Some Design Approaches for Causal Stable IIR and FIR Perfect Reconstruction Filter Banks

A Thesis
Submitted For the Degree of
Master of Science (Engineering)
in the Faculty of Engineering

by

P. Viswanadha Reddy

Department of Electrical Communication Engineering
Indian Institute of Science
Bangalore – 560 012

January 2002

Acknowledgements

I would like to thank my research supervisor Prof. Anamitra Makur for all the guidance and support he has provided. He has always managed to bring more clarity and soundness to any idea that we have discussed together. I have greatly benefited from his academic excellence and his style of thinking. Thank you 'Sir', thanks for everything you have taught me.

I would like to thank Dr. Ashok Rao for his moral support and discussions with him helped me to shape out this thesis. I once again thank him for his kind interest and warm support. I would like to thank Prof. Vittal Rao for his suggestions and doubt clarification sessions. I would like to thank Prof. A. G. Ramakrishnan and Prof. G. V. Anand for their suggestions and support. I would like to thank the instructors of my research training programme who enabled me to acquire the necessary background for my research.

I would like to thank Suresh, Muthu, Selvi, Murthy, Bhavani, Chandra, Galib, Ganni and Karthik for constant encouragement and good feedback. I thankfully recall my good times with my lab-mates and friends, they are really wonderful people and I cannot describe my appreciation and thankfulness towards them with words.

Abstract

Perfect reconstruction filter banks (PRFBs) are widely used for signal decomposition, subband coding, subband adaptive filtering etc. Both IIR and FIR filter banks are used in the applications mentioned above. Causal stable IIR PRFBs are popular as they have good responses and pre processing of input is not necessary. For 2 channel case efficient design methods have been developed in the literature. For M channel case existing design procedures are complicated. General design method for causal stable IIRPRFB received less interest. FIR PRFBs are popular as they are easy to implement. For a class of PRFBs, namely paraunitary PRFBs, complete characterizations and efficient design methods have been developed. However, the problem of complete characterization of general FIRPRFBs received less interest. Recently, complete characterizations of general FIRPRFBs for which the analysis polyphase is of order one has been solved. However, design of a large class of general FIRPRFBs with analysis polyphase order greater than one has not been given. In this work, the problem of design of wider classes of IIR causal stable PRFBs and general FIRPRFBs, with the order of analysis polyphase matrix greater than one, is addressed.

First, some design approaches for M channel IIR causal stable PRFBs are developed. The analysis polyphase matrix is realized in state space form, and minimal characterizations are used to avoid pole zero cancelation. The inverse system is explicitly given if the analysis polyphase matrix is invertible at $z = \infty$. All the design methods are based on forcing the poles of analysis polyphase matrix and its inverse system inside unit circle. A design method based on the concept of function of a matrix is proposed. A design method based on similarity transformation is also developed. Further, factorization based approach is developed where the analysis polyphase matrix is factorized into degree 1 terms using the theorems on factorization of rational matrix functions. Several design examples are provided that compare favorably with existing IIR PRFB designs.

The second part of the work deals with the design of a very wide class of general M channel FIRPRFBs with the order of analysis polyphase matrix greater than one (say l). The analysis polyphase matrix is treated as a regular matrix polynomial so that its inverse exists. The design problem is treated as the inverse problem of constructing a regular matrix polynomial given the spectral data (degree of the determinant of analysis polyphase matrix) of the regular matrix polynomial. Explicit formula of the inverse given spectral data is provided. The proposed design allows restrictions on reconstruction delay and order of the inverse polynomial (synthesis polyphase). Further, near linear phase FBs for which length of all the filters is equal to $(l+1)M$ is developed. Here also the design method allows restrictions on reconstruction delay and order of the inverse polynomial. Lastly, low reconstruction delay is achieved with unimodular matrix polynomials, and a design of low delay FBs is also developed. All the proposed designs are illustrated by few simulation examples.

Contents

List of Tables

List of Figures

Notations and Terminology

Symbol	Represents
$(.)^T$	Transpose
$f^i(.)$	i^{th} derivative of the function $f(.)$
W_M^K	$e^{-j2\pi k/M}$
$(*)(n)$	Discrete time sequences
$\|.\|$	Norm
$ker(.)$	Null space
$(.)^\dagger$	Pseudo inverse
$det(.)$	Determinant
$deg(.)$	Degree
$dim(.)$	Dimension
$diag(.)$	Diagonal or block diagonal matrix with elements inside as diagonal entries
$\mathbf{A} \oplus \mathbf{B}$	$diag(\mathbf{A}, \mathbf{B})$
$col\{\mathbf{AB}^i\}_{i=0}^l$	$[\mathbf{A}^T \ (\mathbf{AB})^T \ (\mathbf{AB}^2)^T \ \cdots (\mathbf{AB}^l)^T]^T$

Lower case bold faced letters represent vectors. Lower case and upper case letters represent constants. Calligraphic letters represent matrix polynomials and rational matrix functions. Upper case bold faced letters represent matrices. $\mathbf{I}$ represents identity matrix. $\mathbf{J}$ represents exchange matrix. In some cases subscripts are skipped if the dimensions are obvious from the context. $A(z)$ represents the z-transform of $a(n)$, where $a(n)$ can be a discrete time sequence or filter response in time domain. $\mathbf{B}_{i,j}$ represents the $(i,j)^{th}$ element of the matrix $\mathbf{B}$. If $\mathcal{A}(z) = \sum_{i=0}^{l} \mathbf{A}_i z^{-i}$ is a matrix polynomial, then it's *dual polynomial* $\tilde{\mathcal{A}}(z)$ is given by $z^{-l} \sum_{i=0}^{l} \mathbf{A}_i z^i$.

$\mathcal{N}(\mathbf{X})$, $\mathcal{R}(\mathbf{X})$ represents null space and range of a arbitrary matrix $\mathbf{X}$. Vector spaces, subspaces and fields are represented by the letters of the type, say $\mathbb{R}$ represents real number field. SVD represents singular value decomposition [49].

The $k \times k$ Jordan block $\mathbf{J}_\lambda^k$ corresponding to eigen value λ is represented by,

$$\mathbf{J}_\lambda^k = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{k \times k}$$

A Jordan matrix $\mathbf{J}$ with *Jordan structure* $[a_1, a_2, \cdots, a_r]$ corresponding to eigen value λ is given by $\mathbf{J} = diag(\mathbf{J}_\lambda^{a_1}, \mathbf{J}_\lambda^{a_2}, \cdots, \mathbf{J}_\lambda^{a_r})$. If all the eigen values of a matrix are zero, then the matrix is nilpotent. *Jordan nilpotent matrix* is a Jordan matrix with zero eigen value. So, for example the Jordan nilpotent matrix $\mathbf{J}$ with Jordan structure [3,4] is represented as

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & & & & \\ & & & 0 & 1 & 0 & 0 \\ & & & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chapter 1

Introduction

1.1 Introduction to multirate signal processing

Within DSP lies the world of multirate signal processing. Sampling of continuous signal leads to the notion of resolution in time. The higher the sampling frequency, the better the resolution of digital approximations of the original signals. This increases storage cost. Multirate signal processing strives to keep that cost at a reasonable level by applying smart signal processing algorithms, involving alterations of sampling rates. Signal decomposition techniques find wide spread applications in modern signal processing. Some applications include subband coding, subband adaptive filtering, signal analysis and restoration etc. Multirate filter banks play a major role in most of these applications. Essential operations of multirate systems are down sampling and up sampling shown in figure (1.1). In words, down sampling keeps every Mth sample while up sampling inserts

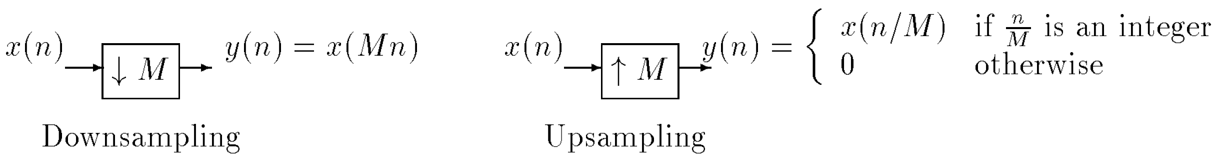


Figure 1.1: Block diagrams for downsampling and upsampling

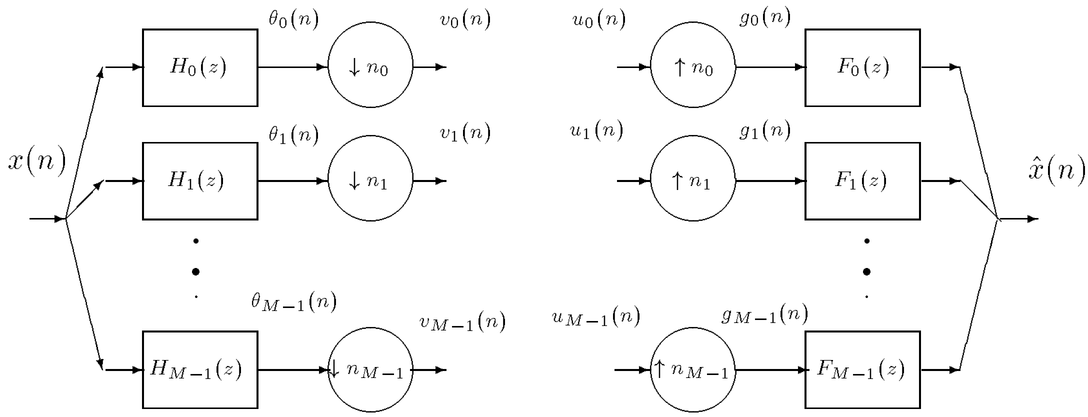


Figure 1.2: M channel analysis and synthesis system

$M-1$ zeros between consecutive samples. In multirate signal processing the signal $x(n)$ is decomposed into M sub signals $\{\theta_i(n)\}_{i=0}^{M-1}$ by a bank of filters $\{H_i(z)\}_{i=0}^{M-1}$, called analysis filters as shown in the figure (1.2). As the number of samples at a time is increased due to M sub signals, entire rate can be reduced by decimating each sub signal by a factor of n_i to obtain signals $\{v_i(n)\}_{i=0}^{M-1}$. On the reconstruction side all the $\{u_i(n)\}_{i=0}^{M-1}$ (changed due to some operations), are up sampled by $\{n_i\}_{i=0}^{M-1}$ to get $\{g_i(n)\}_{i=0}^{M-1}$ and summed up after passing through the stack of filters $\{F_i(z)\}_{i=0}^{M-1}$, called synthesis filters. These filters should be such that if no processing is done on the sub signals, original signal should be retrieved from the sub signals. In other words, if $\{u_i(n)\}_{i=0}^{M-1} = \{v_i(n)\}_{i=0}^{M-1}$, the output $\hat{x}(n)$ can be made delayed and scaled version of $x(n)$. Such a system is called perfect reconstruction (PR) system. If one of the sections (analysis or synthesis) is taken arbitrarily then the other section is constrained. In the present work maximally decimated fitlerbanks are assumed, i.e. $n_i = M \ \forall i = 0, \cdots, M-1$.

1.2 The perfect reconstruction problem

1.2.1 Two channel case

For 2 channel if $\{H_0, H_1\}$ is the analysis filter pair and $\{F_0, F_1\}$ is the synthesis filter pair, then the reconstructed signal $\hat{x}(n)$ can be written as

$$\begin{aligned}\hat{X}(z) &= \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)]X(z) + \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)]X(-z) \\ &= T(z)X(z) + S(z)X(-z) \qquad (1.1)\end{aligned}$$

Aliasing distortion is eliminated by restricting $S(z)$ to zero, which gives the condition $H_0(-z)F_0(z) + H_1(-z)F_1(z) = 0$, and this can be achieved if $F_0(z) = -H_1(-z)$ and $F_1(z) = H_0(-z)$. The reconstructed signal is given as:

$$\hat{X}(z) = \frac{1}{2}[H_1(z)H_0(-z) - H_0(z)H_1(-z)]X(z)$$

Phase and amplitude distortions are eliminated by taking $T(z)$ as a delay.

$$H_1(z)H_0(-z) - H_0(z)H_1(-z) = z^{-d} \qquad (1.2)$$

1.2.2 M channel case

For M channel case perfect reconstruction is achieved by the following two approaches.

1.2.2.1 Alias component approach

The decimated signal $v_i(n)$ can be expressed in terms of $\theta_i(n)$ as follows

$$V_i(z) = \frac{1}{M}\sum_{k=0}^{M-1} \Theta_i(z^{\frac{1}{M}} W_M^k)$$

The up sampled signal $g_i(n)$ can be written as

$$G_i(z) = U_i(z^M).$$

Since $\hat{X}(z) = \sum_{i=0}^{M-1} F_i(z)G_i(z)$, and it is assumed that $u_i(n) = v_i(n)$, the reconstructed signal can be written as

$$\begin{aligned}\hat{X}(z) &= \frac{1}{M}\sum_{i=0}^{M-1} F_i(z) \sum_{k=0}^{M-1} \Theta_i(zW_M^k) \\ &= \frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1} X(zW_M^k)H_i(zW_M^k)F_i(z)\end{aligned}$$

Now $\hat{X}(z)$ can be written compactly as

$$\hat{X}(z) = \frac{1}{M}[X(z),\cdots,X(zW_M^{M-1})] \underbrace{\begin{bmatrix} H_0(z) & \cdots & H_{M-1}(z) \\ H_0(zW_M) & \cdots & H_{M-1}(zW_M) \\ \vdots & \vdots & \vdots \\ H_0(zW_M^{M-1}) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix}}_{\mathcal{H}(z)} \underbrace{\begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix}}_{\mathbf{f}(z)}$$

where $\mathcal{H}(z)$ is called alias component matrix. The above equation can be expanded as

$$\hat{X}(z) = \left(\frac{1}{M}\sum_{k=0}^{M-1} F_k(z)H_k(z)\right) X(z) + \underbrace{\frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=1}^{M-1} H_k(zW_M^k)X(zW_M^k)F_k(z)}_{aliasing\ terms} \tag{1.3}$$

Three kinds of undesirable distortions can be found from the equation (1.3).

Aliasing distortion: The sub sampling process causes aliasing components while up sampling produces images. These aliasing terms can be removed if the following condition is imposed:

$$\mathcal{H}(z)\mathbf{f}(z) = [MT(z), 0, \cdots, 0]^T \tag{1.4}$$

In this case the output is given by

$$\begin{aligned}\hat{X}(z) &= T(z)X(z) \\ T(z) &= \sum_k H_k(z)F_k(z)\end{aligned}$$

Amplitude and phase distortion: After having constrained $\{H_k, F_k\}$ to force the aliasing term to zero, the term $T(z)$ causes amplitude and phase distortion. Since $T(e^{jw}) = |T(e^{jw})|e^{j\phi(w)}$, amplitude distortion can be eliminated if $|T(e^{jw})|$ is a constant and phase deviation can be removed if $\phi(w)$ is linear. For perfect reconstruction $T(z)$ must be a delay, but a more stringent condition can be obtained when polyphase approach is used.

Given analysis filters, synthesis filters can be obtained from equation (1.4) inverting the alias component matrix. In case of FIR filter banks, alias component matrix must have a monomial determinant for the synthesis filters to be FIR, and in case of IIR filter banks, the alias component matrix must have a minimum phase determinant for synthesis filters to be stable and causal. The roots of the alias component matrix are related to analysis filters in a complicated manner. Restricting the analysis filters for the determinant of alias component matrix to be monomial or minimum phase is extreamly difficult, so polyphase approach is used for analyzing PR conditions.

