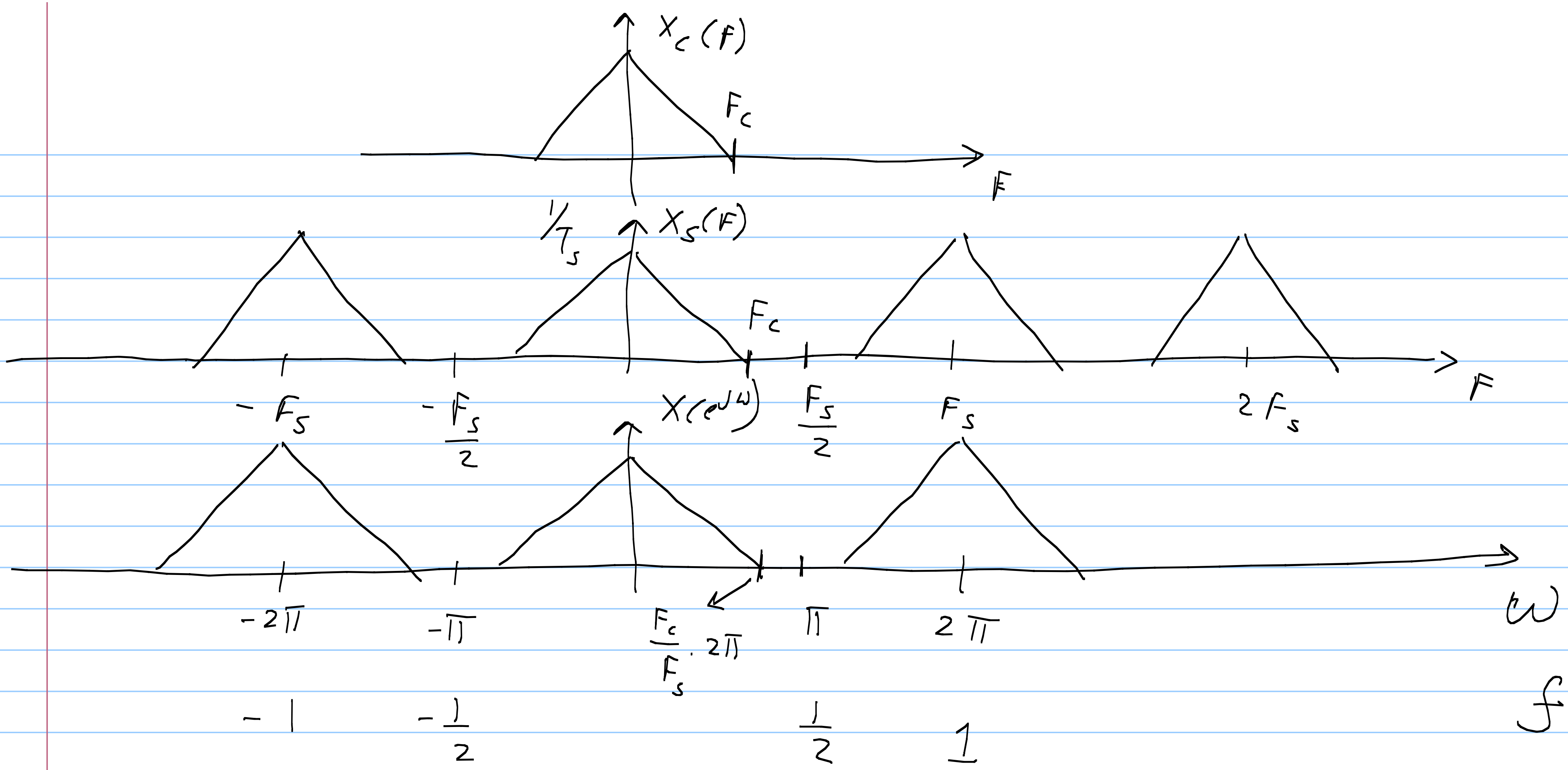
$$X_s(F)\Big|_{F \rightarrow \frac{\omega}{2\pi T_s}} = X(e^{j\omega})$$

