Circuit Analysis Using Fourier and Laplace Transforms

EE2015: Electrical Circuits and Networks

Nagendra Krishnapura
https://www.ee.iitm.ac.in/∼nagendra/

Department of Electrical Engineering
Indian Institute of Technology, Madras
Chennai, 600036, India

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Based on

- $exp(st)$ being an eigenvector of linear systems
  - Steady-state response to $exp(st)$ is $H(s)\exp(st)$ where $H(s)$ is some scaling factor
- Signals being representable as a sum (integral) of exponentials $exp(st)$

Periodic $x(t)$ can be represented as sums of complex exponentials

- $x(t)$ periodic with period T_0
- Fundamental (radian) frequency $\omega_0 = 2\pi/T_0$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \exp(jk\omega_0 t)$$

  - $x(t)$ as a weighted sum of orthogonal basis vectors $\exp(jk\omega_0 t)$
  - Fundamental frequency ω_0 and its harmonics
  - a_k: Strength of k^{th} harmonic

- Coefficients a_k can be derived using the relationship

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) \exp(-jk\omega_0 t) dt$$

  - "Inner product" of $x(t)$ with $\exp(jk\omega_0 t)$

Fourier series

- Alternative form

$$x(t) = a_0 + \sum_{k=1}^{\infty} b_k \cos(k\omega_0 t) + c_k \sin(k\omega_0 t)$$

- Coefficients b_k and c_k can be derived using the relationship

$$b_k = \frac{2}{T_0}\int_0^{T_0} x(t)\cos(k\omega_0 t)dt$$

$$c_k = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(k\omega_0 t)dt$$

- Another alternative form

$$x(t) = a_0 + \sum_{k=1}^{\infty} d_k \cos(k\omega_0 t + \phi_k)$$

- Coefficients b_k and c_k can be derived using the relationship

$$d_k = \sqrt{b_k^2 + c_k^2}$$

$$\phi_k = -\tan^{-1}\left(\frac{c_k}{b_k}\right)$$

If $x(t)$ satisfies the following (Dirichlet) conditions, it can be represented by a Fourier series

- $x(t)$ must be absolutely integrable over a period

$$\int_0^{T_0} |x(t)|\, dt \text{ must exist}$$

- $x(t)$ must have a finite number of maxima and minima in the interval $[0, T_0]$
- $x(t)$ must have a finite number of discontinuities in the interval $[0, T_0]$

- Aperiodic $x(t)$ can be expressed as an integral of complex exponentials

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_\omega(\omega)\exp(j\omega t)d\omega$$

  - $x(t)$ as a weighted sum (integral) of orthogonal vectors $\exp(j\omega t)$
  - Continuous set of frequencies ω
  - $X_\omega(\omega)d\omega$: Strength of the component $\exp(j\omega t)$
  - $X_\omega(\omega)$: Fourier transform of $x(t)$

- $X_\omega(\omega)$ can be derived using the relationship

$$X_\omega(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)dt$$

  - “Inner product” of $x(t)$ with $\exp(j\omega t)$

If $x(t)$ satisfies either of the following conditions, it can be represented by a Fourier transform

- Finite L_1 norm

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

- Finite L_2 norm

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty$$

- Many common signals such as sinusoids and unit step fail these criteria
  - Fourier transform contains impulse functions
  - Laplace transform more convenient

Fourier transform

- $x(t)$ in volts $\Rightarrow X_\omega(\omega)$ has dimensions of volts/frequency
- $X_\omega(\omega)$: Density over frequency
- Traditionally, Fourier transform $X_f(f)$ defined as density per "Hz" (cyclic frequency)
- Scaling factor of $1/2\pi$ when integrated over ω (radian frequency)

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X_f(f) \exp(j2\pi ft) df \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_\omega(\omega) \exp(j\omega t) d\omega \end{aligned}$$

- $X_\omega(\omega) = X_f(\omega/2\pi)$
- $X_f(f)$: volts/Hz (density per Hz) if $x(t)$ is a voltage signal

$$X_f(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) dt$$

$$X_\omega(\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t) dt$$

Fourier transform as a function of $j\omega$

- If $j\omega$ is used as the independent variable

$$x(t) = \frac{1}{j2\pi}\int_{-j\infty}^{j\infty} X(j\omega)\exp(j\omega t)d(j\omega)$$

- $X(j\omega) = X_\omega(\omega)$
- Same function, but $j\omega$ is the independent variable
- Scaling factor of $1/j2\pi$
- With $j\omega$ as the independent variable, the definition is the same as that of the Laplace transform

Fourier transform pairs

- Signals in $-\infty \leq t \leq \infty$

$$
\begin{aligned}
1 &\leftrightarrow 2\pi\delta(\omega) \\
\exp(j\omega_0 t) &\leftrightarrow 2\pi\delta(\omega - \omega_0) \\
\cos(\omega_0 t) &\leftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \\
\sin(\omega_0 t) &\leftrightarrow \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0) \\
\exp(-a|t|) &\leftrightarrow \frac{2a}{a^2 + \omega^2}
\end{aligned}
$$