1.2.2.2 The polyphase approach

The analysis and synthesis filters can be written as

$$
\begin{aligned}
H_i(z) &= \sum_{k=0}^{M-1} z^{-k} E_{i,k}(z^M) \\
E_{i,k}(z) &= \sum_{j=-\infty}^{\infty} h_i(k+Mj)z^{-j}, \qquad 0 \le i,k \le M-1 \\
F_i(z) &= \sum_{k=0}^{M-1} z^{-k} R'_{i,k}(z^M) = \sum_{k=0}^{M-1} z^{-(M-1-k)} R'_{i,M-1-k}(z^M) \\
R'_{i,k}(z) &= \sum_{j=-\infty}^{\infty} f_i(k+Mj)z^{-j}, \qquad 0 \le i,k \le M-1
\end{aligned}
$$

where $E_{i,k}(z)$ and $R'_{i,k}(z)$ are the M fold down sampled versions of $z^k H_i(z)$ and $z^k F_i(z)$ respectively, which are called kth polyphase components of $H_i(z)$ and $F_i(z)$ respectively. The above equations are shown in figure (1.3). The bank of analysis filters can be written

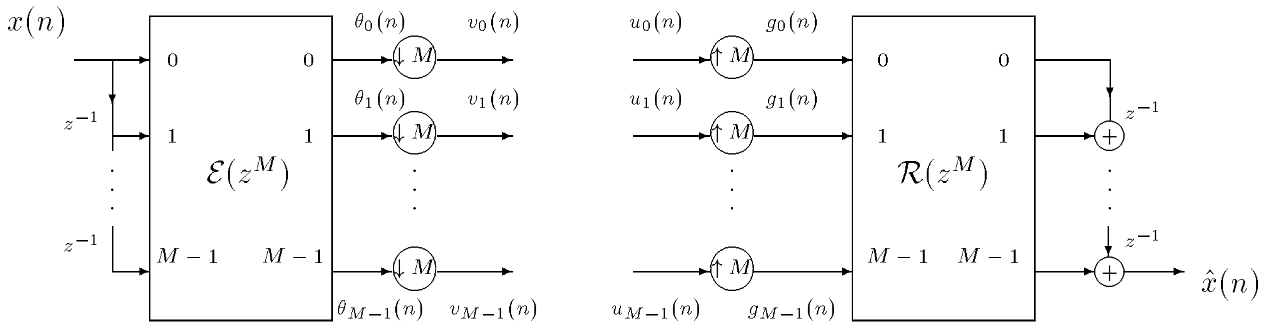


Figure 1.3: Analysis and synthesis sections in polyphase form

as

$$\begin{aligned}
\mathbf{h}(z) &= \mathcal{E}(z^M)\mathbf{e}(z) \\
\mathbf{h}(z) &= [H_0(z), \cdots, H_{M-1}(z)]^T \\
\mathcal{E}(z) &= \begin{bmatrix} E_{0,0}(z) & \cdots & E_{0,M-1}(z) \\ E_{1,0}(z) & \cdots & E_{1,M-1}(z) \\ \vdots & \ddots & \vdots \\ E_{M-1,0}(z) & \cdots & E_{M-1,M-1}(z) \end{bmatrix} \\
\mathbf{e}(z) &= [1, z^{-1}, \cdots, z^{-(M-1)}]^T
\end{aligned}$$

$\mathcal{E}(z)$ is called the analysis polyphase matrix. The synthesis filters are expressed in a different way as

$$\begin{aligned}
\mathbf{f}(z) &= \mathcal{R}'(z^M)\mathbf{e}(z) \\
\mathbf{f}^T(z) &= \mathbf{e}^T(z)\mathcal{R}'^T(z^M) = \mathbf{e}^T(z)JJ\mathcal{R}'^T(z^M) \\
&= \hat{\mathbf{e}}^T(z)\mathcal{R}(z^M) \\
\hat{\mathbf{e}}^T(z) &= [z^{-(M-1)}, \cdots, 1]
\end{aligned}$$

where J is the exchange matrix. Using the noble identities of decimation and interpolation [1], the analysis and synthesis sections can be interchanged as shown in figure (1.4). The

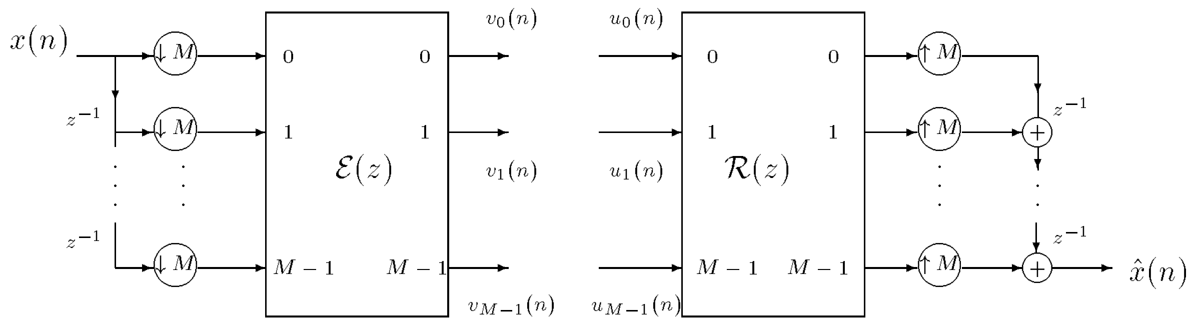


Figure 1.4: Interchanged analysis and synthesis sections

above multirate system is PR if only delays, decimators and interpolators are present. Introducing the matrices $\mathcal{E}(z)$ and $\mathcal{R}(z)$, the PR condition is given by [1]

$$\mathcal{R}(z)\mathcal{E}(z) = cz^{-l_0} \underbrace{\begin{bmatrix} 0 & I_{M-r} \\ z^{-1}I_r & 0 \end{bmatrix}}_{Pseudocirculant\ matrix} \tag{1.5}$$

$$\mathcal{R}(z) = cz^{-l_0} \begin{bmatrix} 0 & I_{M-r} \\ z^{-1}I_r & 0 \end{bmatrix} \mathcal{E}^{-1}(z) \tag{1.6}$$

$$= cz^{-l_0} \begin{bmatrix} 0 & I_{M-r} \\ z^{-1}I_r & 0 \end{bmatrix} \frac{adj(\mathcal{E}(z))}{det(\mathcal{E}(z))} \tag{1.7}$$

The delay due to delay chain and the cascade of the polyphase matrices is $d = l_0M + r + M - 1$. It can be seen that the synthesis fillers are obtained by inverting the analysis polyphase matrix. Thus, synthesis filters have which has a simple relation with the analysis filters, which is difficult to see in alias component approach.

From the above discussion it is obvious that the condition to get synthesis filters is analysis polyphase matrix $\mathcal{E}(z)$ must be invertible. In general, two types of filter banks are popular, namely FIR and IIR. For FIR case both $\mathcal{E}(z)$ and $\mathcal{R}(z)$ must be FIR, and this condition is satisfied if $det(\mathcal{E}(z))$ is monomial, i.e. $det(\mathcal{E}(z)) = cz^{-k}$. For IIR case causal stable synthesis filters are obtained if $det(\mathcal{E}(z))$ is minimum phase with $\mathcal{E}(z)$ being causal, which is obvious from the equation (1.7).

For FIR case, if l_a and l_s are the degrees of $\mathcal{E}(z)$ and $z^{-d}\mathcal{R}(z)$ respectively and if L_a and L_s represent the length of longest filter in analysis and synthesis sections, where d is the delay introduced to make $\mathcal{R}(z)$ causal, then

$$Ml_a < L_a \leq M(l_a + 1)$$
$$Ml_s < L_s \leq M(l_s + 1).$$

In a general design method of a FIRPR system, analysis bank is obtained by first choosing $\mathcal{E}(z)$ with monomial determinant and desired responses for analysis filters, then $\mathcal{R}(z)$ is obtained from the equation (1.7) by chosing proper r and l_0 such that the delay is small and $\mathcal{R}(z)$ is causal. In general r is taken to be zero where the PR condition reduces to $\mathcal{R}(z)\mathcal{E}(z) = cz^{-l_0}I_M$, which is also used for IIR case. In the next section, the design methods for IIR filters is discussed first, and then FIR case is considered.

Paraunitary filter banks: If $\mathcal{E}^{-1}(z) = \mathcal{E}^T(z^{-1})$, then $\mathcal{E}(z)$ is said to be paraunitary or lossless. It can be seen that on the unit circle the inverse system is just flipped and transposed version of $\mathcal{E}(z)$. In this case the length of the synthesis filters is same as the analysis filters. Factorizations of lossless systems are briefly discussed in [4].

Linear phase filter banks: In image processing applications, linear phase filters are preferred because phase information is crucial in an image. The condition for $\mathcal{E}(z)$ to be linear phase is

$$\mathcal{E}(z) = Dz^{-(l-1)}\mathcal{E}(z^{-1})J \tag{1.8}$$

where l is the highest degree term in $\mathcal{E}(z)$, J is $M \times M$ exchange matrix and D is a diagonal matrix with $+1$ and -1 on the diagonal depending upon the number of symmetric and antisymmetric filters. If M is even then it was proved that there exists $M/2$ symmetric and $M/2$ antisymmetric filters, and if M is odd then there exists $(M+1)/2$ symmetric and $(M-1)/2$ antisymmetric filters [15] [38]. As it is known that, only FIR filter can be linear phase and IIR linear phase filters are not practically realizable, FIRPR linear phase filter banks are discussed in the present thesis.

1.3 Review of IIRPRFB designs

There are several design methods for M channel IIR PR Filter Banks, but in most of the cases the synthesis bank becomes anti causal stable. Problem with anti causal synthesis bank is preprocessing of input is necessary to implement anti causal operations at the synthesis section, which is extra burden. Causal stable design doesn't require this. For causal stable design the poles of analysis and synthesis polyphase matrices must be inside the unit circle. The strict condition that the determinant of the analysis polyphase matrix must be minimum phase leads to the design of IIR causal stable FBs. It is possible to obtain PR IIR structures if the synthesis filters are operated in non causal way. In this case, the poles of $\mathcal{R}(z)$ outside unit circle are the stable poles of an anti causal filter, and filtering is performed in a non causal fashion, which is acceptable for image processing. In [2] the filters are run in both causal and anti-causal fashion by introducing the time reversal operations after analysis sections, but this increases the storage cost of the system. In [3] it was shown that for 2 channel case if the filters $\{H_0, H_1\}$ are realized with all-pass filters PR is achieved eliminating the amplitude distortion, but phase distortion cannot be eliminated with this structure. Causal stable solutions are obtained mostly for 2 channel case and few for M channel case. [4] proposed that using all pass filters instead of delays in lossless matrices IIR PRFB can be achieved but synthesis becomes anti causal. First result on IIR causal stable PRFB was given by [5]. In their design first a good low pass filter in the analysis section is designed, then the remaining filters in the analysis section are designed such that the analysis polyphase matrix becomes unimodular, using the *Quillen-Suslin-theorem* on the completion of unimodular matrix polynomials. The design method given by the authors is complex and a general characterization has not been given even though the method covers a large class of causal stable solutions. It is highly likely that other filters in the analysis section may not have good responses as they are constrained by the first filter. In [6] 2 channel causal stable IIR design is given, where special structure for analysis polyphase matrix is assumed. Moreover $\mathcal{E}(z)$ is assumed to be upper triangular with the diagonal elements to be delays and the remaining super diagonal elements are assumed to be a rational function. This may be assumed to be a near FIR polyphase matrix. In [7] 2 channel IIR QMF banks are designed with approximately linear phase analysis and synthesis filters. Similar linear phase case, but with a different method using complex all pass sections, is done by [8]. 2 channel IIR causal stable PR design was done by [9] using constrained optimization. In this design the PR condition is taken as the constraint and special structures are assumed for analysis filters. Orthonormal IIR 2 channel FBs are designed using all pass filters [10]. In [11] M channel causal stable IIR FB is designed by assuming a special structure for $\mathcal{E}(z)$, i.e. $\mathcal{E}(z)$ is assumed to be diagonal with each diagonal filter being minimum phase. So, it can be seen that a general method for the design of IIR causal stable filter bank is difficult, which is the problem of interest of the present thesis. Design of such filters is discussed in second chapter.

1.4 Review of FIRPRFB designs

1.4.1 2 channel case

The first 2 channel FIRPRFB solution was given by [12] by assuming $H_1(z) = z^{-(N-1)}H_0(-z^{-1})$, where N is the length of each filter, there by calling the pair $\{H_0, H_1\}$ as *conjugate quadrature filters*. Then it was shown by [13] that the above imposed condition is indeed a paraunitary solution for 2 channel case, i.e. the polyphase matrix satisfies the paraunitary condition discussed in the previous section. In [14] different choices of analysis filters leading to the PR condition are discussed but a clear characterization method is not given. In [15] the close relation between the continued fraction expansion of functions (CFE) and PRFBs is clearly demonstrated. Low delay filter banks are useful in speech applications where phase information is not crucial, such low delay fiterbank designing methods can be seen in [16]. Design method using linear programming approach can be seen in [17]. Work by [18] investigates the relationship of pade table, CFE and PRFBs. A new lattice structure for general 2 channel PRFB is also given. In [19] 2 channel FBs are designed using frequency domain optimization, where the analysis filters are designed to achieve the frequency specifications for subband coding while the synthesis filters are designed to minimize the reconstruction error in frequency domain.

In the above methods linear phase condition is not considered. Now linear phase designs are presented. Lattice characterization for 2 channel linear phase FB is given in [20]. In this paper the linear phase filters are classified into two types.

1. **Type A.** One filter is symmetric and another is antisymmetric and both the filters are of even length.

2. **Type B.** Both filters are of odd length and symmetric.

If N_1, N_2 are the lengths of the filters in the analysis section then it was proved that the sum $N_1 + N_2$ must be a multiple of 4. In [15] lattice structures are given for 2 channel case. In [6] a special structure for $\mathcal{E}(z)$ is assumed, and lattice structures are given for the linear phase design. In [21] constrained optimization is used, where the PR problem is treated in two ways, one as a quadratic programming with linear constraints and another one with nonlinear constraints. Close form solution for the first approach and iterative solution for the second approach is given. In [22] first one filter is assumed, and the second one is obtained form null space projection approach. In [23] design method for type B filters is given, in [24] type A, of equal and unequal length cases, and type B filters are designed. First the lower length filter is designed, then the second filter is designed using the remaining degrees of freedom, with PR condition. In [25] a design based on least squares sense, minimizing the error energies of analysis filters subjected to PR condition, was done. A large family of type A and type B LPFBs are designed in [26], and PR is ensured structurally in this design. In a recent work [27] the linear phase property and monomial determinant of analysis polyphase matrix are combined and formulated in terms of convolution matrices. Design method for type A filters is given, where first a linear phase filter is assumed, and the second filter is obtained from the solution space of the linear equations framed for PR conditions. A sequential design algorithm using PR condition is derived for both equal and unequal length filter banks.

1.4.2 M channel case

The 2 channel design by [2] is extended to M channel by [13]. It was shown that the earlier design is indeed 2 channel paraunitary design, and that lossless property of alias component matrix is a sufficient condition for PRFB, and indeed it is verified that it is the condition for analysis polyphase matrix to be paraunitary. A structure for the M channel lossless polyphase matrices which corresponds to paraunitary FIRPRFB's has been given. In [28] a structure for paraunitary FIRPRFB's in terms of polyphase matrix in which the filter responses satisfy pairwise mirror image symmetry with respect to $\pi/2$ is given. A complete characterization for paraunitary FIRPRFB's is given in [29] in terms of the polyphase matrix. In [30] the authors give an alternative numerically efficient complete characterization for paraunitary FIRPRFB's. Also the authors discuss about an efficient way of initialization of the parameters. Further developments on paraunitary FIRPRFB's can be seen in [31]. In [14] author presents a method for designing general FIRPRFB. The idea is to first choose $M-1$ filters and solve for the coefficients of the last filter such that the determinant of the polyphase matrix is a monomial. Such a filter is called the *complementary filter*. This leads to a set of linear equations. But a systematic way of parameterization of remaining degrees of freedom after choosing $M-1$ filters is not given. In [3] the author discusses lot of problems regarding the design of general FIRPRFB or *bi orthogonal FB*. Smith form of matrix polynomial [48] is discussed in which any analysis polyphase matrix can be written as $\mathcal{E}(z) = \mathcal{U}(z)\mathcal{D}(z)\mathcal{V}(z)$, where $\mathcal{U}(z)$, $\mathcal{V}(z)$ are unimodular matrix polynomials [1] and $\mathcal{D}(z)$ is a diagonal matrix. But characterization of unimodular matrix polynomials having constant determinant is not given. Idea regarding the order of $\mathcal{E}(z)$ from the above expression is difficult as some terms can become zero. The author discusses some open issues regarding the most general $\mathcal{E}(z)$. Time domain approaches are given in [15] [32], where the reconstruction error is reduced while designing the filters. Filter banks with variable delay is given and it was seen in simulations that very less and very large delays give poor filters.

