Assignment on Linear Block Codes

Error Control Coding

Questions marked (Q) or (F) are questions from previous quizzes or final exams, respecively.

1. Consider two codes with parity-check matrices

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

   respectively.

   (a) List all codewords of the two codes.

   (b) Provide G and H in systematic form for both codes.

2. A code is defined by the following generator matrix.

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

   (a) Find n and k for the code.

   (b) Find the minimum distance of the code.

3. Let $u = [u_1, u_2, \cdots, u_n]$, $v = [v_1, v_2, \cdots, v_n]$ and $w = [w_1, w_2, \cdots, w_n]$ be binary n-tuples. Show the following:

   (a) $d_H(u, v) = \text{wt}(u + v)$.

   (b) If $u*v = (u_1v_1, u_2v_2, \cdots, u_nv_n)$, $d_H(u, v) = \text{wt}(u) + \text{wt}(v) - 2\text{wt}(u*v)$.

   (c) If u and v have even weight, $u + v$ has even weight.

   (d) $d_H(u, v) \leq d_H(u, w) + d_H(w, v)$.

   (e) $\text{wt}(u + v) \geq \text{wt}(u) - \text{wt}(v)$.

4. Provide H of a $(n, 8, 4)$ binary linear code with minimum n. How about $(n, 16, 4)$ and $(n, 32, 4)$ codes?

5. (a) Find the dual of the $(n, 1, n)$ repetition code.

   (b) Find the dual of the $(n, n-1, 2)$ even weight code.

6. If a code C has an invertible generator matrix, what is C?

7. G_1 and G_2 are generator matrices for $[n_1, k, d_1]$ and $[n_2, k, d_2]$ codes, respectively.

   (a) What are the three parameters of the code with $G = [G_1|G_2]$?

   (b) What are the three parameters of the code with

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}?$$

8. If C is a linear code with both even and odd weight codewords, show that the number of even-weight codewords is equal to the number of odd-weight codewords. Show that the even-weight codewords form a linear code.

9. Let C be an $[n, k]$ code whose generator matrix G contains no all-zero column. Arrange all the codewords of C in a $2^k \times n$ array. Show that each column contains 2^{k-1} ones and 2^{k-1} zeros.

10. Design a maximum-likelihood decoder for a code with parity-check matrix

$$H = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

    over a BSC. Find the probability of codeword error.

11. Cosntruct the standard array for a code with parity-check matrix

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

    Find the probability of message bit error.

12. Consider the $[15, 11, 3]$ Hamming code.

    (a) Give H for the code.

    (b) Encode the message 11111100000.

    (c) Decode 111000111000111.

13. If C is a linear binary code and $u \notin C$, show that $C \cup (u + C)$ is also a linear code.

14. Show that

    (a) $(C^\perp)^\perp = C$.

    (b) Let $C+D = \{u+v : u \in C,\ v \in D\}$. Show that $(C+D)^\perp = C^\perp \cap D^\perp$.

15. Show that, in each of the following cases, the generator matrices G and G' generate equivalent codes.

    (a)
$$G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad G' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

    (b)
$$G = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad G' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

16. (Q) A $(7, 3)$ code C has the following parity-check matrix with two missing columns.
$$H = \begin{bmatrix} 1 & \_ & \_ & 1 & 0 & 0 & 0 \\ 0 & \_ & \_ & 0 & 1 & 0 & 0 \\ 1 & \_ & \_ & 0 & 0 & 1 & 0 \\ 1 & \_ & \_ & 0 & 0 & 0 & 1 \end{bmatrix}.$$

    Provide possible bits in the missing columns given that [0110011] is a codeword of C and the minimum distance of C is 4.

17. (Q) A code C has generator matrix
$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

    (a) What is the minimum distance of C?
    (b) Decode the received word [111111111111].

18. (F) A code C has only odd-weight codewords. Say "possible" or "impossible" for the following with reasons.

    (a) C is linear.
    (b) Minimum distance of C is 5.
    (c) C is self-orthogonal i.e. $C \subseteq C^{\perp}$.