$$\frac{\omega}{2\pi T_s} = \frac{\omega}{2\pi} F_s = f F_s$$



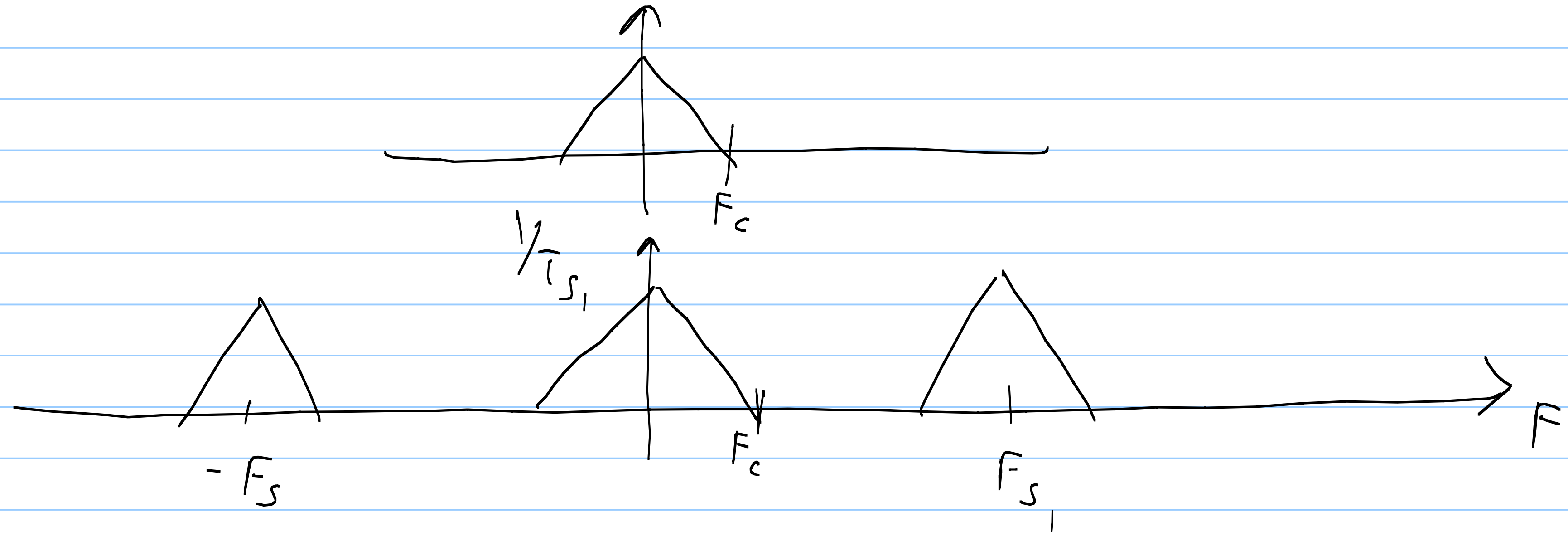


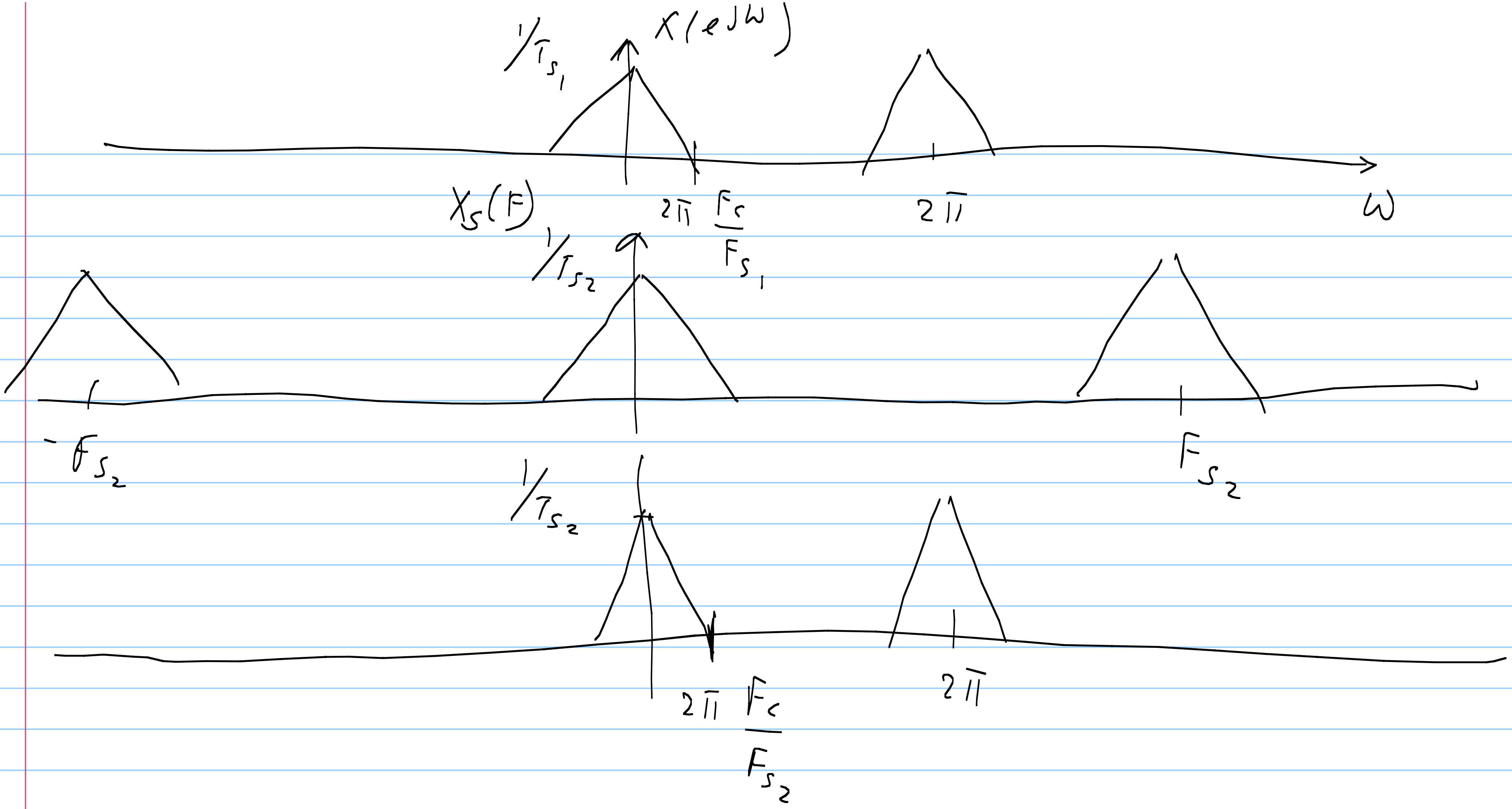
$$y(t) = x_c(t \cdot T_s) \longleftrightarrow \frac{1}{T_s} X_c(F/T_s) = \frac{1}{T_s} X_c(F \cdot F_s)$$


$X_c(F)$
F_c
F
$1/T_s$
$X_s(F)$
F_c
$-F_s$
$-\frac{F_s}{2}$
$\frac{F_s}{2}$