In [33] the author presents some ideas for capturing all FIR QMFs with lossless and

[1] matrix polynomials with constant determinant

unimodular matrix polynomials. The author presents a class of systems that has the following factorization structure for $\mathcal{E}(z)$, if $det(\mathcal{E}(z)) = cz^{-l_a}$ and degree [2] (μ) of $\mathcal{E}(z)$ is l_a,

$$\mathcal{E}(z) = R_{l_a}\Lambda(z)R_{l_a-1}\Lambda(z)\cdots R_1\Lambda(z)R_0 \tag{1.9}$$

where R_i is some nonsingular matrix. If R_i is unitary the above system becomes lossless. It has been shown by the author that if $\mathcal{E}_L(z)$ is the analysis polyphase matrix with $det(\mathcal{E}_L(z)) = z^{-L}$ then every $\mathcal{E}_L(z)$ can be factorized into a form $\mathcal{G}(z)\mathcal{U}(z)$, where $\mathcal{G}(z)$ is FIR with degree L and $\mathcal{U}(z)$ is unimodular. Paraunitary $\mathcal{G}(z)$ is one of the possible cases. Then the problem reduces to the parameterization of unimodular matrices.

A class of FIRPRFB's for which the inverse of the polyphase matrix has only anti causal terms i.e. only powers of z is considered in [34] [35]. Note that this class of FIRPRFB's includes paraunitary systems. Generalization of this design is given recently by [36]. In [37] the authors address the problem of given K, filters finding $M-K$ filters such that they form a FIRPRFB. The problem is referred as (M, K) problem. First the authors derive a condition that polyphase components of first K filters should satisfy for the existence of complementary filters. Then characterization for the complementary filters is derived. But the parameterization is given only for $K = M-1$. In [46] FIRPRFB with low delay is designed using nilpotent matrices, where specific structures for $\mathcal{E}(z)$ are assumed. In [47] the authors present the design method of FIRFB's with arbitrary length analysis and synthesis filters, but the FBs are not PR.

Characterizations for M channel linear phase paraunitary FIRPRFB's can be found in [38] [40]. Generally, while designing the M channel filter bank, it is assumed that all the filters have same length, but this is not always so. Different filters can have different lengths, and its easy to incorporate this in linear phase conditions. Design and factorization of FIR paraunitary filter banks given several analysis filters is given in [39]. Linear phase case has been dealt, and PR is maintained structurally. In [41] a general algorithm for design of linear phase paraunitary PRFBs is given. In [42] a lattice structure for three channel linear phase FIRPRFB's is discussed. But completeness of the structure

[2] Degree is the minimum number of delays elements to realize a system, for more information refer [1]

is not provided. An approach for extending the structure for M channel case when M is odd is provided. In [44], given K linear phase filters, an approach for finding $M-K$ linear phase filters forming a linear phase FIRPRFB is discussed. But completeness of the approach has not been verified. In [45] the authors prove that given K linear phase filters, there always exists $M-K$ linear phase filters forming a FIRPRFB if the $K \times M$ polyphase matrix of first K filters has the rank K for all z^{-1} except $z^{-1}=0$. Given such K filters, an approach similar to the one given in [44] is provided for finding $M-K$ filters.

1.5 The scope of the thesis

1.5.1 Proposed causal stable IIRFB design

In chapter (2) the problem of characterization of M channel IIR PR causal stable filter banks is addressed. The IIR causal stable condition demands the minimum phase condition of $det(\mathcal{E}(z))$, which is very difficult to impose. As described in the section (1.3), in most of the previous works this condition is achieved by assuming special structures for $\mathcal{E}(z)$. In the proposed characterization $\mathcal{E}(z)$ is written in the state space form as follows.

$$\mathcal{E}(z) = D[I + C(zI - A)^{-1}B]$$

Here D is an $M \times M$ invertible matrix. It is known that the eigen values of the state transition matrix (A) are the poles of the transfer function (written in state space form). Stability demands eigenvalues of the state transition matrix to be inside the unit circle. Explicit inverse for $\mathcal{E}(z)$ can be given if $\mathcal{E}(z)$ is assumed to be invertible at $z=\infty$ as

$$\mathcal{E}^{-1}(z) \quad = \quad [I + C(zI - A^{*})^{-1}B]D^{-1}$$

Where $A^{*} = A - BC$. The eigen values of the state transition matrix of the inverse system (A^{*}) decides the stability of the inverse system. Minimal structures are preferred to avoid nonunique representations of $\mathcal{E}(z)$. Conditions for minimality were imposed on $\mathcal{E}(z)$. Given $\mathcal{E}(z)$ satisfying the minimal conditions, additional constraints have to be imposed on the matrices used in the state space form of $\mathcal{E}(z)$ for forcing the eigen values of A^* to be in side unit circle.

A is constructed by constraining its eigen values to be inside unit circle, and assuming a Jordan from. Matrices B and C can be constrained to force eigen values of A^* inside unit circle. The spectrum of A^* is constructed using a mapping function from the spectrum of A to some arbitrary points inside the unit circle, using the concept of *function of a matrix*, so that A^* is completely known once A and the mapping function are defined. Some more results are given in chapter (2) to obtain matrices B and C if rank of matrix $(A - A^*)$ is known.

Next a design method based on factorization of a minimal system into degree one factors is presented. Conditions for decomposing a large system into smaller systems is presented, and based on these ideas a cascade structure for causal stable IIR FB's is given. Special cases where change of sign in the structure gives inverse system are also presented.

1.5.2 Proposed FIRPRFB design

In section (1.4), review of several FIR PRFB designs were presented. The open issues discussed by [33] [3] regarding general structures for FIR $\mathcal{E}(z)$ or *bi orthogonal filter banks* received less interest. In most of the methods discussed in section (1.4) the synthesis polyphase is of same degree as that of analysis for an $M \times M$ order l_a matrix polynomial $\mathcal{E}(z)$. Synthesis filters, however, can be longer or shorter than analysis filters. Moreover there is no flexibility of having arbitrary system delay. All these issues can be dealt using theory of matrix polynomials.

In chapter (3), spectral theory of matrix polynomials is used for the design of FIR-PRFB's. FIRPRFB's constrain $det(\mathcal{E}(z))$ to have a monomial determinant, and $\mathcal{E}(z)$ to be regular [3]. The eigen values of the matrix polynomial $\mathcal{E}(z)$ [4] must be zero to satisfy the monomial determinant condition. Methods to obtain spectral data of a matrix polynomial is given. The concept of decomposable pair, which is crucial in the construction of matrix

[3]Invertible matrix polynomials

[4]Refer [48] for details regarding the eigen values of a matrix polynomial

polynomial, is discussed. The inverse problem, i.e. given the spectral data, construction of a regular matrix polynomial is given. Some theorems, which are instrumental in finding the minimum delay to be introduced to make the synthesis section causal, are given. The system delay can be controlled in the proposed design. Length of the synthesis filters can be varied by using different Jordan forms of the spectral data. It can be shown that for the same $(deg(det(\mathcal{E}(z))))$, lot of structures exist and can be obtained by simply changing the spectral data corresponding to $\mathcal{E}(z)$.

Conditions on $\mathcal{E}(z)$ to be linear phase are given. Some conditions on the spectrum of the matrix polynomial are given so that it becomes an analysis polyphase matrix of a linear phase filter bank. Some results regarding the $deg(det(\mathcal{E}(z)))$ and restriction of order of $\mathcal{E}(z)$, given the parity of M, are given. Due to difficulty in solving one of the equations, near linear phase condition is obtained instead. An error measure is defined for near LPPRFB, and filters are designed to reduce this error besides frequency specifications of filters. In this case also, there is a flexibility to alter the length of synthesis filters.

Unimodular matrix polynomials provide zero delay. Characterization of unimodular matrix polynomials is given by giving the conditions under which a matrix polynomial becomes unimodular. Here also the order of synthesis polyphase matrix can be varied.

Chapter 2

Design of M channel causal stable IIRPRFB

2.1 Introduction

Let $\mathcal{E}(z)$ and $\mathcal{R}(z)$ be the $M \times M$ analysis and synthesis polyphase matrices. For perfect reconstruction condition $\mathcal{R}(z)\mathcal{E}(z)=cz^{-k}\mathbf{I}_M$. Assuming $c = 1$ and $k = 0$ i.e. $\mathcal{R}(z) = \mathcal{E}^{-1}(z)$, $\mathcal{E}(z)$ and $\mathcal{R}(z)$ are $M \times M$ rational matrix functions. Then $\mathcal{R}(z)$ is given by

$$\begin{aligned} \mathcal{R}(z) &= (\mathcal{E}(z))^{-1} \\ &= \frac{1}{det(\mathcal{E}(z))} adj(\mathcal{E}(z)). \end{aligned} \tag{2.1}$$

If $\mathcal{E}(z)$ is causal stable, $\mathcal{R}(z)$ is causal stable if and only if $det(\mathcal{E}(z))$ is minimum phase. Characterization of a rational matrix function with minimum phase determinant is difficult, so state space formulation is assumed.

2.2 State space realization of $\mathcal{E}(z)$

A rational matrix function can be realized as a state space system with realization $\mathcal{E}(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$. Assuming $\mathcal{E}(z)$ is invertible at $z = \infty$ [56] implies $\mathbf{D}$ is an invertible

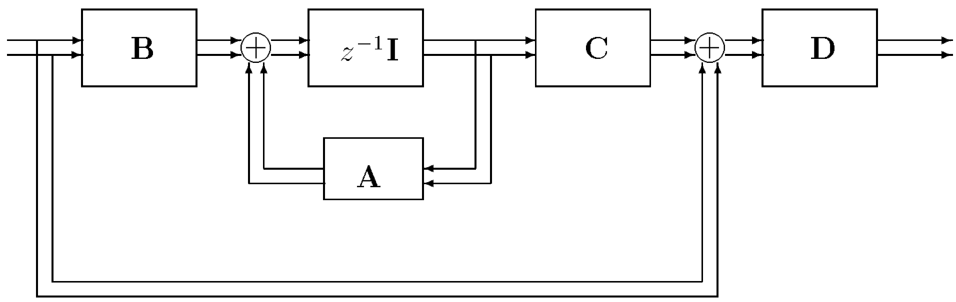


Figure 2.1: State space representation of $\mathcal{E}(z)$

matrix. Thus $\mathcal{E}(z)$ can be taken as

$$\mathcal{E}(z) = \mathbf{D}(\mathbf{I} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}) \tag{2.2}$$

where $\mathbf{A}$[1], $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are matrices with sizes $m \times m$, $m \times M$, $M \times m$ and $M \times M$ respectively. The system is shown in the figure (2.1). The inverse system $\mathcal{R}(z)$ is realized as

$$\mathcal{R}(z) = (\mathbf{I} - \mathbf{C}(z\mathbf{I} - \mathbf{A}^{*})^{-1}\mathbf{B})\mathbf{D}^{-1} \tag{2.3}$$

where $\mathbf{A}^{*} = \mathbf{A} - \mathbf{BC}$. The poles of $\mathcal{E}(z)$ are the eigenvalues of $\mathbf{A}$ [56], similarly the poles of inverse system are given by the eigen values of matrix $\mathbf{A}^{*}$. For causal stable analysis and synthesis filter banks magnitude of eigenvalues of $\mathbf{A}$ and $\mathbf{A}^{*}$ must be less than one. One possible solution was described by Basu et al [5] for M-channel case. In their work, first a good low pass filter was assumed, which forms first row of the analysis polyphase matrix. Remaining rows of the analysis polyphase matrix are filled such that the resultant matrix is unimodular with its poles inside unit circle. As the determinant of the unimodular matrix is constant, it is a subclass of a generic class with minimum phase determinant.

2.2.1 Minimal systems

Before proposing the design, importance of minimal characterizations is discussed. A rational transfer function is minimal if it can be represented with minimal number of

[1]called as *state transition matrix* in control literature.

delays [56]. Equivalent definitions are given for minimal systems in literature [51]. A minimal system is a consequence of pole zero cancellation. Non-unique combinations of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ matrices are avoided in minimal systems. Given $\mathcal{E}(z)$, it can be verified for minimality by checking the following two conditions, namely:

- $rank(\mathcal{C}(\mathbf{A},\mathbf{B})) = m$, where $\mathcal{C}(\mathbf{A},\mathbf{B}) = [\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \cdots\ \mathbf{A}^k\mathbf{B}]$, where m is the dimension of $\mathbf{A}$ and k is the index of nil potency of $\mathbf{A}$, this condition is called *controllability* condition.

- $rank(\mathcal{O}^T(\mathbf{C},\mathbf{A})) = m$, where $\mathcal{O}^T(\mathbf{C},\mathbf{A}) = [\mathbf{C}^T\ \mathbf{A}^T\mathbf{C}^T\ \cdots\ (\mathbf{A}^T)^k\mathbf{C}^T]$, where this condition is called *observability* condition.

Once $\mathcal{E}(z)$ is ensured to be minimal, its inverse $\mathcal{R}(z)$ is minimal [56]. In the present design some assumptions are made keeping in mind the minimality of $\mathcal{E}(z)$. Since $\mathbf{B}$ and $\mathbf{C}$ are $m \times M$ and $M \times m$ matrices respectively, $\mathbf{C}(\mathbf{A},\mathbf{B})$ and $\mathcal{O}(\mathbf{C},\mathbf{A})$ will be $m \times kM$ and $kM \times m$ matrices. If $m \leq M$, full rank matrices $\mathbf{B}$ and $\mathbf{C}$ ensure the minimality conditions, which is not a necessary condition but a sufficient one. So, the dimension of the matrix $\mathbf{A}$ never exceeds the number of channels M and $rank(\mathbf{B}) = rank(\mathbf{C}) = m$ in the present design.

2.3 Proposed function/transformation based design

From equation (2.3) matrix $\mathbf{BC}$ can be written as

$$\begin{aligned} \mathbf{BC} &= \mathbf{A} - \mathbf{A}^* \\ &= \mathbf{\Delta_A} \end{aligned} \tag{2.4}$$

Now the rank of the matrix $\mathbf{BC}$ is given by the expression

$$rank(\mathbf{BC}) = rank(\mathbf{C}) - dim(\mathcal{N}(\mathbf{B}) \cap \mathcal{R}(\mathbf{C})) \tag{2.5}$$

where $\mathcal{N}(\mathbf{B})$ and $\mathcal{R}(\mathbf{B})$ represents the null space and range of matrix $\mathbf{B}$. If $rank(\mathbf{BC}) = r$ (say), then

$$dim(\mathcal{N}(\mathbf{B}) \cap \mathcal{R}(\mathbf{C})) = m - r \tag{2.6}$$

Even if $\mathbf{B}$ and $\mathbf{C}$ are full rank (m here), the product of the matrices ($\mathbf{BC}$) can have a rank less than m. Now the bounds on rank of a matrix, generated by the product of two full rank matrices is discussed in the following theorem.

Theorem 1 *If $\mathbf{B}$ and $\mathbf{C}$ are full rank matrices, and if r is the rank of their product, then r is bounded as*

$$\begin{aligned} 2m - M \;\leq\; r \;\leq\; m, &\quad \textit{if } m \geq M/2 \\ 0 \;\leq\; r \;\leq\; m, &\quad \textit{other wise} \end{aligned}$$

Proof: From equation (2.6) matrix $\mathbf{C}$ has $m - r$ linearly independent columns forming a subspace (say $\mathcal{N}'(\mathbf{B})$) of the space $\mathcal{N}(\mathbf{B})$. The dimension of $\mathcal{N}(\mathbf{B})$ is $M - m$. Since $\mathcal{N}'(\mathbf{B})$ is a subspace of $\mathcal{N}(\mathbf{B})$, $dim(\mathcal{N}'(\mathbf{B})) \leq dim(\mathcal{N}(\mathbf{B}))$ which implies

$$\begin{aligned} m - r &\leq M - m \\ r &\geq 2m - M \end{aligned} \tag{2.7}$$

which is the lower bound on r. The upper bound on r is m which is obvious from the equation (2.6), which happens when $\mathcal{N}(\mathbf{B}) \cap \mathcal{R}(\mathbf{C}) = \phi$. If $m < M/2$ then the lower bound on r can become negative, but since rank cannot be negative lower bound by default becomes zero. □

Given an $m \times m$ matrix with rank r ($r \leq m$), which is known to be a product of two full rank matrices (not given) of sizes $m \times M$ and $M \times m$ with ($m \leq M$), and if r lies in the bounds given by the theorem (1), then the following theorem gives the way to construct the matrices whose product is the given matrix.