- Not very useful in circuit analysis

Fourier transform pairs

- Signals in $0 \leq t \leq \infty$

$$
\begin{aligned}
u(t) &\leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega} \\
\exp(j\omega_0 t)u(t) &\leftrightarrow \pi\delta(\omega - \omega_0) + \frac{1}{j(\omega - \omega_0)} \\
\cos(\omega_0 t)u(t) &\leftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) + \frac{j\omega}{\omega_0^2 - \omega^2} \\
\sin(\omega_0 t)u(t) &\leftrightarrow \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0) + \frac{\omega_0}{\omega_0^2 - \omega^2} \\
\exp(-at)u(t) &\leftrightarrow \frac{1}{j\omega + a}
\end{aligned}
$$

- Useful for analyzing circuits with inputs starting at $t = 0$

Circuit analysis using the Fourier transform

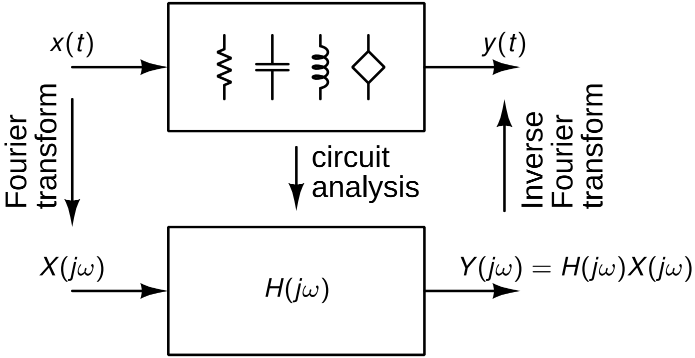
- For an input $\exp(j\omega t)$, steady state output is $H(j\omega)\exp(j\omega t)$
- A general input $x(t)$ can be represented as a sum (integral) of complex exponentials $\exp(j\omega t)$ with weights $X(j\omega)d\omega/2\pi$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\exp(j\omega t)d\omega$$

- Linearity $\Rightarrow$ steady-state output $y(t)$ is the superposition of responses $H(j\omega)\exp(j\omega t)$ with the same weights $X(j\omega)d\omega/2\pi$

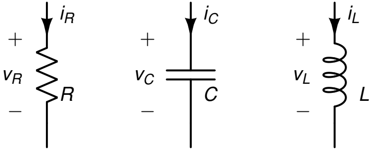
$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overbrace{X(j\omega)H(j\omega)}^{Y(j\omega)}\exp(j\omega t)d\omega$$

- Therefore, $y(t)$ is the inverse Fourier transform of $Y(j\omega) = H(j\omega)X(j\omega)$



- Calculate $X(j\omega)$
- Calculate $H(j\omega)$
  - Directly from circuit analysis
  - From differential equation, if given
- Calculate (look up) the inverse Fourier transform of $H(j\omega)X(j\omega)$ to get $y(t)$

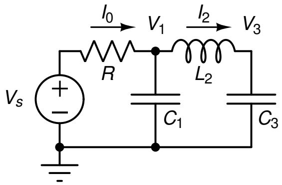
- In steady state with an input of $\exp(j\omega t)$, "Ohms law" also valid for L, C



		$v(t)$	$i(t)$	$v(t)/i(t)$
Resistor	$v_R = Ri_R$	$RI_R \exp(j\omega t)$	$I_R \exp(j\omega t)$	R
Inductor	$v_L = L\,(di_L/dt)$	$j\omega LI_L \exp(j\omega t)$	$I_L \exp(j\omega t)$	$j\omega L$
Capacitor	$i_C = C\,(dv_C/dt)$	$V_C \exp(j\omega t)$	$j\omega CV_C \exp(j\omega t)$	$1/\,(j\omega C)$

- I_R, I_L, V_C: Phasors corresponding to i_R, i_L, v_C
- Use analysis methods for resistive circuits with dc sources to determine $H(j\omega)$ as ratio of currents or voltages
  - e.g. Nodal analysis, Mesh analysis, etc.
- No need to derive the differential equation

Example: Calculating the transfer function



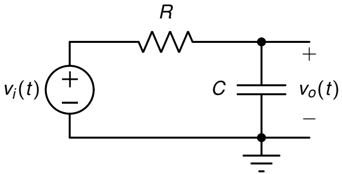
- Mesh analysis with currents I_0, I_2

$$\begin{bmatrix} R + \frac{1}{j\omega C_1} & -\frac{1}{j\omega C_1} \\ -\frac{1}{j\omega C_1} & j\omega L_2 + \frac{1}{j\omega C_1} + \frac{1}{j\omega C_3} \end{bmatrix} \begin{bmatrix} I_0 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix}$$

$$\frac{I_0\left(j\omega\right)}{V_s\left(j\omega\right)} = \frac{(j\omega)^3 C_1 C_3 L_2 + (j\omega)\left(C_3 + C_1\right)}{(j\omega)^3 C_1 C_3 L_2 + (j\omega)^2 C_3 L_2 + (j\omega)\left(C_3 + C_1\right) R + 1}$$

$$\frac{I_2\left(j\omega\right)}{V_s\left(j\omega\right)} = \frac{(j\omega)\, C_3}{(j\omega)^3 C_1 C_3 L_2 + (j\omega)^2 C_3 L_2 + (j\omega)\left(C_3 + C_1\right) R + 1}$$

