Assignment 2

DMCs, Differential Entropy, Channel Capacity

Date Assigned: Mar 16

1. *Preprocessing the output.* One is given a communication channel with channel transition probabilities $p(y|x)$ and channel capacity $C = \max_{p(x)} I(X;Y)$. A helpful statistician preprocesses the output by forming $\widehat{Y} = g(Y)$. He claims that this will strictly improve the capacity.

   (a) Show that he is wrong.

   (b) Under what condition does he not strictly decrease the capacity.

2. *An additive noise channel.* Find the channel capacity of the discrete memoryless channel, $Y = X + Z$, where $\Pr\{Z = 0\} = \Pr\ \{Z = a\} = \frac{1}{2}$ and the alphabet for X is $A = \{0, 1\}$. Assume that Z is independent of X. Observe that the channel capacity depends on the value of a.

3. *Channel capacity.* Consider the discrete memoryless channel $Y = X + Z(\mod 11)$, where $Z \in \{1, 2, 3\}$ with uniform probability and $X \in \{0, 1, \ldots, 10\}$. Assume that Z is independent of X.

   (a) Find the capacity.

   (b) What is the maximizing input distribution $p^*(x)$?

4. *Cascade of binary symmetric channels.* Show that a cascade of n identical binary symmetric channels,

$$X_0 \rightarrow \boxed{\text{BSC}\#1} \rightarrow X_1 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow \boxed{\text{BSC}\#n} \rightarrow X_n,$$

   each with a transition probability p is equivalent to a single BSC with error probability $\frac{1}{2}(1-(1-2p)^n)$ and hence that $\lim_{n\to\infty} I(X_0; X_n) = 0 \,\text{if}\, p \neq 0, 1$. No encoding or decoding take place at the intermediate terminals $X_1, \ldots, X_{n-1}$. Thus the capacity of the cascade tends to zero.

5. *Time varying channels.* Consider a time-varying discrete memoryless channel with input vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and output vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$. The conditional distribution is given by $p(y|x) = \prod_{i=1}^{n} P_i(y_i|x_i)$, where

$$P_i(y_i|x_i) = \begin{bmatrix} 1-p_i & p_i \\ p_i & 1-p_i \end{bmatrix}.$$

   Find $\max_{p(x)} I(\mathbf{X}; \mathbf{Y})$.

6. *Differential entropy.* Evaluate the differential entropy $h(X) = -\int f \log f$ for the following:

   (a) The exponential density, $f(x) = \lambda e^{-\lambda x}, x \geq 0$.

   (b) The Laplace density, $f(x) = \frac{1}{2} e^{-\lambda|x|}$.

   (c) The sum of X_1 and X_2, where X_1 and X_2 are independent normal random variables with means μ_i and variance σ_i^2, $i = 1, 2$.

7. *Mutual information for correlated normals* Find the mutual information I(X;Y), where

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2 \left(\mathbf{0}, \begin{bmatrix} \sigma^2 & p\sigma^2 \\ p\sigma^2 & \sigma^2 \end{bmatrix} \right).$$

   Evaluate I(X;Y) for $\rho = 1, \rho = 0$, and $\rho = -1$ and comment.

8. *Uniformly distributed noise.* Let the input random variable X for a channel be uniformly distributed over the interval $-1/2 \leq x \leq +1/2$. Let the output of the channel be $Y = X + Z$, where the noise random variable is uniformly distributed over the interval $-a/2 \leq z \leq +a/2$.

   (a) Find $I(X;Y)$ as a function of a.

(b) For $a = 1$ find the capacity of the channel when the input X is peak limited, that is, the range of X is limited to $-1/2 \leq x \leq 1/2$. What probability distribution on X maximizes the mutual information $I(X;Y)$?

(c) Find the capacity of the channel for all values of a, again assuming that the range of X is limited to $-1/2 \leq x \leq +1/2$.

9. *A channel with two independent looks at Y.* Let Y_1 and Y_2 be conditionally-independent and conditionally-identically distributed given X.

   (a) Show $I(X;Y_1,Y_2) = 2I(X;Y_1) - I(Y_1;Y_2)$.

   (b) Conclude that the capacity of the channel

$$X \rightarrow \boxed{\text{Channel1}} \rightarrow (Y_1, Y_2)$$

   is less than twice the capacity of the channel

$$X \rightarrow \boxed{\text{Channel2}} \rightarrow (Y_1).$$

10. *The two-look Gaussian Channel.*

$$X \rightarrow \boxed{\text{Channel1}} \rightarrow (Y_1, Y_2)$$

   Consider the ordinary Gaussian channel with two correlated looks at X, i.e., $Y = (Y_1, Y_2)$ where $Y_1 = X + Z_1$ and $Y_2 = X + Z_2$ with a power constraint P on X, and $(Z_1, Z_2) \sim N_2(0, K)$, where

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}.$$

   Find the capacity C for

   (a) $\rho = 1$

   (b) $\rho = 0$

   (c) $\rho = -1$

11. Find the capacity and optimizing input probability assignment for the DMC's with transition matrices given below:

(a) $p(y|x) = \begin{bmatrix} 1-\epsilon-\delta & \delta & \epsilon \\ \epsilon & \delta & 1-\epsilon-\delta \end{bmatrix}$ (b) $p(y|x) = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}$

(c) $p(y|x) = \begin{bmatrix} 1-\epsilon & \epsilon & 0 \\ 0 & 1-\epsilon & \epsilon \\ \epsilon & 0 & 1-\epsilon \end{bmatrix}$ (d) $p(y|x) = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \end{bmatrix}$

(e) $p(y|x) = \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{bmatrix}$ (f) $p(y|x) = \begin{bmatrix} 1-\epsilon & \epsilon \\ \delta & 1-\delta \end{bmatrix}$