Theorem 2 *If $\mathbf{X}$ is an $m \times m$ matrix with rank r and it is known to be the product of* *two full rank matrices say* $\mathbf{Y}$ *and* $\mathbf{Z}$*, with sizes* $m \times M$ *and* $M \times m$ *respectively, and if* r *lies in the bounds given by theorem (1), then* $\mathbf{Y}$ *and* $\mathbf{Z}$ *are given by*

$$\begin{aligned} \mathbf{Y} &= \mathbf{U}\mathfrak{B} \\ \mathbf{Z} &= \begin{bmatrix} \mathfrak{B}^{\dagger} \begin{bmatrix} \mathbf{\Sigma}_r \\ 0 \end{bmatrix} & \mathcal{N}(\mathfrak{B})\mathbf{\Gamma} \end{bmatrix} \mathbf{V}^T \end{aligned}$$

where $\mathfrak{B}$ *is an* $m \times M$ *full rank matrix,* $\mathbf{\Gamma}$ *is an* $(M-m)\times(m-r)$ *full rank matrix,* $\mathbf{\Sigma}_r$ *is an* $r \times r$ *diagonal matrix,* $\mathfrak{B}^{\dagger}$ *is a full rank matrix and is the pseudo inverse of* $\mathfrak{B}$ *with size* $M \times m$*, and* $\mathbf{U}$ *and* $\mathbf{V}$ *are* $m \times m$ *unitary matrices.*

Proof: Taking the SVD of the matrix $\mathbf{X}$ we have,

$$\mathbf{X} = \mathbf{U}\mathbf{\Sigma}_{\mathbf{X}}\mathbf{V}^T$$

where $\mathbf{U}$ and $\mathbf{V}$ are $m \times m$ unitary matrices and $\mathbf{\Sigma}_{\mathbf{X}}$ is an $m \times m$ diagonal matrix with first r diagonal entries nonzero. So, $\mathbf{\Sigma}_{\mathbf{X}}$ can be written as $\begin{bmatrix} \mathbf{\Sigma}_r & 0 \\ 0 & 0 \end{bmatrix}$, where $\mathbf{\Sigma}_r$ is the nonzero block of $\mathbf{\Sigma}_{\mathbf{X}}$. If $\mathfrak{B}$ is any $m \times M$ full rank matrix and $\mathcal{N}(\mathfrak{B})$ is the null space of $\mathfrak{B}$, then the product $\mathfrak{B}\mathcal{N}(\mathfrak{B})\mathbf{\Gamma}$ is zero for any full rank matrix $\mathbf{\Gamma}$ (this fact is used below). Now the matrix $\mathbf{X}$ is written as

$$\begin{aligned} \mathbf{X} &= \mathbf{U} \begin{bmatrix} \mathbf{\Sigma}_r & 0 \\ 0 & 0 \end{bmatrix} \mathbf{V}^T \\ &= \mathbf{U} \begin{bmatrix} \mathfrak{B}\mathfrak{B}^{\dagger} \begin{pmatrix} \mathbf{\Sigma}_r \\ 0 \end{pmatrix} & 0_{m\times(m-r)} \end{bmatrix} \mathbf{V}^T \\ &= \mathbf{U} \begin{bmatrix} \mathfrak{B}\mathfrak{B}^{\dagger} \begin{pmatrix} \mathbf{\Sigma}_r \\ 0 \end{pmatrix} & \mathfrak{B}\mathcal{N}(\mathfrak{B})_{M\times(M-m)}\mathbf{\Gamma}_{(M-m)\times(m-r)} \end{bmatrix} \mathbf{V}^T \\ &= \mathbf{U}\mathfrak{B} \underbrace{\begin{bmatrix} \mathfrak{B}^{\dagger} \begin{pmatrix} \mathbf{\Sigma}_r \\ 0 \end{pmatrix} & \mathcal{N}(\mathfrak{B})\mathbf{\Gamma} \end{bmatrix}}_{\mathbf{\Upsilon}} \mathbf{V}^T \end{aligned} \tag{2.8}$$

Since $\mathfrak{B}\mathfrak{B}^{\dagger} = \mathbf{I}_m$, using the rank of product concept

$$\begin{aligned} rank(\mathfrak{B}\mathfrak{B}^{\dagger}) &= rank(\mathbf{I}_m) \\ rank(\mathfrak{B}^{\dagger}) - dim(\mathcal{N}(\mathfrak{B}) \cup \mathcal{R}(\mathfrak{B}^{\dagger})) &= m \\ dim(\mathcal{N}(\mathfrak{B}) \cup \mathcal{R}(\mathfrak{B}^{\dagger})) &= 0 \\ \mathcal{N}(\mathfrak{B}) \cup \mathcal{R}(\mathfrak{B}^{\dagger}) &= \phi \end{aligned} \tag{2.9}$$

Form the equation (2.9), the space spanned by the columns of $\mathfrak{B}^{\dagger}$ and $\mathcal{N}(\mathfrak{B})$ are independent, so the block matrix $\left[\mathfrak{B}^{\dagger} \begin{pmatrix} \boldsymbol{\Sigma}_r \\ 0 \end{pmatrix} \mathcal{N}(\mathfrak{B})\boldsymbol{\Gamma} \right]$ is full rank if $\begin{pmatrix} \boldsymbol{\Sigma}_r \\ 0 \end{pmatrix}$ and $\boldsymbol{\Gamma}$ are full rank. Thus the equation (2.8) can be written as the product of two full rank matrices, $\mathbf{X} = \mathbf{YZ}$, with $\mathbf{Y} = \mathbf{U}\mathfrak{B}$ and $\mathbf{Z} = \left[\mathfrak{B}^{\dagger} \begin{pmatrix} \boldsymbol{\Sigma}_r \\ 0 \end{pmatrix} \quad \mathcal{N}(\mathfrak{B})\boldsymbol{\Gamma} \right] \mathbf{V}^T$. □

Theorem (2) will be used to construct $\mathbf{B}$ and $\mathbf{C}$ from $\boldsymbol{\Delta}_{\mathbf{A}}$. Now we propose how $\mathbf{A}$ and $\mathbf{A}^*$ may be designed. The state transition matrices $\mathbf{A}$ and $\mathbf{A}^*$ can have either same or different eigen values with different Jordan forms. But both the matrices must have eigen values inside unit circle for causal stable condition of IIRPRFB. Let us assume $\mathbf{A}^*$ is obtained by an operation $\mathcal{O}(\mathbf{A})$ acting on $\mathbf{A}$,

$$\mathcal{O}(\mathbf{A}) = \mathbf{A}^* \tag{2.10}$$

changing it's spectrum. Now $\mathbf{A}^*$ can have different spectrum and different Jordan form [59] compared to $\mathbf{A}$. The operation $\mathcal{O}(.)$ should preserve the causal stable condition. If $\mathcal{J}_{\mathbf{A}}$ and $\mathcal{J}_{\mathbf{A}^*}$ are the Jordan forms of matrices $\mathbf{A}$ and $\mathbf{A}^*$ then in the present design following two cases are assumed.

- $\mathbf{A}$ and $\mathbf{A}^*$ have different or same eigen values with different Jordan forms, i.e, $\mathcal{J}_{\mathbf{A}} \neq \mathcal{J}_{\mathbf{A}^*}$, even if all the eigen values of $\mathbf{A}$ and $\mathbf{A}^*$ are same.
- $\mathbf{A}$ and $\mathbf{A}^*$ have same eigen values with same Jordan form, i.e. $\mathcal{J}_{\mathbf{A}} = \mathcal{J}_{\mathbf{A}^*}$.

2.3.1 Case 1: $\mathcal{J}_{\mathbf{A}} \neq \mathcal{J}_{\mathbf{A}}^*$ — function of a matrix approach

For this case, let $\mathcal{J}_{\mathbf{A}}$ be known, then $\mathbf{A}$ can be constructed as

$$\mathbf{A} = \mathbf{P_A}^{-1} \mathcal{J}_{\mathbf{A}} \mathbf{P_A} \tag{2.11}$$

where $\mathbf{P_A}$ is an invertible matrix. Here the eigen values of $\mathbf{A}$ are taken to be inside the unit circle. The eigen values of $\mathbf{A}^*$ must be inside unit circle for causal stable condition. Spectrum of $\mathbf{A}$ and $\mathbf{A}^*$ can be disjoint or intersecting or one can be contained in another. Once $\mathcal{J}_{\mathbf{A}}$ and $\mathbf{P_A}$ are given, one possible way to construct $\mathbf{A}^*$ is by the concept of *function of a matrix* approach.

In this case the operation $\mathcal{O}(.)$ acting on $\mathbf{A}$ can be taken as $f(\mathbf{A})$, where $f(.)$ is an analytic [2] function. If $f(.)$ is analytic and $\mathbf{A}$ is given by equation (2.11) then $f(\mathbf{A})$ is given as [3]

$$\mathbf{A}^* = \mathcal{O}(\mathbf{A}) = f(\mathbf{A}) = \mathbf{P_A}^{-1} f(\mathcal{J}_{\mathbf{A}}) \mathbf{P_A}. \tag{2.12}$$

In this case $\mathcal{J}_{\mathbf{A}^*} = f(\mathcal{J}_{\mathbf{A}})$.

2.3.1.1 Construction of the function $f(.)$

Detailed discussion on the structure of $f(\mathcal{J}_{\mathbf{A}})$ and ways to construct the function $f(.)$ are given in Appendix (A). If $\{\lambda_1, \cdots, \lambda_k\}$ are the eigen values of $\mathbf{A}$, then $\{f(\lambda_1), \cdots, f(\lambda_k)\}$ are the eigen values of $f(\mathbf{A})$ (refer Appendix (A)). Now assuming the spectrum of $\mathbf{A}$ and $f(\mathbf{A})$ are known, i.e. the above discussed eigen values are known, then $f(.)$ can be constructed (refer Appendix (A)) such that $rank(\mathbf{A} - f(\mathbf{A}))$ lies with in the bounds specified in theorem (1) for a given m. So, in simple words the mapping function is constructed from the sample points and the matrix $f(\mathbf{A})$ is determined completely.

[2] must be infinitely differentiable

[3] Refer [59] for a detailed discussion on function of a matrix

2.3.1.2 Construction of B and C

For an M channel filter bank, once $\mathbf{A}$ is fixed (with eigen values and Jordan from), a suitable mapping function $f(.)$ is defined (Appendix (A)), so that $rank(\mathbf{A}-\mathbf{A}^*)$ lies with in the bounds given by theorem (1). Now matrices $\mathbf{B}$ and $\mathbf{C}$ are constructed as follows.

$$\begin{aligned}\mathbf{BC} &= \mathbf{A}-\mathbf{A}^* = \mathbf{A}-f(\mathbf{A})\\ &= \mathbf{P}_{\mathbf{A}}^{-1}\mathcal{J}_{\mathbf{A}}\mathbf{P}_{\mathbf{A}} - \mathbf{P}_{\mathbf{A}}^{-1}f(\mathcal{J}_{\mathbf{A}})\mathbf{P}_{\mathbf{A}}\\ &= \mathbf{P}_{\mathbf{A}}^{-1}(\mathcal{J}_{\mathbf{A}}-f(\mathcal{J}_{\mathbf{A}}))\mathbf{P}_{\mathbf{A}}\\ &= \mathbf{P}_{\mathbf{A}}^{-1}\mathbf{\Delta}_{\mathcal{J}_{\mathbf{A}}}\mathbf{P}_{\mathbf{A}}\end{aligned}$$

As $rank(\mathbf{BC})$ is r so is the rank of $\mathbf{\Delta}_{\mathcal{J}_{\mathbf{A}}}$. As it is assumed that r lies with in the bounds given by theorem (1), $\mathbf{\Delta}_{\mathcal{J}_{\mathbf{A}}}$ can be written as the product of two full rank matrices from theorem (2).

$$\begin{aligned}\mathbf{BC} &= \mathbf{P}_{\mathbf{A}}^{-1}\mathbf{\Delta}_{\mathcal{J}_{\mathbf{A}}}\mathbf{P}_{\mathbf{A}}\\ &= \underbrace{\mathbf{P}_{\mathbf{A}}^{-1}\mathbf{U}_{\mathcal{J}_{\mathbf{A}}}\mathfrak{B}}_{\mathbf{B}}\underbrace{\overbrace{\begin{bmatrix}\mathfrak{B}^{\dagger}\begin{pmatrix}\mathbf{\Sigma}_r\\ 0\end{pmatrix} & \mathcal{N}(\mathfrak{B})\mathbf{\Gamma}\end{bmatrix}}^{\mathbf{\Upsilon}_{\mathcal{J}_{\mathbf{A}}}}\mathbf{V}_{\mathcal{J}_{\mathbf{A}}}^T\mathbf{P}_{\mathbf{A}}}_{\mathbf{C}}\end{aligned} \tag{2.13}$$

where $\mathbf{U}_{\mathcal{J}_{\mathbf{A}}}$ and $\mathbf{V}_{\mathcal{J}_{\mathbf{A}}}$ are the unitary matrices obtained from the SVD of $\mathbf{\Delta}_{\mathcal{J}_{\mathbf{A}}}$ as $\mathbf{U}_{\mathcal{J}_{\mathbf{A}}}\mathbf{\Sigma}_{\mathcal{J}_{\mathbf{A}}}\mathbf{V}_{\mathcal{J}_{\mathbf{A}}}^T$, $\mathbf{\Sigma}_{\mathcal{J}_{\mathbf{A}}} = \begin{bmatrix}\mathbf{\Sigma}_r & 0\\ 0 & 0\end{bmatrix}$ and $\mathfrak{B}$ is any $m\times M$ full rank matrix and $\mathbf{\Gamma}$ is any $(M-m)\times(m-r)$ full rank matrix.