- $V_1 = \left(I_0 - I_2\right) / \left(j\omega C_1\right)$, $V_3 = I_2 / \left(j\omega C_3\right)$

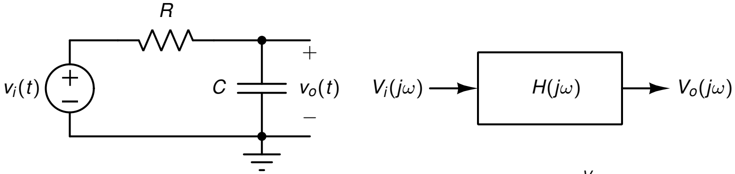


$$v_i(t) = V_p \exp(-at)u(t)$$

- From direct time-domain analysis, with zero initial condition

$$v_o(t) = \overbrace{\frac{V_p}{1-aCR}\exp(-at)u(t)}^{\text{Steady-state response}} - \overbrace{\frac{V_p}{1-aCR}\exp(-t/RC)u(t)}^{\text{Transient response}}$$

Example: Calculating the response of a circuit



$v_i(t) = V_p \exp(-at)u(t)$

$V_i(j\omega) = \frac{V_p}{a+j\omega}$

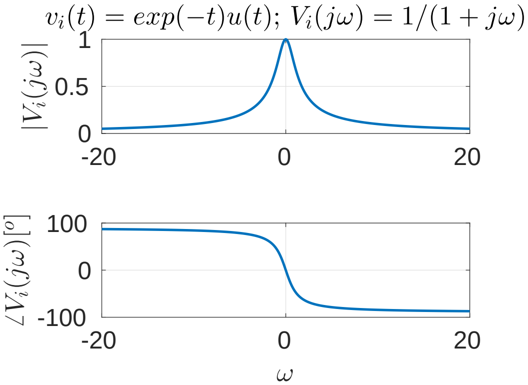
- Using Fourier transforms and transfer function

$$\begin{aligned} V_o(j\omega) &= \frac{V_p}{a+j\omega}\frac{1}{1+j\omega CR} \\ &= \frac{V_p}{1-aCR}\frac{1}{a+j\omega} - \frac{V_p}{1-aCR}\frac{CR}{1+j\omega CR} \end{aligned}$$

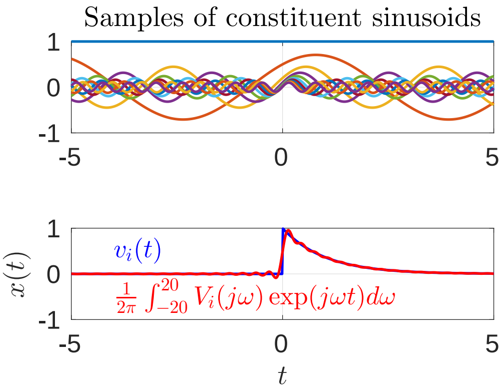
- From the inverse Fourier transform

$$v_o(t) = \overbrace{\frac{V_p}{1-aCR}\exp(-at)u(t)}^{\text{Steady-state response}} - \overbrace{\frac{V_p}{1-aCR}\exp(-t/RC)u(t)}^{\text{Transient response}}$$

- We get both steady-state and transient responses with zero initial condition



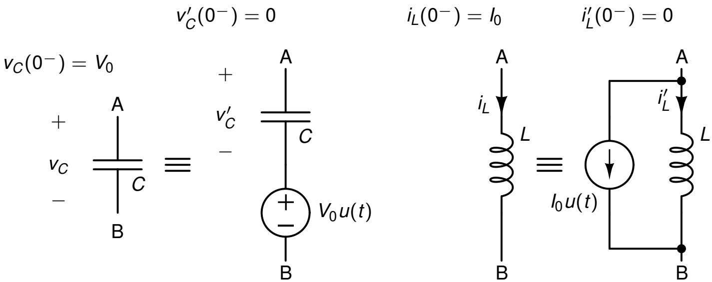
- Fourier transform magnitude and phase ($V_p = 1$, $a = 1$)
- Shown for $-20 \leq \omega \leq 20$



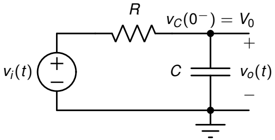
- Fourier transform components $V_i(j\omega)d\omega \cdot \exp(j\omega t)$: Sinusoids from $t = -\infty$ to ∞
  - A small number of sample sinusoids shown above
- The integral is close, but not exactly equal to $x(t)$
- Extending the frequency range improves the representation

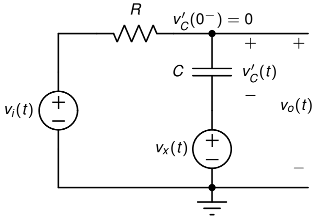
How do we get the total response by summing up steady-state responses?