F_s
$2F_s$
F
$X(e^{j\omega})$
-2π
$-\pi$
$\frac{F_c}{F_s} \cdot 2\pi$
π
2π
ω
f
-1
$-\frac{1}{2}$
$\frac{1}{2}$
1

$$X_s(F) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c(F - kF_s)$$

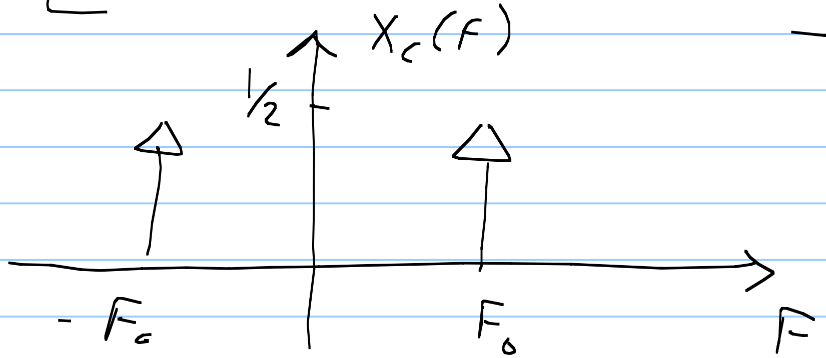
$$X(e^{j\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{2\pi T_s} - kF_s\right)$$