Theorem 3 *If $\mathbf{A}^*$ is obtained from $\mathbf{A}$ as $f(\mathbf{A})$ then, the matrix $\mathbf{P}_{\mathbf{A}}$ plays no role in the design, i.e. information regarding $\mathcal{J}_{\mathbf{A}}$ is enough for the design.*

Proof: Substituting the expressions for the matrices $\mathbf{B}$ and $\mathbf{C}$ as in equation (2.13) in the analysis polyphase matrix expression given in the equation (2.2), we have

$$\mathcal{E}(z) = \mathbf{D}[\mathbf{I}+\mathbf{\Upsilon}_{\mathcal{J}_{\mathbf{A}}}\mathbf{V}_{\mathcal{J}_{\mathbf{A}}}^T\mathbf{P}_{\mathbf{A}}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{P}_{\mathbf{A}}^{-1}\mathbf{U}_{\mathcal{J}_{\mathbf{A}}}\mathfrak{B}]$$

$$
\begin{aligned}
&= \mathbf{D}[\mathbf{I} + \boldsymbol{\Upsilon}_{\mathcal{J}_\mathbf{A}} \mathbf{V}^T_{\mathcal{J}_\mathbf{A}} \mathbf{P}_\mathbf{A} (z\mathbf{P}^{-1}_\mathbf{A}\mathbf{P}_\mathbf{A} - \mathbf{P}^{-1}_\mathbf{A}\mathcal{J}_\mathbf{A}\mathbf{P}_\mathbf{A})^{-1}\mathbf{P}^{-1}_\mathbf{A}\mathbf{U}_{\mathcal{J}_\mathbf{A}}\mathfrak{B}] \\
&= \mathbf{D}[\mathbf{I} + \boldsymbol{\Upsilon}_{\mathcal{J}_\mathbf{A}} \mathbf{V}^T_{\mathcal{J}_\mathbf{A}} \mathbf{P}_\mathbf{A}\mathbf{P}^{-1}_\mathbf{A} (z\mathbf{I} - \mathcal{J}_\mathbf{A})^{-1}\mathbf{P}_\mathbf{A}\mathbf{P}^{-1}_\mathbf{A}\mathbf{U}_{\mathcal{J}_\mathbf{A}}\mathfrak{B}] \\
&= \mathbf{D}[\mathbf{I} + \boldsymbol{\Upsilon}_{\mathcal{J}_\mathbf{A}} \mathbf{V}^T_{\mathcal{J}_\mathbf{A}} (z\mathbf{I} - \mathcal{J}_\mathbf{A})^{-1}\mathbf{U}_{\mathcal{J}_\mathbf{A}}\mathfrak{B}]. \qquad (2.14)
\end{aligned}
$$

Thus $\mathbf{P}_\mathbf{A}$ plays no role in the construction of $\mathcal{E}(z)$. □

The inverse structure is given by the equation (2.3) and using the equation (2.13) we have

$$
\begin{aligned}
\mathcal{R}(z) &= [\mathbf{I} - \boldsymbol{\Upsilon}_{\mathcal{J}_\mathbf{A}} \mathbf{V}^T_{\mathcal{J}_\mathbf{A}} (z\mathbf{I} - (\mathcal{J}_\mathbf{A} - \mathbf{U}_{\mathcal{J}_\mathbf{A}}\mathfrak{B}\boldsymbol{\Upsilon}_{\mathcal{J}_\mathbf{A}}\mathbf{V}^T_{\mathcal{J}_\mathbf{A}}))^{-1}\mathbf{U}_{\mathcal{J}_\mathbf{A}}\mathfrak{B}]\mathbf{D}^{-1} \\
&= [\mathbf{I} - \boldsymbol{\Upsilon}_{\mathcal{J}_\mathbf{A}} \mathbf{V}^T_{\mathcal{J}_\mathbf{A}} (z\mathbf{I} - (\mathcal{J}_\mathbf{A} - (\mathcal{J}_\mathbf{A} - f(\mathcal{J}_\mathbf{A}))))^{-1}\mathbf{U}_{\mathcal{J}_\mathbf{A}}\mathfrak{B}]\mathbf{D}^{-1} \\
&= [\mathbf{I} - \boldsymbol{\Upsilon}_{\mathcal{J}_\mathbf{A}} \mathbf{V}^T_{\mathcal{J}_\mathbf{A}} (z\mathbf{I} - f(\mathcal{J}_\mathbf{A}))^{-1}\mathbf{U}_{\mathcal{J}_\mathbf{A}}\mathfrak{B}]\mathbf{D}^{-1} \qquad (2.15)
\end{aligned}
$$

Free variables: Since real coefficient filters are of interest, $\mathbf{A}$ is constructed by assuming some structure for $\mathcal{J}_\mathbf{A}$. k distinct eigen values [4] are used as variables for $\mathcal{J}_\mathbf{A}$. Then function $f(.)$ is selected such that r lies in the rank bounds given by theorem (1). To construct $\mathbf{B}$ and $\mathbf{C}$, full rank matrices $\mathfrak{B}$ and $\boldsymbol{\Gamma}$ require Mm and $(M-m)(m-r)$ free variables. Invertible matrix $\mathbf{D}$ requires M^2 free variables. Then $(M^2 + Mm + (M-m)(m-r) + k)$ free variables are required for this design.

2.3.2 Case 2: $\mathcal{J}_\mathbf{A} = \mathcal{J}_{\mathbf{A}^*}$ — similarity transformation approach

This approach forces $(\mathbf{A} - f(\mathbf{A}))$ to zero, since $\mathcal{J}_\mathbf{A}$ and $\mathcal{J}_{\mathbf{A}^*}$ are same. Before proceeding further a lemma is given which is used later in this section.

Lemma 1 *If* $\mathbf{G}$ *and* $\mathbf{H}$ *are* $M \times M$ *matrices, and if* $\mathbf{H} = \mathbf{T}^{-1}\mathbf{G}\mathbf{T}$*, i.e.* $\mathbf{H}$ *is obtained by the similarity transformation of* $\mathbf{G}$*, where* $\mathbf{T}$ *is an* $M \times M$ *nonsingular matrix, then* $\mathbf{G}$ *and* $\mathbf{H}$ *have same Jordan forms.*

Proof: If $\mathbf{G} = \mathbf{P}^{-1}_\mathbf{G}\mathcal{J}_\mathbf{G}\mathbf{P}_\mathbf{G}$ is the Jordan decomposition of $\mathbf{G}$, then for the assumption

$$
\mathbf{H} = \mathbf{P}^{-1}_\mathbf{G}\mathbf{T}^{-1}\mathcal{J}_\mathbf{G}\mathbf{T}\mathbf{P}_\mathbf{G}
$$

[4]can be complex, but conjugate pairs should exist for real coefficient filters

$$= (\mathbf{TP_G})^{-1}\mathcal{J}_\mathbf{G}(\mathbf{TP_G})$$

So, $(\mathbf{TP_G})$ is the Jordan chain matrix of $\mathbf{H}$, and $\mathcal{J}_\mathbf{H} = \mathcal{J}_\mathbf{G}$. □

If the eigen values of $\mathbf{A}$ and $\mathbf{A}^*$ are same with same Jordan forms, $\mathbf{A}^*$ can be obtained from $\mathbf{A}$ by a similarity transformation. So, the operation $\mathcal{O}(\mathbf{A})$ is the similarity transformation.

$$\mathcal{O}(\mathbf{A}) = \mathbf{A}^* = \mathbf{T}^{-1}\mathbf{AT} \tag{2.16}$$

But so constructed $\mathbf{A}^*$ must ensure that r lies in the bounds given by the theorem (1), which implies only some pairs of $\{\mathbf{A}, \mathbf{T}\}$ exist. Characterizing such pairs for a given r is very difficult. Assuming that we are working with such a pair, the matrices $\mathbf{B}$ and $\mathbf{C}$ can be constructed from theorem (2) as

$$\begin{aligned}\mathbf{BC} &= \mathbf{A} - \mathbf{T}^{-1}\mathbf{AT} \\ &= \mathbf{U_A \Sigma_A V_A}^T\end{aligned}$$

and $\mathbf{B} = \mathbf{U_A}\mathfrak{B}$ and $\mathbf{C} = \mathbf{\Upsilon_A V_A}^T$, where $\mathbf{\Upsilon_A}$ is analogous to one given in equation (2.8). The analysis polyphase matrix is given by

$$\mathcal{E}(z) = \mathbf{D}[\mathbf{I} + \mathbf{\Upsilon_A V_A}^T(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{U_A}\mathfrak{B}] \tag{2.17}$$

and the synthesis polyphase matrix is given by

$$\mathcal{R}(z) = [\mathbf{I} - \mathbf{\Upsilon_A V_A}^T(z\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{U_A}\mathfrak{B}]\mathbf{D}^{-1} \tag{2.18}$$

Matrices $\mathbf{A}$ and $\mathbf{T}$ are selected such that r is lies in required bounds, but characterizing all such pairs is difficult. So specific cases are taken while designing the PRFB. Examples for the special cases are given below.

2.3.2.1 Case where $rank(\mathbf{\Delta_A}) = m$

For this case $rank(\mathbf{BC}) = m$. Since $\mathbf{C}$ is chosen to be a full rank $M \times m$ matrix, rows of $\mathbf{B}$ must be the span of columns of $\mathbf{C}$ for the above condition to be satisfied. So $\mathbf{B} = \mathbf{\Psi C}^T$, where $\mathbf{\Psi}$ is an $m \times m$ full rank matrix. Then

$$\begin{aligned} \mathbf{\Psi C}^T\mathbf{C} &= \mathbf{\Delta_A} \\ \mathbf{\Psi} &= \mathbf{\Delta_A}(\mathbf{C}^T\mathbf{C})^{-1} \end{aligned}$$

Here $(\mathbf{C}^T\mathbf{C})^{-1}$ is an $m \times m$ full rank symmetric matrix. Now $\mathbf{A}$ and $\mathbf{T}$ are selected such that $\mathbf{\Delta_A}$ is full rank. One possible choice is assuming $\mathbf{A}$ to be a diagonal matrix with distinct eigen values and $\mathbf{T}$ to be a circulant matrix.

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix} \quad \mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Here $\lambda_i \neq \lambda_j$ for $i \neq j$. Then $\mathbf{\Delta_A}$ is given by

$$\mathbf{\Delta_A} = \begin{pmatrix} \triangle_{\lambda_1} & 0 & \cdots & 0 \\ 0 & \triangle_{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \triangle_{\lambda_m} \end{pmatrix}$$

$$\begin{aligned} \triangle_{\lambda_i} &= \lambda_{i+1} - \lambda_i \neq 0 \quad \forall i = 1, 2, \cdots, m-1 \\ \triangle_{\lambda_m} &= \lambda_m - \lambda_1 \neq 0 \end{aligned}$$

From the above inequalities $\mathbf{\Delta_A}$ becomes a nonsingular matrix.

Free variables: Diagonal matrix $\mathbf{A}$, full rank matrices $\mathbf{C}_{M\times m}$ and $\mathbf{D}_{M\times M}$ require m, Mm and M^2 free variables respectively. Thus in total $(M^2 + Mm + m)$ free variables are required.

2.3.2.2 Case where $rank(\mathbf{\Delta_A}) = 0$

For this case $rank(\mathbf{BC}) = 0$, $\mathbf{A} - \mathbf{T}^{-1}\mathbf{AT} = 0$, then matrix **A** is called *reflexive matrix* and **T** is called *reflection matrix*. **B** must be in the *null space* of **C** and vice versa. Since **B** and **C** are full rank matrices, the dimension of $\mathcal{N}(\mathbf{B})$ is $M - m$, and columns of **C** span a subspace of $\mathcal{N}(\mathbf{B})$, implying $M - m \geq m$ leading to $m \leq M/2$. So, this case is possible only when $m \leq M/2$.

Free variables: The number of free variables depends on the matrices **A** and **T**. Matrices **B** and **C** are rank m matrices and are irrespective of the choice of **A** and **T**. mM free variables are required for matrix $\mathbf{B}_{m\times M}$. Then columns of matrix **C** spans $\mathcal{N}(\mathbf{B})$. m linearly independent vectors are generated from $\mathcal{N}(\mathbf{B})$, $\mathbf{C}_{M\times m} = \mathcal{N}(\mathbf{B})_{M\times M-m}\mathbf{\Theta}_{(M-m)\times m}$, where $\mathbf{\Theta}$ is a rank m matrix. $\mathbf{\Theta}$ requires $m \times M - m$ free variables. Depending upon choice of **A** and **T** some more free variables can be added.

2.4 Factorization of $\mathcal{E}(z)$

Cascade approach for the design of filter banks received lot of attention in the filter bank design community. Filter banks based on factorization of rational lossless systems [4] is popular in filter bank designs. The propositions discussed in this section are general, i.e. matrices **B** and **C** need not be full rank and size of matrix **A** is unrestricted.

2.4.1 Factorization of rational matrix functions

The analysis polyphase matrix given in equation (2.2) can be written as

$$\mathcal{E}(z) = \mathbf{D}\mathcal{E}'(z)$$

where $\mathcal{E}'(z) = \mathbf{I} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, and the synthesis polyphase matrix $\mathcal{R}(z)$ can be written as

$$\mathcal{R}(z) = \mathcal{R}'(z)\mathbf{D}^{-1}$$

where $\mathcal{R}'(z) = \mathbf{I} - \mathbf{C}(z\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{B}$. Since $\mathcal{R}(z)\mathcal{E}(z) = \mathbf{I}$, this implies $\mathcal{R}'(z)\mathcal{E}'(z) = \mathbf{I}$. It is sufficient to focus on factorization of primed matrices. The problem boils down to factorization of *rational matrix functions*, exhaustively dealt in [56]. All the theorems described in this chapter are taken from [56], proofs are not given here. Here the main idea is to decompose a minimal system of degree m into a product of minimal systems with smaller degrees.

If $\mathcal{E}'(z)$ is a minimal realization then the *McMillan degree*[5] of $\mathcal{E}'(z)$, denoted by $\delta(\mathcal{E}')$, is the size of matrix $\mathbf{A}$ in the minimal realization. There are several equivalent definitions of *McMillan degree* given in control texts [51].

Consider the factorization

$$\mathcal{E}'(z) = \mathcal{E}'_1(z)\mathcal{E}'_2(z)\cdots\mathcal{E}'_p(z) \tag{2.19}$$

where, for $j = 1, 2, \cdots, p$, $\mathcal{E}'_j(z)$ are $M \times M$ rational matrix functions with minimal realizations $\mathcal{E}'_j(z) = \mathbf{I} + \mathbf{C}_j(z\mathbf{I} - \mathbf{A}_j)^{-1}\mathbf{B}_j$. Then realization for $\mathcal{E}'(z)$ is

$$\mathcal{E}'(z) = \mathbf{I} + [\mathbf{C}_1\ \mathbf{C}_2\ \cdots\ \mathbf{C}_p] \left(z\mathbf{I} - \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1\mathbf{C}_2 & \cdots & \mathbf{B}_1\mathbf{C}_p \\ 0 & \mathbf{A}_2 & \cdots & \mathbf{B}_2\mathbf{C}_p \\ 0 & 0 & & \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_p \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_p \end{bmatrix} \tag{2.20}$$

This realization is not necessarily minimal, and in general

$$\delta(\mathcal{E}') \leq \delta(\mathcal{E}'_1) + \delta(\mathcal{E}'_2) + \cdots + \delta(\mathcal{E}'_p).$$

[5] minimum number of delays required to realize the system

The factorization given in the equation (2.19) is minimal if the equality is satisfied. Minimality of the above factorization means that zero-pole cancellation does not occur between the factors $\mathcal{E}_j(z)$. The McMillan degree of a rational matrix function (with $\mathbf{I}$ at infinity) and its inverse are same [56], so if the above factorization is minimal then the corresponding factorization for the inverse matrix function is also minimal.

$$\mathcal{E}'(z)^{-1} = \mathcal{R}'(z) = \mathcal{E}'_p(z)^{-1}\mathcal{E}'_{p-1}(z)^{-1}\cdots\mathcal{E}'_1(z)^{-1}$$

Now a question arises that on what conditions do a larger system, which is minimal, can be decomposed into product of several smaller minimal systems. For lossless systems, which are orthogonal on the unit circle, a specific factorization based upon projections is given by Gohberg et al [50], and a different approach is followed by Vaidyanathan et al [4]. The following proposition gives the conditions under which a minimal system can be factorized into several minimal systems.