- Fourier transform components $V_i(j\omega)d\omega \cdot \exp(j\omega t)$: Sinusoids from $t = -\infty$ to ∞
- For any $t > -\infty$, the output is the steady-state response $H(j\omega)V_i(j\omega)d\omega \cdot \exp(j\omega t)$
- Sum (integral) of Fourier transform components produces the input $x(t)$ (e.g. $\exp(-at)u(t)$) which starts from $t = 0$
- Sum (integral) of steady-state responses produces the output including the response to changes at $t = 0$, i.e. including the transient response
- Inverse Fourier transform of $V_i(j\omega)H(j\omega)$ is the total zero-state response



- A capacitor cannot be distinguished from a capacitor in series with a constant voltage source
- An inductor cannot be distinguished from an inductor in parallel with a constant current source
- Initial conditions reduced to zero by inserting sources equal to initial conditions
- Treat initial conditions as extra step inputs and find the solution
  - Step inputs because they start at $t = 0$ and are constant afterwards





$$
\begin{aligned}
V_o(j\omega) &= V_i(j\omega)\overbrace{\frac{1}{1+j\omega CR}}^{H(j\omega)} + V_x(j\omega)\overbrace{\frac{j\omega CR}{1+j\omega CR}}^{H_x(j\omega)} \\
&= \frac{V_p}{a+j\omega}\frac{1}{1+j\omega CR} + V_0\left(\pi\delta(\omega)+\frac{1}{j\omega}\right)\frac{j\omega CR}{1+j\omega CR} \\
&= \frac{V_p}{1-aCR}\left(\frac{1}{a+j\omega} - \frac{CR}{1+j\omega CR}\right) + V_0\frac{CR}{1+j\omega CR} \\
v_o(t) &= \frac{V_p}{1-aCR}\exp(-at)u(t) + \left(V_o - \frac{V_p}{1-aCR}\right)\exp(-t/RC)u(t)
\end{aligned}
$$

- Impulse vanishes because $\delta(\omega)H_x(j\omega) = \delta(\omega)H_x(0)$, and $H_x(0) = 0$

- Contains impulses for some commonly used signals with infinite energy
- e.g. $u(t)$, $\cos(\omega_0 t)u(t)$
- Even more problematic for signals like the ramp—Contains impulse derivative

$\rightarrow$ Laplace transform eliminates these problems

- Problem with Fourier transform of $x(t)$ (zero for $t < 0$)

$$\int_{0-}^{\infty} x(t)\exp(-j\omega t)dt \text{ may not converge}$$

- Multiply $x(t)$ by $\exp(-\sigma t)$ to turn it into a finite energy signal[1]
- Fourier transform of $x(t)\exp(-\sigma t)$

$$X_{\sigma,j\omega}(j\omega) = \int_{0-}^{\infty} x(t)\exp(-\sigma t)\exp(-j\omega t)dt$$

- Inverse Fourier transform of $X_{\sigma,j\omega}(j\omega)$ yields $x(t)\exp(-\sigma t)$

$$x(t)\exp(-\sigma t) = \frac{1}{j2\pi}\int_{-j\infty}^{j\infty} X_{\sigma,j\omega}(j\omega)\exp(j\omega t)d(j\omega)$$

- To get $x(t)$, multiply by $\exp(\sigma t)$

$$x(t) = \frac{1}{j2\pi}\int_{-j\infty}^{j\infty} X_{\sigma,j\omega}(j\omega)\exp(\sigma t)\exp(j\omega t)d(j\omega)$$

[1] Allowable values of σ will be clear later

- Defining $s = \sigma + j\omega$

$$x(t) = \frac{1}{j2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \exp(st)\, ds$$

    - s: complex variable
    - Integral carried out on a line parallel to imaginary axis on the s-plane
- Representation of $x(t)$ as a weighted sum of $\exp(st)$ where $s = \sigma + j\omega$
    - s was purely imaginary in case of the Fourier transform
- Well defined weighting function $X(s)$ for a suitable choice of σ
- $X(s)$ (same as $X_{\sigma,j\omega}(j\omega)$ with $s = \sigma + j\omega$) given by

$$X(s) = \int_{0^-}^{\infty} x(t) \exp(-st) dt$$

- This is the Laplace transform of $x(t)$
- Same definition as the Fourier transform expressed as a function of $j\omega$