1/Ts1
X(eJW)
2π Fc/Fs1
2π
ω
Xs(F)
1/Ts2
-Fs2
Fs2
1/Ts2
2π Fc/Fs2
2π

$$\text{Cos}\ 2\pi F_0 t \longleftrightarrow \frac{1}{2}\left[\delta(F-F_0) + \delta(F+F_0)\right]$$



$$x[n] = x_c(nT_s) = \text{Cos}\ 2\pi F_0 n T_s$$

$$= \text{Cos}\ 2\pi \frac{F_0}{F_s} n$$

$$= \text{Cos}\ 2\pi f_0 n$$

$$= \cos w_0 n$$

$$\cos w_0 n \longleftrightarrow \pi \left[\delta(w - w_0) + \delta(w + w_0) \right] \quad -\pi \le w < \pi$$

$$X_s(F) \longleftrightarrow \frac{1}{2T_s} \left[\delta(F - F_0) + \delta(F + F_0) \right] \quad -\frac{F_s}{2} \le F < \frac{F_s}{2}$$

$$X_s(F) \Big|_{F \to \frac{w}{2\pi T_s}} = X(e^{jw})$$

$$\frac{1}{2T_s}\delta(F - F_0) \longrightarrow \frac{1}{2T_s}\delta\left(\frac{w}{2\pi T_s} - F_0\right)$$

$$\delta(at+b) = \frac{1}{|a|}\delta(t + b/a)$$

$$\frac{1}{2T_s}\delta\left(\frac{w}{2\pi T_s} - F_0\right) = \frac{1}{2T_s} 2\pi T_s \, \delta(w - 2\pi T_s F_0)$$

$$= \pi \, \delta\left(w - 2\pi \frac{F_0}{F_s}\right)$$

$$= \pi \, \delta(w - 2\pi f_0)$$

$$= \pi \, \delta(\omega - \omega_0)$$

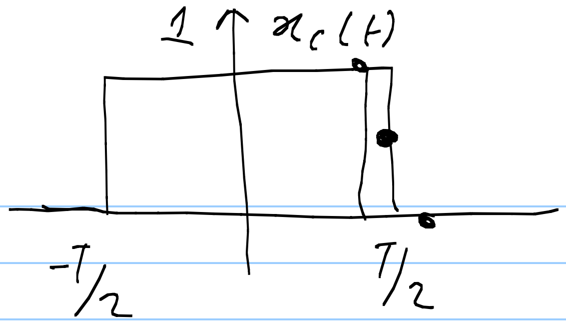
$\cos 2\pi F_0 t$

(i) $F_0 = 8 \text{ kHz}$ $F_s = 24 \text{ kHz}$

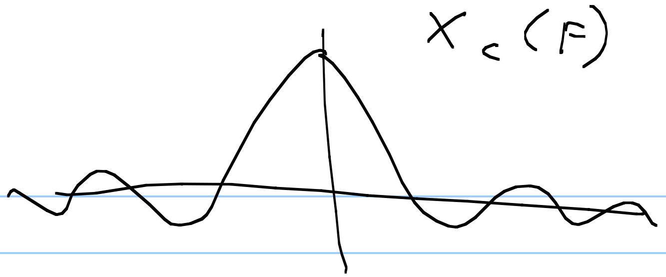
$$x[n] = x_c(nT_s) = \cos 2\pi \frac{8 \times 10^3}{24 \times 10^3} n = \cos \frac{2\pi n}{3}$$

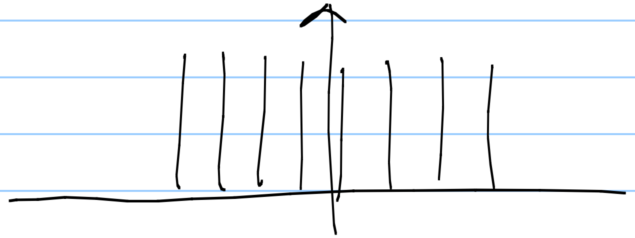
(ii) $F_0 = 16 \text{ kHz}$ $F_s = 48 \text{ kHz}$

$$x[n] = x_c(nT_s) = \cos 2\pi \frac{16 \times 10^3}{48 \times 10^3} n = \cos \frac{2\pi n}{3}$$