Proposition 1 *Let m be the size of $\mathbf{A}$ in $\mathcal{E}'(z)$, and let*

$$\mathbb{C}^m = \mathbb{L}_1 + \cdots + \mathbb{L}_p \tag{2.21}$$

where the chain

$$\mathbb{L}_1 \subset \mathbb{L}_1 + \mathbb{L}_2 \subset \cdots \subset \mathbb{L}_1 + \mathbb{L}_2 + \cdots + \mathbb{L}_{p-1} \tag{2.22}$$

consists of $\mathbf{A}$-invariant subspaces [6] *whereas the chain*

$$\mathbb{L}_p \subset \mathbb{L}_p + \mathbb{L}_{p-1} \subset \cdots \subset \mathbb{L}_p + \mathbb{L}_{p-1} + \cdots + \mathbb{L}_2 \tag{2.23}$$

consists of $\mathbf{A}^*$-invariant subspaces. Then $\mathcal{E}'(z)$ admits the minimal factorization

$$\mathcal{E}'(z) = [\mathbf{I} + \mathbf{C}\pi_1(z\mathbf{I} - \mathbf{A})^{-1}\pi_1\mathbf{B}]\cdots[\mathbf{I} + \mathbf{C}\pi_p(z\mathbf{I} - \mathbf{A})^{-1}\pi_p\mathbf{B}] \tag{2.24}$$

[6] *A subspace $\mathbb{W}$ is $\mathbf{A}$-invariant if for any $\mathbf{x} \in \mathbb{W}$, $\mathbf{Ax} \in \mathbb{W}$*

where π_j *is the projector* [7] *on* $\mathbb{L}_j$ *along* $\mathbb{L}_1 + \cdots + \mathbb{L}_{j-1} + \mathbb{L}_{j+1} + \cdots + \mathbb{L}_p$.
Conversely, for every minimal factorization

$$\mathcal{E}'(z) = \mathcal{E}'_1(z)\mathcal{E}'_2(z)\cdots\mathcal{E}'_p(z) \tag{2.25}$$

where $\mathcal{E}'_j(z)$ *are rational* $M \times M$ *matrix functions with* $\mathcal{E}'_j(\infty) = \mathbf{I}$*, there exists a unique direct sum decomposition (2.21) with the property that the chains (2.22) and (2.23) consist of invariant subspaces for* $\mathbf{A}$ *and* $\mathbf{A}^*$*, respectively, such that*

$$\mathcal{E}'_j(z) = \mathbf{I} + \mathbf{C}\pi_j(z\mathbf{I} - \mathbf{A})^{-1}\pi_j\mathbf{B}, \qquad j = 1, \cdots, p$$

Detailed proof of the above theorem is given in [56] . The factorization (2.24) implies the minimal factorization for $\mathcal{E}'(z)^{-1}$ as

$$\mathcal{E}'(z)^{-1} = [\mathbf{I}-\mathbf{C}\pi_p(z\mathbf{I}-\mathbf{A}^*)^{-1}\pi_p\mathbf{B}]\,[\mathbf{I}-\mathbf{C}\pi_{p-1}(z\mathbf{I}-\mathbf{A}^*)^{-1}\pi_{p-1}\mathbf{B}]\cdots[\mathbf{I}-\mathbf{C}\pi_1(z\mathbf{I}-\mathbf{A}^*)^{-1}\pi_1\mathbf{B}]$$

2.4.2 Factorization to degree one systems

The problem of factorizing a minimal system $\mathcal{E}'(z)$ with $\delta(\mathcal{E}') = m$ into degree one systems is considered, i.e.

$$\mathcal{E}'(z) = \mathcal{E}'_1(z)\mathcal{E}'_2(z)\cdots\mathcal{E}'_m(z) \tag{2.26}$$

The sufficient condition (not a necessary condition) for the existence of the above minimal factorization is given by the following lemma and proposition.

Lemma 2 *Let* $\mathbf{A}$, $\mathbf{A}^* : \mathbb{C}^m \to \mathbb{C}^m$ *be transformations and assume that at least one of them is diagonable. Then there exists a direct sum decomposition*

$$\mathbb{C}^m = \mathbb{L}_1 + \cdots + \mathbb{L}_m \tag{2.27}$$

[7] *A matrix* $\mathbf{P}$ *is a projector onto a space* $\mathbb{W}$*, if* $\mathbf{P}^2 = \mathbf{P}$ *and* $\mathcal{R}(\mathbf{P}) = \mathbb{W}$

with one dimensional spaces $\mathbb{L}_j,\ j = 1, \cdots, m$*, such that the complete chains*

$$\mathbb{L}_1 \subset \mathbb{L}_1 + \mathbb{L}_2 \subset \cdots \subset \mathbb{L}_1 + \mathbb{L}_2 + \cdots + \mathbb{L}_{m-1} \tag{2.28}$$

and

$$\mathbb{L}_m \subset \mathbb{L}_m + \mathbb{L}_{m-1} \subset \cdots \subset \mathbb{L}_m + \mathbb{L}_{m-1} + \cdots + \mathbb{L}_2 \tag{2.29}$$

consist of $\mathbf{A}$*-invariant and* $\mathbf{A}^*$*-invariant subspaces, respectively.*

Proof of the lemma is given in [56]. A sufficient condition for minimal factorization of a rational matrix function $\mathcal{E}'(z)$ into the product of degree 1 nontrivial factors is given in the following proposition.

Proposition 2 *Let* $\mathcal{E}'(z)$ *be a rational* $M \times M$ *matrix function with a minimal realization* $\mathcal{E}'(z) = \mathbf{I} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ *and assume that at least one of the matrices* $\mathbf{A}$ *and* $\mathbf{A} - \mathbf{BC}$ *is diagonable. Then* $\mathcal{E}'(z)$ *admits a minimal factorization of the form*

$$\mathcal{E}'(z) = [\mathbf{I} + (z - z_1)^{-1}\mathbf{R}_1] \cdots [\mathbf{I} + (z - z_m)^{-1}\mathbf{R}_m] \tag{2.30}$$

where $z_1, \cdots, z_m$ *are complex numbers and* $\mathbf{R}_1, \cdots, \mathbf{R}_m$ *are* $M \times M$ *matrices of rank* 1.

Proof: The proof is obtained by combining proposition (1) and lemma (2). If at least one of $\mathbf{A}$ and $\mathbf{A}^*$ is diagonable then from lemma (2), there exists a unique direct sum decomposition and complete chains of $\mathbf{A}$-invariant and $\mathbf{A}^*$-invariant subspaces. From proposition (1) and from the above conditions a minimal factorization of a rational matrix function exists. □

In the propositions discussed above no assumptions were made regarding $\mathbf{B}$, $\mathbf{C}$ and size of state space matrix $\mathbf{A}$ except that $\mathcal{E}'(z)$ is a minimal system.

2.5 Proposed factorization based design

The analysis polyphase matrix $\mathcal{E}(z) = \mathbf{D}(\mathbf{I} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B})$ is assumed to be a minimal system, so the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ must satisfy the conditions for minimality discussed earlier. Rank deficient **B** and **C** can make the system minimal and can have a minimal factorization. This case is clearly explained with an example in [56] (ref. ch7 pp.230). In the present design, matrices **B** and **C** are assumed to be full rank and $m \leq M$. The analysis polyphase matrix $\mathcal{E}(z) = \mathbf{D}\mathcal{E}'(z)$ with a McMillan degree $\delta(\mathcal{E}) = m$ is to be factorized into degree 1 terms. From proposition (1), that if there exists a direct sum decomposition (2.27) with each component space being one dimensional, and if there exists **A**-invariant and $\mathbf{A}^*$-invariant chains (2.28) and (2.29), then minimal factorization is given by equation (2.30). From lemma (1) diagonable matrix **A** will satisfy this condition, but this is just a sufficient condition. In order to simplify the calculation of invariant subspaces, matrices **A** and $\mathbf{A}^*$ are constrained to be triangular.

2.5.1 Construction of invariant subspaces

In this section we discuss the design method for decomposing a minimal system into degree 1 systems. In the previous sections it was discussed that the factorization requires information regarding the chain of invariant spaces with respect to **A** and $\mathbf{A}^*$. Now it will be shown that the choice of the matrix **A** being upper triangular eases the calculation of invariant subspaces.

Theorem 4 *If* **A** *is an* $m \times m$ *upper triangular matrix and* $\mathbf{e}_j$ *is an* m *dimensional vector with* j^{th} *element as unity and others being zero, then there exists an* **A***-invariant chain of subspaces*

$$\mathbb{S}_{\mathrm{e}_1} \subset \mathbb{S}_{\mathrm{e}_1} + \mathbb{S}_{\mathrm{e}_2} \subset \cdots \subset \mathbb{S}_{\mathrm{e}_1} + \mathbb{S}_{\mathrm{e}_2} + \cdots + \mathbb{S}_{\mathrm{e}_m} \tag{2.31}$$

where $\mathbb{S}_{\mathrm{e}_j}$ *is the space spanned by the vector* $\mathbf{e}_j$*.*

Proof: Let **A** be an $m \times m$ upper triangular matrix of the form

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & k_1^2 & k_1^3 & \cdots & k_1^m \\ 0 & \lambda_2 & k_2^3 & \cdots & k_2^m \\ 0 & 0 & \lambda_3 & \cdots & k_3^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_m \end{bmatrix} \tag{2.32}$$

Here the diagonal elements, which are the eigen values, need not be distinct. Let $\mathbf{x} \in \mathbb{M}_i$, where $\mathbb{M}_i$ is the $Span\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_i\}$, then

$$\mathbf{x} = \sum_{j=1}^{j=i} \alpha_j \mathbf{e}_j \qquad where \quad \alpha_j \in \mathbb{C}$$

Now the vector

$$\begin{aligned} \mathbf{Ax} &= \sum_{j=1}^{j=i} \alpha_j \{\mathbf{A}\mathbf{e}_j\} \\ &= \sum_{j=1}^{j=i} \alpha_j \mathbf{a}_j \end{aligned}$$

where $\mathbf{a}_j$, the j^{th} column of the matrix $\mathbf{A}$, can be written as

$$\mathbf{a}_j = \sum_{m=1, j\neq 1}^{m=j-1} k_m^j \mathbf{e}_m + \lambda_j \mathbf{e}_j$$

Using this in the previous expression

$$\mathbf{Ax} = \sum_{j=1}^{j=i} \lambda_i \alpha_i \mathbf{e}_i + \sum_{j=1}^{j=i} \sum_{m=1, j\neq 1}^{m=j-1} k_m^j \alpha_j \mathbf{e}_m$$

It was assumed that $\mathbb{M}_i$ is the $Span\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_i\}$, and $\forall \mathbf{x} \in \mathbb{M}_i$ it was shown that $\mathbf{Ax} \in \mathbb{M}_i$, so $\mathbb{M}_i$ is $\mathbf{A}$-invariant, and this is true $\forall i = 1, 2, \cdots, m$. So, if $\mathbf{A}$ is an $m \times m$ upper triangular matrix, then there exists an $\mathbf{A}$-invariant subspace of the form $Span\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_i\} \quad \forall i = 1, 2, \cdots, m$. Since the space $\mathbb{S}_{\mathbf{e}_1} + \mathbb{S}_{\mathbf{e}_2} + \cdots + \mathbb{S}_{\mathbf{e}_j}$ is the $Span\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_j\}$ $\forall\ j = 1, \cdots, m$, $\mathbb{S}_{\mathbf{e}_1} + \mathbb{S}_{\mathbf{e}_2} + \cdots + \mathbb{S}_{\mathbf{e}_j}$ is an $\mathbf{A}$-invariant subspace. Thus there exists a chain of $\mathbf{A}$-invariant subspaces $\mathbb{S}_{\mathbf{e}_1} \subset \mathbb{S}_{\mathbf{e}_1} + \mathbb{S}_{\mathbf{e}_2} \subset \cdots \subset \mathbb{S}_{\mathbf{e}_1} + \mathbb{S}_{\mathbf{e}_2} + \cdots + \mathbb{S}_{\mathbf{e}_{m-1}}$. □

It can be shown with the similar principles that if $\mathbf{A}^*$ is a lower triangular matrix, then there exists an $\mathbf{A}^*$-invariant subspace $Span\{\mathbf{e}_m, \mathbf{e}_{(m-1)}, \cdots, \mathbf{e}_i\} \quad \forall i = m, m-1, \cdots, 1$ and a chain of $\mathbf{A}^*$-invariant subspaces $\mathbb{S}_{\mathbf{e}_m} \subset \mathbb{S}_{\mathbf{e}_m} + \mathbb{S}_{\mathbf{e}_{m-1}} \subset \cdots \subset \mathbb{S}_{\mathbf{e}_m} + \mathbb{S}_{\mathbf{e}_{m-1}} + \cdots + \mathbb{S}_{\mathbf{e}_2}$.

Using propositions (1) and (2) and above choice of matrices $\mathbf{A}$ and $\mathbf{A}^*$, there exists projector matrices π_j on the space $\mathbb{S}_{\mathbf{e}_j}$ along the space $\mathbb{S}_{\mathbf{e}_1}+\cdots+\mathbb{S}_{\mathbf{e}_{j-1}}+\mathbb{S}_{\mathbf{e}_{j+1}}+\cdots+\mathbb{S}_{\mathbf{e}_m}$. Since the spaces $\mathbb{S}_{\mathbf{e}_j}$ are defined on the standard ordered basis, the projector matrix π_j is given by

$$\begin{aligned}\{\pi_j\}_{k,m} &= 1 \quad \text{if } k=m=j \\ &= 0 \quad \text{otherwise.}\end{aligned} \tag{2.33}$$

Thus π_j has all the elements as zeros except element in the j^{th} row and j^{th} column as 1.

2.5.2 Construction of matrices B and C

All the eigen values of the upper triangular $\mathbf{A}$ are assumed to be inside unit circle. Lower triangular $\mathbf{A}^*$ is selected such that all it's eigen values are also inside the unit circle leading to causal stable synthesis polyphase matrix $\mathcal{R}(z)$, and $rank(\mathbf{A}-\mathbf{A}^*)$ lies in the bounds specified by theorem (1). If matrices $\mathbf{A}$, $\mathbf{A}^*$ are known, matrices $\mathbf{B}$ and $\mathbf{C}$ are constructed using the theorem (2) as follows,

$$\mathbf{B} = \mathbf{U_A}\mathfrak{B}$$
$$\mathbf{C} = \boldsymbol{\Upsilon}_\mathbf{A}\mathbf{V}_\mathbf{A}^T$$

where $\mathbf{U_A}$ and $\mathbf{V_A}$ are the unitary matrices obtained by taking the SVD of $\mathbf{A}$ as $\mathbf{A} = \mathbf{U_A}\boldsymbol{\Sigma}_\mathbf{A}\mathbf{V}_\mathbf{A}^T$ and the matrices $\mathfrak{B}$ and $\boldsymbol{\Upsilon}_\mathbf{A}$ are similar to the ones described in theorem (2) and understood from the context.

If $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ are the eigen values (need not be distinct) of $\mathbf{A}$ (here diagonal elements), and $\{\lambda'_1, \lambda'_2, \cdots, \lambda'_m\}$ are the eigen values (need not be distinct) of $\mathbf{A}^*$ then the analysis polyphase matrix can be decomposed as

$$\begin{aligned}\mathcal{E}(z) &= \mathbf{D}(\mathbf{I}+\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}) \\ &= \mathbf{D}(\mathbf{I}+\mathbf{C}\pi_1(z\mathbf{I}-\mathbf{A})^{-1}\pi_1\mathbf{B})(\mathbf{I}+\mathbf{C}\pi_2(z\mathbf{I}-\mathbf{A})^{-1}\pi_2\mathbf{B})\cdots(\mathbf{I}+\mathbf{C}\pi_m(z\mathbf{I}-\mathbf{A})^{-1}\pi_m\mathbf{B})\end{aligned}$$

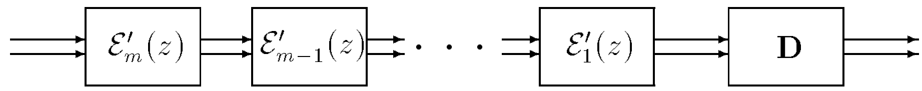


Figure 2.2: Factorization structure of $\mathcal{E}(z)$

The factorization structure for $\mathcal{E}(z)$ is given in figure (2.2). From the structure of projector matrix given in equation (2.33), each factor can be simplified as

$$\begin{aligned}\mathcal{E}'_i(z) &= (\mathbf{I} + \mathbf{C}\pi_i(z\mathbf{I} - \mathbf{A})^{-1}\pi_i\mathbf{B}) \\ &= (\mathbf{I} + \mathbf{c}_i(z - \lambda_i)^{-1}\mathbf{b}_i)\end{aligned}$$

where $\mathbf{c}_i$ is the i^{th} column of matrix $\mathbf{C}$ and $\mathbf{b}_i$ is the i^{th} row of matrix $\mathbf{B}$. The analysis polyphase matrix is then given as

$$\mathcal{E}(z) = \mathbf{D}(\mathbf{I} + \mathbf{c}_1(z - \lambda_1)^{-1}\mathbf{b}_1)(\mathbf{I} + \mathbf{c}_2(z - \lambda_2)^{-1}\mathbf{b}_2)\cdots(\mathbf{I} + \mathbf{c}_m(z - \lambda_m)^{-1}\mathbf{b}_m) \qquad (2.34)$$

The synthesis polyphase matrix $\mathcal{R}(z)$ is decomposed as

$$\begin{aligned}\mathcal{R}(z) &= (\mathbf{I} - \mathbf{C}\pi_1(z\mathbf{I} - \mathbf{A}^*)^{-1}\pi_1\mathbf{B})(\mathbf{I} - \mathbf{C}\pi_2(z\mathbf{I} - \mathbf{A}^*)^{-1}\pi_2\mathbf{B})\cdots(\mathbf{I} - \mathbf{C}\pi_m(z\mathbf{I} - \mathbf{A}^*)^{-1}\pi_m\mathbf{B})\mathbf{D}^{-1} \\ &= (\mathbf{I} - \mathbf{c}_m(z - \lambda'_m)^{-1}\mathbf{b}_m)(\mathbf{I} - \mathbf{c}_{m-1}(z - \lambda'_{m-1})^{-1}\mathbf{b}_{m-1})\cdots(\mathbf{I} - \mathbf{c}_1(z - \lambda'_1)^{-1}\mathbf{b}_1)\mathbf{D}^{-1}\end{aligned} \qquad (2.35)$$

The factors $\mathcal{E}'_i(z)$ and $\mathcal{R}'_i(z)$ are represented as shown in the figures (2.3) and (2.4) respectively.