- e.g. $x(t) = u(t)$

$$\int_{0-}^{\infty} x(t)\exp(-j\omega t)dt \text{ does not converge}$$

$$\int_{0-}^{\infty} x(t)\exp(-st)dt \text{ converges to } \frac{1}{s} \text{ for } \sigma > 0$$

- If σ is such that Fourier transform of $x(t)\exp(-\sigma t)$ converges, $x(t)$ can be written as sum (integral) of complex exponentials with that σ

$$x(t) = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\exp(st)\,ds$$

- Steady-state response to $\exp(st)$ is $H(s)\exp(st)$, so proceed as with Fourier transform

Circuit analysis using the Laplace transform

- For an input $\exp(st)$, steady state output is $H(s)\exp(st)$
- A general input $x(t)$ represented as a sum (integral)[2] of complex exponentials $\exp(st)$ with weights $X(s)ds/j2\pi$

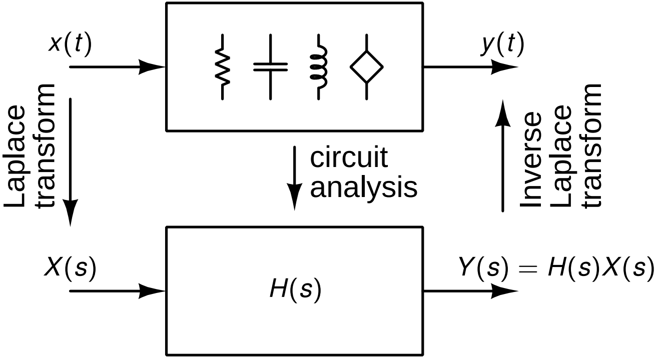
$$x(t) = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\exp(st)ds$$

- By linearity, steady-state $y(t)$ is the superposition of responses $H(s)\exp(st)$ with weights $X(s)ds/j2\pi$

$$y(t) = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} \overbrace{X(s)H(s)}^{Y(s)}\exp(st)ds$$

- Therefore, $y(t)$ is the inverse Laplace transform of $Y(s) = H(s)X(s)$

[2] σ is some value with which $X(s)$ can be found; Value not relevant to circuit analysis as long as it exists.



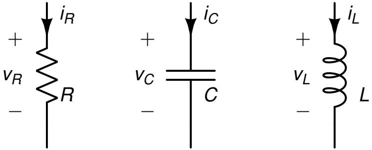
- Calculate $X(s)$
- Calculate $H(s)$
  - Directly from circuit analysis
  - From differential equation, if given
- Calculate (look up) the inverse Laplace transform of $H(s)X(s)$ to get $y(t)$

Laplace transform pairs

- Signals in $0 \leq t \leq \infty$

$$
\begin{aligned}
u(t) &\leftrightarrow \frac{1}{s} \\
tu(t) &\leftrightarrow \frac{1}{s^2} \\
\exp(j\omega_0 t)u(t) &\leftrightarrow \frac{1}{s - j\omega_0} \\
\cos(\omega_0 t)u(t) &\leftrightarrow \frac{s}{s^2 + \omega_0^2} \\
\sin(\omega_0 t)u(t) &\leftrightarrow \frac{\omega_0}{s^2 + \omega_0^2} \\
\exp(-at)u(t) &\leftrightarrow \frac{1}{s + a} \\
t\exp(-at)u(t) &\leftrightarrow \frac{1}{(s + a)^2}
\end{aligned}
$$

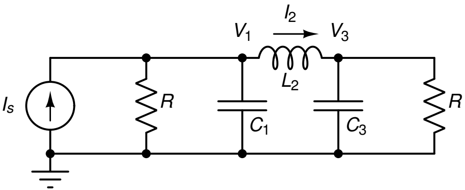
- In steady-state with $\exp(st)$ input, "Ohms law" also valid for L, C



		$v(t)$	$i(t)$	$v(t)/i(t)$
Resistor	$v_R = Ri_R$	$RI_R \exp(st)$	$I_R \exp(st)$	R
Inductor	$v_L = L\,(di_L/dt)$	$sLI_L \exp(st)$	$I_L \exp(st)$	sL
Capacitor	$i_C = C\,(dv_C/dt)$	$V_C \exp(st)$	$sCV_C \exp(st)$	$1/\,(sC)$

- Use analysis methods for resistive circuits with dc sources to determine $H(s)$ as ratio of currents or voltages
  - e.g. Nodal analysis, Mesh analysis, etc.
- No need to derive the differential equation