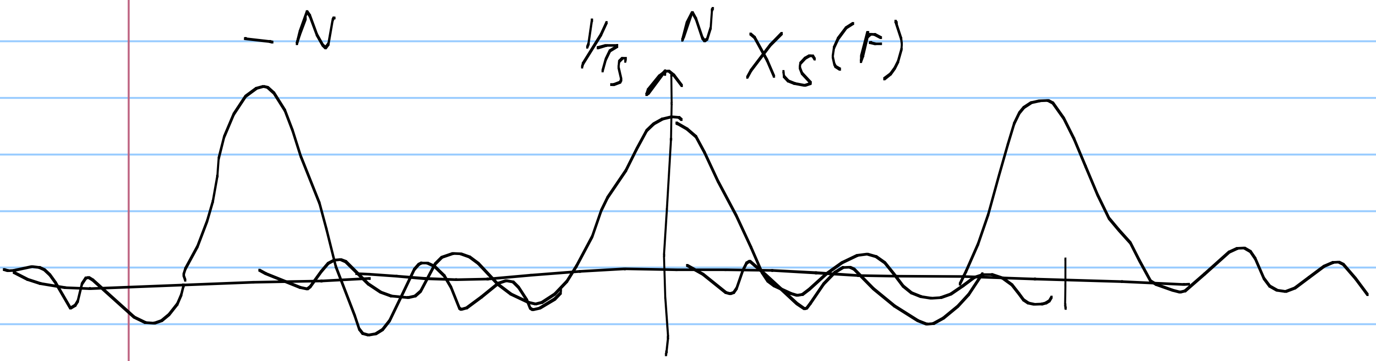
$\longleftrightarrow$





$\longleftrightarrow$

$$\frac{\sin (2N+1)\,\omega/2}{\sin \omega/2}$$



EC 5142 Nov. 17 2011

Note Title 17-11-2011

FFT Algorithm

$$\underline{X} = \underline{W}\,\underline{x}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ & & & & \\ 1 & & & & W_N^{(N-1)(N-1)} \end{bmatrix}}_{N \times N} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}}_{N \times 1} = \underbrace{\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix}}_{N \times 1}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\uparrow$$

$$— \quad e^{-j\frac{2\pi kn}{N}} \longrightarrow n = 0, 1, \dots, N-1$$

$$\Big|_{k = 0, 1, \dots N-1}$$

Requires roughly N^2 mult. & N^2 add

Decimation in Time Algorithm

$$X[k] = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} \qquad k = 0, 1, \dots, N-1$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Split the i/p seq into odd & even indices

$$= \sum_{l=0}^{\frac{N}{2}-1} x[2l]\, e^{-j\frac{2\pi}{N}(2l)k} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, e^{-j\frac{2\pi k}{N}(2r+1)}$$

$$= \sum_{l=0}^{\frac{N}{2}-1} g[l]\, e^{-j\frac{2\pi l k}{N/2}} + \underbrace{e^{-j\frac{2\pi k}{N}}}_{} \sum_{r=0}^{\frac{N}{2}-1} h[r]\, e^{-j\frac{2\pi k r}{N/2}}$$

$$= \sum_{l=0}^{\frac{N}{2}-1} g[l]\, W_{\frac{N}{2}}^{lk} + \underbrace{e^{-j\frac{2\pi k}{N}}}_{} \sum_{r=0}^{\frac{N}{2}-1} h[r]\, W_{\frac{N}{2}}^{kr}$$

$k = 0, 1, \ldots, N-1$

$G[k]$ W_N^k $H[k]$

$$N^2 \longrightarrow N + 2\left(\frac{N}{2}\right)^2$$

$N = 8$ $N^2 = 64$

$8 + 2 * 16 = 40$

$$\left(\frac{N}{2}\right)^2 \longrightarrow \frac{N}{2} + 2\left(\frac{N}{4}\right)^2$$

$$N^2 \longrightarrow N + 2\left[\frac{N}{2} + 2\left(\frac{N}{4}\right)^2\right]$$