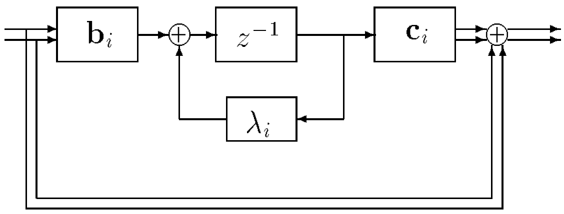


Figure 2.3: Realization of $\mathcal{E}_i'(z)$

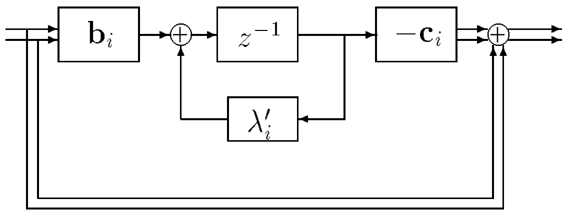


Figure 2.4: Realization of $\mathcal{R}_i'(z)$

2.5.3 Case with diagonal matrix

The matrices $\mathbf{A}$ and $\mathbf{A}^*$ are to be selected such that $rank(\mathbf{A}-\mathbf{A}^*)$ lies within the bounds given by theorem (1) because $(\mathbf{A}-\mathbf{A}^*)$ has to be written as the product of two full rank matrices $\mathbf{B}$ and $\mathbf{C}$. There exist some pairs of $\{\mathbf{A},\mathbf{A}^*\}$ which satisfy the above condition, but there is no easy way to characterize all such pairs. One possible case is assuming one of the matrices $\mathbf{A}$ or $\mathbf{A}^*$ to be diagonal, such that the $rank(\mathbf{A}-\mathbf{A}^*)$ may be easily found.

2.5.4 Case with triangular matrix

If $\mathbf{A}^*$ is taken such that $\lambda_i = \lambda_i' \;\; \forall i = 1, 2, \cdots, m$, then

$$\mathcal{E}_i'(z) = \mathbf{I} + \mathbf{c}_i(z-\lambda_i)^{-1}\mathbf{b}_i$$
$$\mathcal{R}_i'(z) = \mathbf{I} - \mathbf{c}_i(z-\lambda_i)^{-1}\mathbf{b}_i$$

i.e. each factor in the factorization of $\mathcal{R}(z)$ is just a sign changed version of $\mathcal{E}(z)$. The diagonal elements in the difference matrix $(\mathbf{A}-\mathbf{A}^*)$ are zero, which implies $\mathbf{b}_i\mathbf{c}_i = 0 \;\; \forall i = 1, 2, \cdots, m$. So, the i^{th} row of matrix $\mathbf{B}$ is orthogonal to i^{th} column of matrix $\mathbf{C}$. Thus if $\mathbf{A}$ is upper triangular with the structure shown in the equation (2.32) and $\mathbf{A}^*$ is a lower triangular matrix with the following structure

$$\mathbf{A}^* = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ l_2^1 & \lambda_2 & 0 & \cdots & 0 \\ l_3^1 & l_3^2 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_m^1 & l_m^2 & l_m^3 & \cdots & \lambda_m \end{bmatrix}$$

then matrix $\mathbf{BC} = \mathbf{A} - \mathbf{A}^*$ has all the diagonal elements as zero and the elements

$$\begin{aligned} \mathbf{b}_i \mathbf{c}_j &= k_i^j \quad \forall j > i \\ &= 0 \quad \forall j = i \\ &= -l_i^j \quad \forall j < i \end{aligned}$$

So, full rank matrices $\mathbf{B}$ and $\mathbf{C}$ with $\mathbf{b}_i \mathbf{c}_i = 0$ and $\|\lambda_i\| < 1 \ \ \forall i = 1, 2, \cdots, m$ is enough to design this special case. The design method is as follows.

2.5.4.1 Choice of matrices B and C

In the present design $\mathbf{B}$ and $\mathbf{C}$ are full rank for minimality of $\mathcal{E}(z)$. Condition $\mathbf{b}_i \mathbf{c}_i = 0$ implies $\mathbf{b}_i$ (i^{th} row of $\mathbf{B}$) is in the space which is orthogonal complement to $\mathbf{c}_i$. If a full rank matrix $\mathbf{C}$ is assumed and $\mathbf{b}_i$'s are constructed with the above condition, it is not guaranteed that the constructed matrix $\mathbf{B}$ will be full rank, and there is always a possibility for $\mathbf{B}$ to be rank deficient. If an $M \times M$ matrix $\mathbf{C}_\alpha = [\mathbf{c}_1, \cdots, \mathbf{c}_m, \cdots, \mathbf{c}_M]$ is assumed to be orthogonal, and it's first m columns are taken as matrix $\mathbf{C}$, then $\mathbf{b}_i$ will be in span of $\mathbf{c}_j^T$'s for $i \neq j \ \ \forall\ i = 1, \cdots, m$ and $j = 1, \cdots, M$, i.e. $\mathbf{b}_i = \sum_{j=1}^{m}{}^{j \neq i} \alpha_i^j \mathbf{c}_j^T$, where

α_i^j is a constant. Then matrix $\mathbf{B}$ has the following structure.

$$\mathbf{B} = \underbrace{\begin{bmatrix} 0 & \alpha_1^2 & \alpha_1^3 & \cdots & \alpha_1^m & \alpha_1^{m+1} & \cdots & \alpha_1^M \\ \alpha_2^1 & 0 & \alpha_2^3 & \cdots & \alpha_2^m & \alpha_2^{m+1} & \cdots & \alpha_2^M \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_m^1 & \alpha_m^2 & \alpha_m^3 & \cdots & 0 & \alpha_m^{m+1} & \cdots & \alpha_m^M \end{bmatrix}}_{\mathbf{B}_\alpha} \mathbf{C}_\alpha^T \tag{2.36}$$

$rank(\mathbf{B}) = rank(\mathbf{B}_\alpha \mathbf{C}_\alpha^T) = rank(\mathbf{C}_\alpha^T) - dim(\mathcal{N}(\mathbf{B}_\alpha) \cap \mathcal{R}(\mathbf{C}_\alpha^T))$. Since $\mathbf{C}_\alpha^T$ is an orthogonal matrix $rank(\mathbf{C}_\alpha^T) = M$. So, $rank(\mathbf{B}) = M - (M - rank(\mathbf{B}_\alpha)) = rank(\mathbf{B}_\alpha)$. If $\mathbf{B}_\alpha$ is full rank, then $\mathbf{B}$ is also full rank. Characterizing $\mathbf{B}_\alpha$ to be full rank is difficult, and in simulations it was observed that most of the times it ends up to be full rank matrix.

Free variables: For invertible matrix $\mathbf{D}$ in $\mathcal{E}(z)$ $M \times M$ free variables are required. $M(M-1)/2$ are required for orthogonal matrix $\mathbf{C}_\alpha$, and $Mm - m$ are for $\mathbf{B}_\alpha$, and m parameters are required for the diagonal elements of matrix $\mathbf{A}$. When the number of channels increases the free variables required increases, thus optimization becomes difficult. So, a sequential method of building the cascade structure is discussed. In this method, first a single block is taken and optimized for the given response, then another block is added keeping the previous optimized values unchanged, and so forth.

2.5.4.2 Sequential design

When the parameter space increases it is highly unlikely that a good filter bank can be designed. So, it is better to first optimize a small parameter space, and then to optimize the remaining keeping the previous ones constant. This type of procedure is adopted for design of IIR filter banks with factorization approach.

In case of factorization approach, the filter bank can be realized as a cascade of blocks. So, first block is optimized for the desired response. Then the next block is cascaded and the parameters for this block are optimized to reduce the error achieved with the previous stage intact. This process is done till the desired number of blocks are added.

In the present design, analysis polyphase matrix $\mathcal{E}(z) = \mathbf{D}\mathcal{E}'(z)$. First matrix $\mathbf{D}$ is optimized, then the first block of $\mathcal{E}'(z)$ are optimized, i.e. vectors $\mathbf{c}_1$, $\mathbf{b}_1$ and λ_1 for the feedback are optimized, such that $\mathbf{b}_1\mathbf{c}_1 = 0$. The design method is as follows.

First the invertible matrix $\mathbf{D}$ is optimized to the desired response, and then for first and subsequent vectors matrix $\mathbf{C}_\alpha$ is constructed, from which m columns are selected for $\mathbf{C}$, and from (2.36) vector $\mathbf{b}_1$ is constructed as

$$\mathbf{b}_1 = \underbrace{\begin{bmatrix} 0 & \alpha_1^2 & \alpha_1^3 \cdots & \alpha_1^m & \cdots & \alpha_1^M \end{bmatrix}}_{\mathbf{B}_\alpha^1} \mathbf{C}_\alpha^T$$

Thus vector $\mathbf{b}_1$ is constructed and the free variables are optimized for desired response. By the above assumption matrix $\mathbf{C}$ is fixed by taking the first m columns of $\mathbf{C}_\alpha$. For next stage $\mathbf{b}_2$ is constructed by appending a row $\begin{bmatrix} \alpha_2^1 & 0 & \alpha_2^3 \cdots & \alpha_2^m & \cdots & \alpha_2^M \end{bmatrix}$ to the matrix $\mathbf{B}_\alpha^1$ such that the composite matrix $\mathbf{B}_\alpha^2$ is full rank (most of the times from the structure of the matrix, the composite matrix becomes full rank), and the feedback variable for this block λ_2 is seen not be equal to the previous one, λ_1. This procedure is repeated till last row is appended to form full rank $\mathbf{B}_\alpha^m$. Thus from the above construction, full rank matrix $\mathbf{B}$ is obtained. Here except for first optimization step, each step requires $M-1$ free variables. Free variables included in the process is $M \times M$ for $\mathbf{D}$, $M(M-1)/2$ for $\mathbf{C}_\alpha$ and $mM - m$ for construction of $\mathbf{B}$, i.e. construction of vector $\mathbf{b}_i$'s where $i = 1, \cdots, m$.

2.6 Simulation results

In this section the simulation results are presented. The cost function used in the optimization is based on passband and stopband shaping. Cost function is given explicitly as:

$$\sum_{i=0}^{M-1} \left(\alpha_p \int_{\text{Passband}_i} \|1 - H_i(w)\|^2 dw + \alpha_s \int_{\text{Stopband}_i} \|H_i(w)\|^2 dw \right)$$

where $H_i(w)$ is the i^{th} analysis filter, and α_p and α_s are appropriately chosen passband and stopband error weights. Matlab constrained optimization routine *fmincon* is used

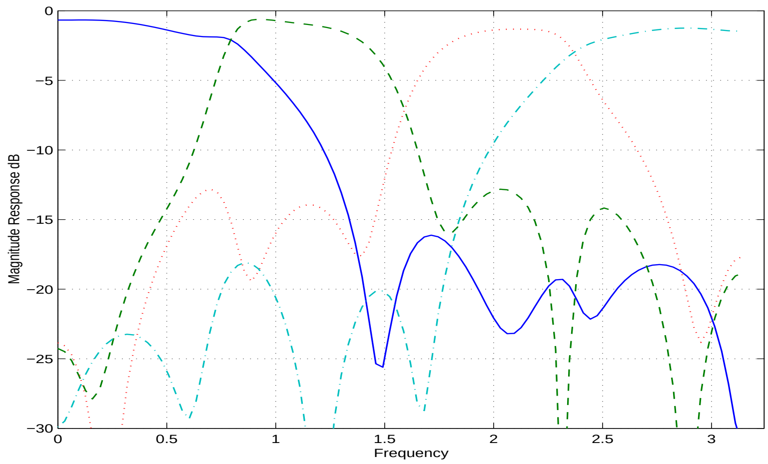


Figure 2.5: 4 channel design based on function of a matrix approach

to constrain the eigen values of $\mathbf{A}$ and $\mathbf{A}^*$ inside unit circle. All the full rank matrices required in the design are generated using SVD. Simulations are done for the first design method based on the concept of function of a matrix discussed in section (2.3.1). Here 4 channel case with $m = 3$ is considered, then the bounds on r are $2 \leq r \leq 3$. Eigen values of $\mathbf{A}$ are assumed to be $\{\lambda_1, \lambda_1, \lambda_2\}$, i.e. two distinct eigen values are assumed with Jordan form

$$\mathcal{J}_{\mathbf{A}} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

A mapping function $f(x) = 2x - \lambda_1$ is taken using Lagrange interpolation method (refer Example -1 in Appendix - A), then

$$f(\mathcal{J}_{\mathbf{A}}) = \begin{bmatrix} \lambda_1 & 2 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & 2\lambda_2 - \lambda_1 \end{bmatrix}.$$

Here $(\mathcal{J}_{\mathbf{A}} - f(\mathcal{J}_{\mathbf{A}}))$ has a rank 2 if $\lambda_2 \neq \lambda_1$, so in optimization it is made sure that $-1 < \lambda_1, \lambda_2 < 1$ and $\lambda_2 \neq \lambda_1$. Matrices $\mathbf{B}$ and $\mathbf{C}$ are constructed as per the method given in section (2.3.1.2). Figure (2.5) shows the optimized filter responses. Both analysis and synthesis filters are of order (15/12).

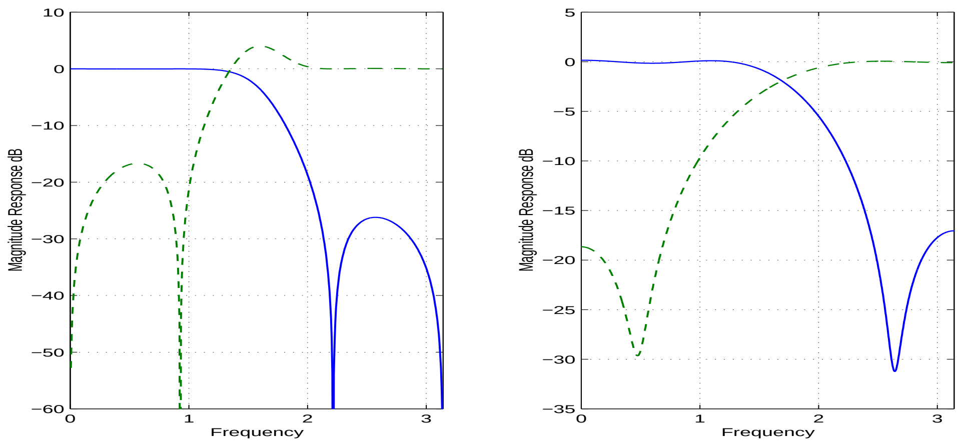


Figure 2.6: Comparison of 2 channel design based on Phoong et al [6] (left) and the proposed similarity transformation approach (right)

Next simulations are done for 2 channel and 3 channel cases for the second design scheme discussed in section (2.3.2) based on similarity transformation method. The case of section (2.3.2.1) where $rank(\mathbf{\Delta_A}) = m$ is considered. $\mathbf{T}$ is assumed to be a circulant matrix and $\mathbf{A}$ is assumed to be a diagonal matrix with distinct real eigen values. For the 2 channel case $m = 2$ is assumed, so for the proposed method low pass and high pass filters are of same order (5/4). Figure (2.6) shows a comparison of 2 channel IIR causal stable FB design by Phoong et al [6] and the proposed method. The orders of low pass and high pass filters are 4/2 and 7/4 respectively for the former design. While there is a 4dB bump in the passband of the high pass filter of [6], for the proposed method the filters have flat passband. Further, the filters designed in [6] are of different orders, while the proposed design leads to equal orders. The number of multipliers and adders required for both the methods are nearly same. For 3 channel case $m = 3$ is assumed and simulations are done. Figure (2.7) shows the analysis filter responses, where the third filter is almost the mirror image of the first one. Both analysis and synthesis filters have order (11/9).