Example: Calculating the transfer function

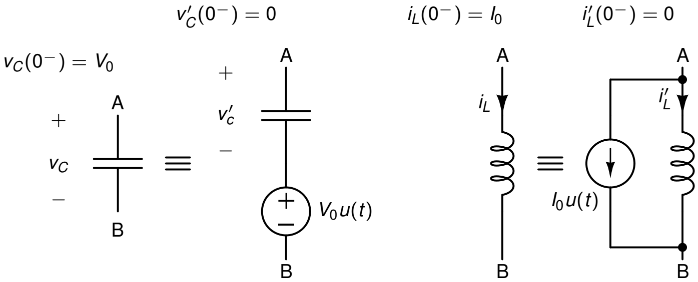


- Nodal analysis with voltages V_1, V_2

$$\begin{bmatrix} \frac{1}{R} + sC_1 + \frac{1}{sL_2} & -\frac{1}{sL_2} \\ -\frac{1}{sL_2} & \frac{1}{sL_2} + sC_3 + \frac{1}{R} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$$

$$\frac{V_1}{I_s} = R\frac{s^2C_3L_2 + sL_2/R + 1}{s^3C_1C_3L_2R + s^2\left(C_1 + C_3\right)L_2 + s\left((C_1 + C_3)R + L_2/R\right) + 2}$$

$$\frac{V_2}{I_s} = R\frac{1}{s^3C_1C_3L_2R + s^2\left(C_1 + C_3\right)L_2 + s\left((C_1 + C_3)R + L_2/R\right) + 2}$$

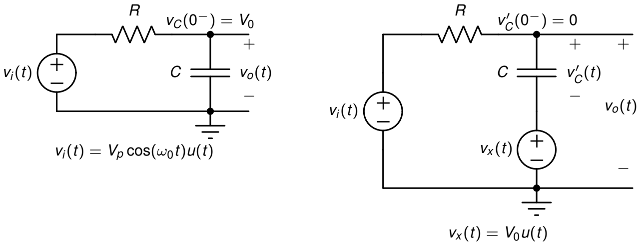


- Initial conditions reduced to zero; extra step inputs
- Circuit interpretation of the derivative operator

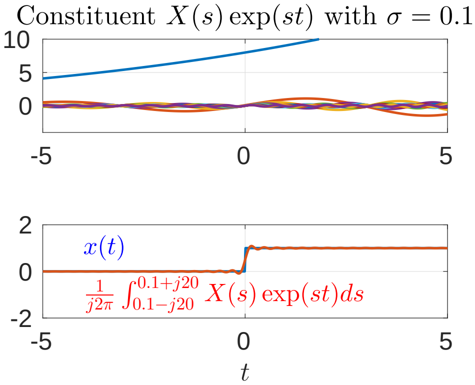
$$\frac{dx}{dt} \leftrightarrow sX(s) - x(0^-)$$

$$\frac{dx}{dt} \leftrightarrow s\left(X(s) - \frac{x(0^-)}{s}\right)$$

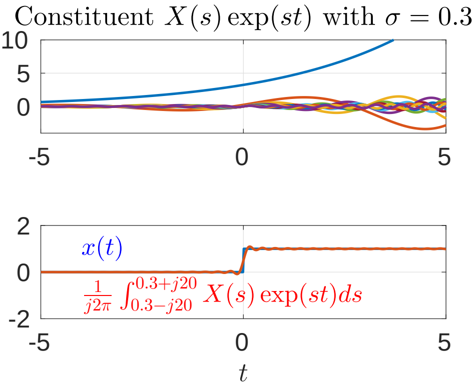
- Extra step input $x(0^-)/s$



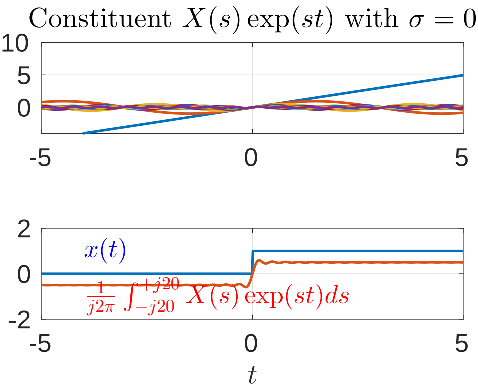
$$
\begin{aligned}
V_o(s) &= V_p \frac{s}{s^2 + \omega_0^2} \frac{1}{1 + sCR} + \frac{V_0}{s} \frac{sCR}{1 + sCR} \\
&= \frac{V_p}{1 + (\omega_0 CR)^2} \frac{s + (\omega_0 CR)\,\omega_0}{s^2 + \omega_0^2} + \left(V_0 - \frac{V_p}{1 + (\omega_0 CR)^2} \right) \frac{CR}{1 + sCR} \\
v_o(t) &= \overbrace{\frac{V_p}{\sqrt{1 + (\omega_0 CR)^2}} \cos\left(\omega_0 t - \phi\right) u(t)}^{\text{Steady-state response}} + \overbrace{\left(V_0 - \frac{V_p}{1 + (\omega_0 CR)^2} \right) \exp(-t/RC) u(t)}^{\text{Transient response}} \\
\phi &= \tan^{-1}\left(\omega_0 CR\right)
\end{aligned}
$$