$$= N + N + 4\left(\frac{N}{4}\right)^2$$

$$N = 2^L \Rightarrow L = \log_2 N$$

We have L stages

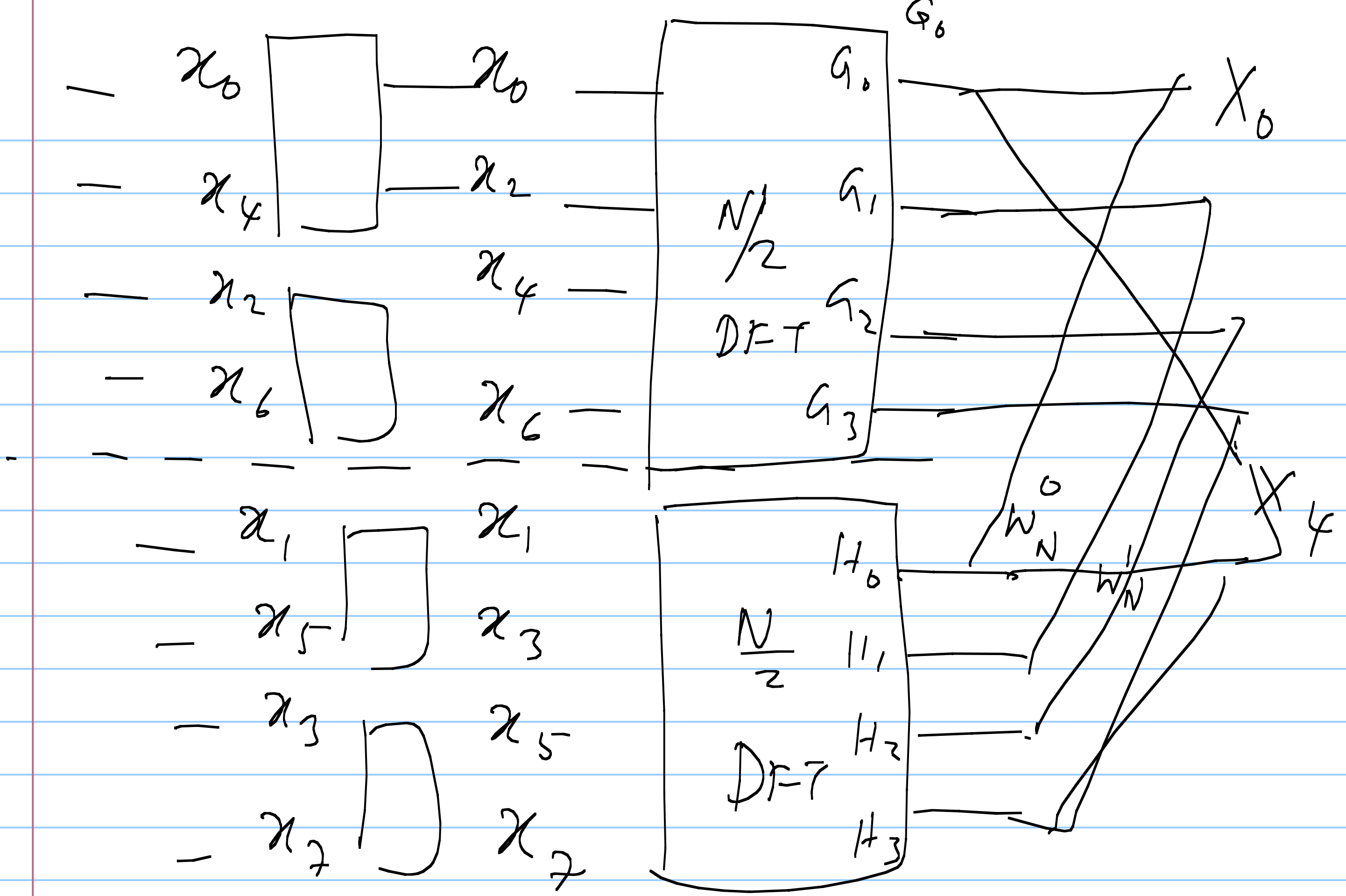
$$N^2 \longrightarrow NL = N\log_2 N$$

$\underline{N=8}$

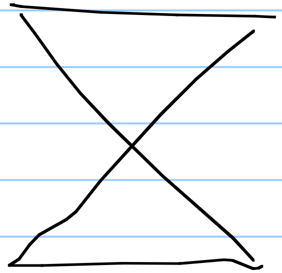
2 pt. DFT

$x_0 + x_1$

$x_0 - x_1$



x_0
x_4
x_2
x_6
x_1
x_5
x_3
x_7
x_0
x_2
x_4
x_6
x_1
x_3
x_5
x_7
$N/2$
DFT
$\frac{N}{2}$
DFT
G_0
G_0
G_1
G_2
G_3
H_0
H_1
H_2
H_3
W_N^0
W_N^1
X_0
X_4



x_0	000	→	000	0
x_4	001		100	4
x_2	010		010	2
x_6	011		110	6
x_1	100		001	1
x_5	101		101	5
x_3	110		011	3
x_7	111		111	7