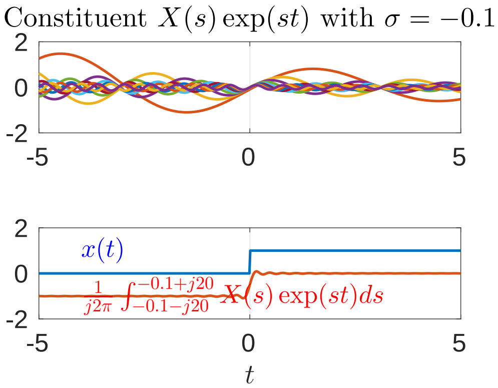
- $x(t) = u(t)$; $X(s) = 1/s$
- Sum of exponentially modulated sinusoids with $\sigma = 0.1$ converges to the unit step



- $x(t) = u(t)$; $X(s) = 1/s$
- Sum of exponentially modulated sinusoids with $\sigma = 0.3$ converges to the unit step
- Any σ in the region of convergence (ROC) would do
- For $u(t)$, ROC is $\sigma > 0$

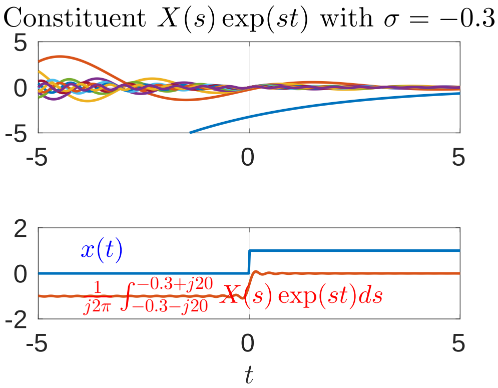


- $x(t) = u(t)$; $X(s) = 1/s$
- For $u(t)$, ROC is $\sigma > 0$
- Sum of exponentially modulated sinusoids with $\sigma = 0$ does not converge to the unit step
- This is the Fourier transform of $u(t)$ with $\pi\delta(\omega)$ missing
- Zero dc part in the sum



- $x(t) = u(t)$; $X(s) = 1/s$
- For $u(t)$, ROC is $\sigma > 0$
- Sum of exponentially modulated sinusoids with $\sigma = -0.1$ does not converge to $u(t)$, but converges of $-u(-t)$!
- Inverse Laplace transform formula uniquely defines the function only if the ROC is also specified
- Inverse Laplace transform of $X(s) = 1/s$ with ROC of $\sigma < 0$ is $-u(-t)$

Laplace transform: exp(st) components and convergence



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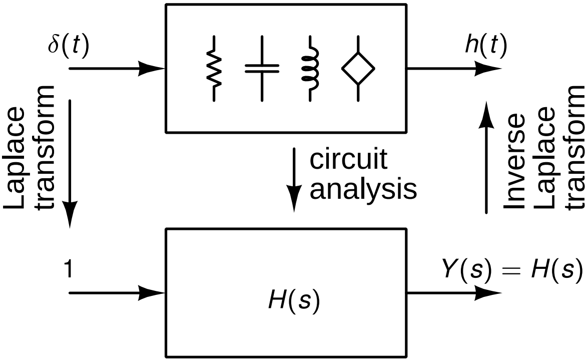
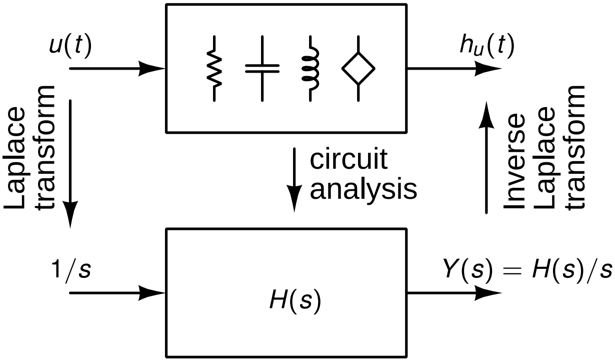
- Laplace transform $F(s)$ uniquely defines the function only if the ROC is also specified
- Inverse Laplace transform of $F(s)$ can be $f(t)u(t)$ (a right-sided or causal signal) as well as $-f(t)u(-t)$ (a left-sided or anti-causal signal) depending on the choice of σ
- Speficying causality or the ROC removes the ambiguity
- One-sided ($0 \leq t \leq \infty$) Laplace transform applies only to causal signals



- Laplace transform of $\delta(t)$ is 1
- Transfer function $H(s)$: Laplace transform of the impulse response $h(t)$
- Impulse response usually calculated from the Laplace transform



- Laplace transform of $u(t)$ is $1/s$
- $H(s)/s$: Laplace transform of the unit step response $h_u(t)$
- Step response usually calculated from the Laplace transform

- Transfer function: Rational polynomial in s
  - Transfer function from any voltage or current $x(t)$ to any voltage or current $y(t)$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \ldots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0}$$

  - $H(s)$ of the form $N(s)/D(s)$ where $N(s)$ and $D(s)$ are polynomials in s
- Differential equation relating y and x

$$a_N \frac{d^N y}{dt^N} + a_{N-1} \frac{d^{N-1} y}{dt^{N-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y \quad = \\ b_M \frac{d^M x}{dt^M} + b_{M-1} \frac{d^{M-1} x}{dt^{M-1}} + \ldots + b_1 \frac{dx}{dt} + b_0 x$$

- $D(s)$ corresponds to LHS of the differential equation
  - Highest power of s in $D(s)$: Order of the transfer function
- $N(s)$ corresponds to RHS of the differential equation
- Transfer function: Convenient way of getting the differential equation

- Transfer function: Rational polynomial in s

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \ldots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0}$$

  - Convenient form for finding dc gain b_0/a_0, high frequency behavior $(b_M/a_N)\, s^{M-N}$

- Transfer function: Factored into first and second order polynomials

$$H(s) = \frac{N(s)}{D(s)} = \frac{N_1(s) N_2(s) \cdots N_K(s)}{D_1(s) D_2(s) \cdots D_L(s)}$$

  - $K = M/2$ (even M), $K = (M+1)/2$ (odd M); $L = N/2$ (even N), $L = (N+1)/2$ (odd N)
  - $N_k(s)$: All second order (even M) or one first order and the rest second order (odd M); Similarly for $D_l(s)$
  - Convenient for realizing as a cascade; combining different N_k and D_l

Transfer function: Factored into terms with zeros and poles

- Transfer function: zero, pole, gain form

$$H(s) = \frac{N(s)}{D(s)} = k\frac{(s - z_1)(s - z_2)\cdots(s - z_M)}{(s - p_1)(s - p_2)\cdots(s - p_N)}$$

  - Zeros z_k, poles p_k, gain k
  - Convenient for seeing poles and zeros

- Transfer function: Alternative zero, pole, gain form[3]

$$H(s) = \frac{N(s)}{D(s)} = k_0\frac{\left(1 - \frac{s}{z_1}\right)\left(1 - \frac{s}{z_2}\right)\cdots\left(1 - \frac{s}{z_M}\right)}{\left(1 - \frac{s}{p_1}\right)\left(1 - \frac{s}{p_2}\right)\cdots\left(1 - \frac{s}{p_N}\right)}$$

  - Zeros z_k, poles p_k, gain k
  - k_0: dc gain
  - Convenient for seeing poles and zeros
  - Convenient for drawing Bode plots

[3] Cannot use when poles or zeros are at the origin

- Transfer function: Partial fraction expansion

$$
\begin{aligned}
H(s) &= \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \ldots + \frac{c_N}{s-p_N} \\
h(t) &= c_1 \exp(p_1 t) + c_2 \exp(p_2 t) + \ldots c_N \exp(p_N t)
\end{aligned}
$$

  - Convenient for finding the impulse response (natural response)
  - Shown for distinct poles; Modified for repeated roots
  - Terms for complex conjugate poles can be combined to get responses of type $\exp(p_{1r} t) \cos(p_{1i} t + \phi)$

Applicability of Laplace transforms to circuit analysis

Circuits with lumped R, L, C and controlled sources

- Causal, with natural responses of the type exp(pt)
- Laplace transform of the impulse response converges with σ greater than the largest real part among all the poles

$\therefore$ Can be used for analyzing the total response of any circuit (even unstable ones) with inputs which have well-defined Laplace transform

- Don't have to worry about ROC while using the Laplace transform to analyze circuits with lumped R, L, C and controlled sources

Analysis using the Laplace transform

- Solve for the complete response including initial conditions
- Determine the poles and zeros, evaluate stability
- Write down the differential equation
- Get the Fourier transform (when it exists without impulses) by substituting $s = j\omega$
- Get the sinusoidal steady-state response
  - Response to $\cos(\omega_0 t + \theta)$ is $|H(j\omega_0)| \cos(\omega_0 t + \theta + \angle H(j\omega_0))$
- Not convenient for analysis of energy/power
  - Have to use time domain or Fourier transform

- Only sinusoidal steady-state
- Convenient for fixed-frequency (e.g. power) or narrowband(e.g. RF) signals
- Easier to see cancellation of reactances etc., than with Laplace transform
  - Laplace transform requires finding zeros of polynomials
- Maybe easier to see other types of impedance transformation

- Exact analysis can be tedious
- Provides a lot of intuition
- Can handle nonlinearity
- For some problems frequency-domain analysis can be unwieldy whereas time-domain analysis is very easy
    - e.g. steady-state response of a first order RC filter to a square wave—try using the Fourier series and transfer functions at the fundamental frequency and its harmonics!
    - Response to $\sum_k a_k \exp(jk\omega_0 t)$ is $\sum_k a_k H(jk\omega_0) \exp(jk\omega_0 t)$

Practice all techniques on a large number of problems so that you can attack any problem