Simulations are done for the factorization method discussed in section (2.5). First the case in section (2.5.3) with diagonal matrix is considered. 3 channel case is considered with $m = 3$. $\mathbf{A}$ is assumed to be upper triangular matrix and $\mathbf{A}^*$ is assumed to be a diagonal matrix with distinct eigen values. For this case the rank bounds are $3 \leq r \leq 3$. Matrices $\mathbf{B}$ and $\mathbf{C}$ are constructed as per the design methods given in section (2.5.2).

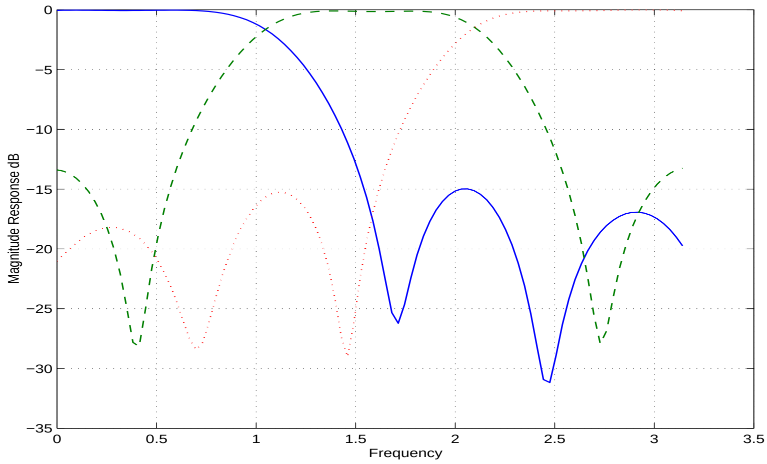


Figure 2.7: 3 channel design based on similarity transformation approach

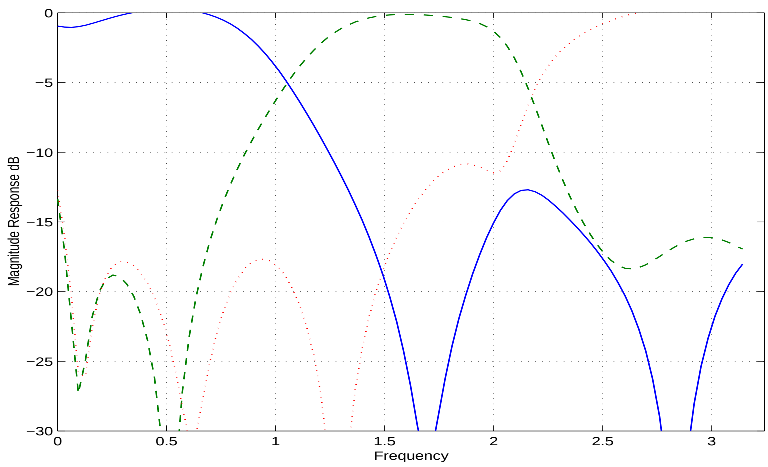


Figure 2.8: 3 channel design based on factorization with diagonal matrix case

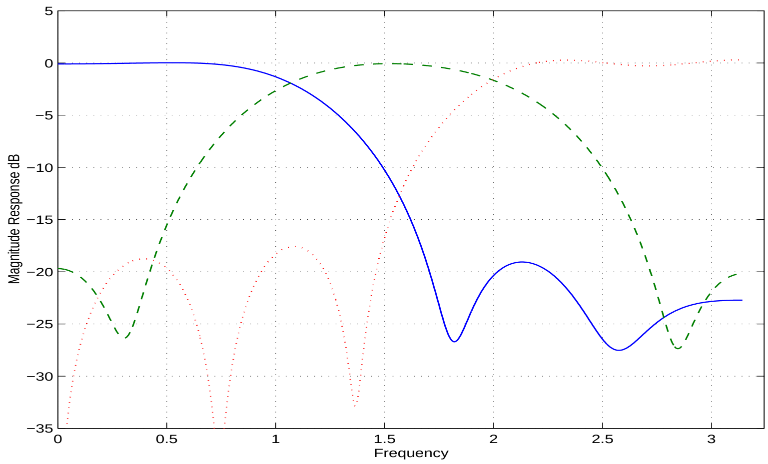


Figure 2.9: 3 channel design based on factorization with triangular matrix case

Figure (2.8) shows the analysis filter responses for this case.

Simulations are done for the case with triangular matrix of section (2.5.4), where the diagonal elements of $\mathbf{A}$ and $\mathbf{A}^*$ are same. 3 channel case with $m = 3$ is assumed. Both analysis and synthesis filters are of order (11/9). Figure (2.9) shows the analysis filter responses for this case. The designed filters have flat pass band and stop band attenuation of -20dB, which is better than the design method given by [5]. To reduce the optimization time, simulations were repeated for 3 channel using sequential method discussed in section (2.5.4.2). Figure (2.10) shows the analysis filter responses during various stages. It can be seen that the filter responses are becoming better with the cascade of more blocks.

To summerize, after doing simulations following observations are made. The optimization time increases with the number of channels and with the number of free variables. 2 channel design is faster than the 3 and 4 channel design. As the number of free variables increases the filter performances are good, so the 3 channel design based on factorization with triangular matrix case gives good filters compared to the 3 channel design based on similarity transformation approach. It was observed in simulations that all the designs are robust to initialization for optimization. With the sequential design approach good filters are obtained at a faster rate of convergence, but at the cost of filter responses, which is evident from the figures (2.10) and (2.9). In the former case all the free variables are optimized at once, but in the later case few variables are optimized keeping others

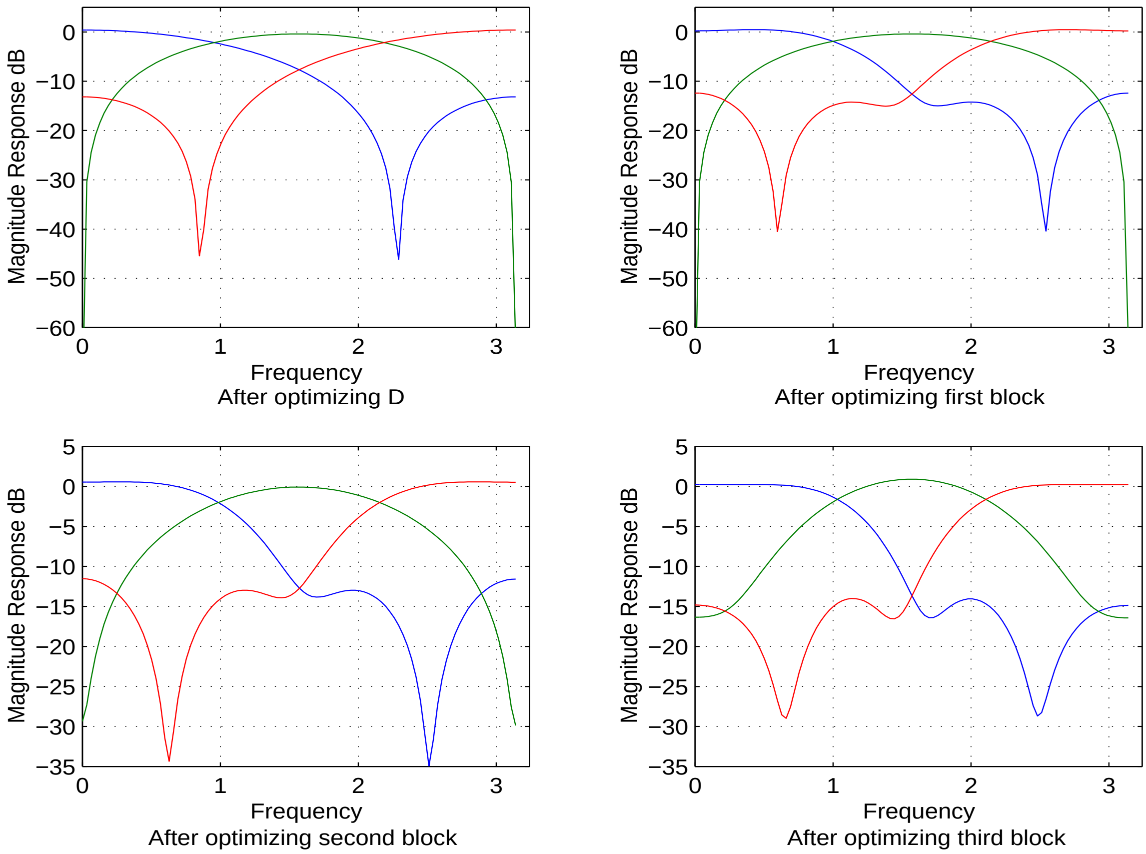


Figure 2.10: 3 channel design based on factorization with triangular matrix case with sequential design

constant. The stop band attenuation is -20 dB for the former case and -15 dB for the later case.

Chapter 3

Design of M channel FIRPRFB

3.1 Introduction

In this chapter some characterizations for FIR bi orthogonal perfect reconstruction filter banks are discussed. If $\mathcal{E}(z)$ and $\mathcal{R}(z)$ are the $M \times M$ analysis and synthesis polyphase matrices, then for perfect reconstruction $\mathcal{R}(z)\mathcal{E}(z) = cz^{-l_0}\mathbf{I}_M$, where c and l_0 are the constants. Given the analysis polyphase matrix, the synthesis polyphase matrix may be expressed as

$$\begin{aligned} \mathcal{R}(z) &= cz^{-l_0}\mathcal{E}^{-1}(z) \\ &= cz^{-l_0}\frac{adj(\mathcal{E}(z))}{det(\mathcal{E}(z))}. \end{aligned}$$

Even if $\mathcal{E}(z)$ is of finite degree, $\mathcal{R}(z)$ can be rational. FIR synthesis polyphase imposes a condition that $det(\mathcal{E}(z)) = cz^{-k}$, which is obvious from the above equation. Since it is known from system theory that any matrix polynomial [51] can be realized in the state space form, $\mathcal{E}(z)$ can be written as

$$\mathcal{E}(z) = \mathbf{D}_M + \mathbf{C}_{M\times m}(z\mathbf{I}_m - \mathbf{A}_{m\times m})^{-1}\mathbf{B}_{m\times M}.$$

For $\mathcal{E}(z)$ to be of finite degree, the state transition matrix $\mathbf{A}$ must be nilpotent [1]. Since $\mathcal{R}(z)$ involves the inversion of a matrix polynomial, explicit inversion formula given $\mathcal{E}(z)$

involves calculation of spectral information of the matrix polynomial $\mathcal{E}(z)$ [48] [58]. For FIR systems with FIR inverse, the synthesis polyphase matrix has a similar state space form with the state transition matrix being nilpotent.

If $\mathbf{E}_0, \mathbf{E}_1, \cdots, \mathbf{E}_l$ are $M \times M$ complex matrices, then the matrix valued function defined on the complex numbers by $\mathcal{E}(\rho) = \sum_{i=0}^{l} \mathbf{E}_i \rho^i$ is called a *matrix polynomial.* Such polynomials are encountered while studying systems of ordinary differential equations with constant coefficient, particularly while studying theory of vibrating systems [54]. The development of the theory of matrix polynomials is due to the significant work by Lancaster [52] on vibrating system problem being analyzed using the theory of matrix polynomials. Quite independent of application of the theory of matrix polynomials to vibration problem, the Russian mathematician M. V. Keldysh viewed the vibration problem as an operator polynomial solution to a class of non self adjoint linear operators and relied heavily on the theory of analytic functions [53]. Subsequently, rapid progress was made following the work by Lancaster, Gohberg and Rodman [54] [55]. A complete book evolved in 1982 [48]. Much of the work discussed in the following sections are from the sections of this book. Another book relating factorizations of matrix polynomials evolved in 1986 by the same authors [56]. Similar kind of work can be seen in linear multi variable control scenario. Significant work by Vardulakis on polynomial matrix dynamics can be seen in [58].

For FIR bi orthogonal filter banks with FIR inverse, the analysis polyphase matrix is treated as a matrix polynomial of finite degree. The synthesis polyphase matrix involves the inversion of the analysis polyphase matrix. Matrix polynomials with non zero determinant are termed as *regular.* In the following sections some theorems concerning the spectrum of matrix polynomials are presented. Explicit inversion formula of a regular matrix polynomial, given the spectral data, is given. The inverse problem of construction of a regular matrix polynomial, given spectral information, becomes the basis for the characterization of FIR bi orthogonal filter banks with FIR inverse, as discussed in this chapter.

3.1.1 Inversion of regular matrix polynomials

In this section the inversion formula for a regular matrix polynomial is given. The problem of inverting a matrix polynomial is a well known one [58] [57], and dealt in the case of finding the solutions of linear matrix differential equations of the form

$$\mathcal{E}(\rho)\mathbf{x}(t) = \mathcal{B}(\rho)\mathbf{u}(t) \tag{3.1}$$

where $\rho = \frac{d}{dt}$, $\mathcal{E}(\rho)$ is an $M \times M$ regular matrix polynomial, $\mathcal{B}(\rho)$ is an $M \times N$ matrix polynomial and $\mathbf{x}(t)$ is a vector valued function to be found out, given $N \times 1$ vector valued input function $\mathbf{u}(t)$. Special case of (3.1) with the from

$$(\rho\mathbf{E} - \mathbf{A})\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t) \tag{3.2}$$

is called *generalized state space system* (GSSS). The solution for GSSS is well known from the work of Gantmacher [59] and Verghese et al [60]. Most of these results are based on Gantmacher's [59] analysis of the canonical form of the *matrix pencil* $\rho\mathbf{E} - \mathbf{A}$, called the *Weierstrass canonical form.* More recently the solution of (3.1) was given by Gohberg *et al.* [48] and Vardulakis [58]. The solution of the above equation involves the inversion of the matrix polynomial $\mathcal{E}(\rho)$. It was shown in [48] [58], that solution space of (3.1) is the sum of smooth and impulsive parts which are controlled by the spectrum of the matrix polynomial. Mainly the solution at $t = 0$, impulsive behavior, is controlled by the *zeros at infinity* of $\mathcal{E}(\rho)$.

3.1.2 Jordan chains and solutions of matrix differential equations

If $\mathcal{E}(\rho) = \mathbf{E}_0 + \mathbf{E}_1\rho + \cdots + \mathbf{E}_l\rho^l$ is a regular matrix polynomial, and if λ_0 is a zero of $\mathcal{E}(\rho)$, i.e. the root of $det(\mathcal{E}(\rho))$, then the homogeneous solution is [58] [48] [53]

$$\mathbf{x}(t) = \left[\frac{t^{k_0-1}}{(k_0-1)!}\mathbf{x}_0 + \frac{t^{k_0-2}}{(k_0-2)!}\mathbf{x}_1 + \cdots + \frac{t}{1!}\mathbf{x}_{k_0-2} + \mathbf{x}_{k_0-1}\right] e^{\lambda_0 t}$$

and $\mathbf{x}_0 \neq 0$ satisfies equation (3.1), iff the condition

$$\sum_{j=0}^{i} \frac{1}{j!}\mathcal{E}^{(j)}(\lambda_0)\mathbf{x}_{i-j} = 0, \qquad i = 0, 1, \cdots, k_0 - 1, \tag{3.3}$$

is satisfied. The sequence $\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k_0-1}$ for which the equalities in (3.3) hold is called a *Jordan chain of length* k_0 for $\mathcal{E}(\rho)$ corresponding to the number λ_0. It can be seen from (3.3) that $\mathbf{x}_0 \in ker(\mathcal{E}(\lambda_0))$. The length of the Jordan chain k_0 depends upon the multiplicity of the eigen value λ_0 in $det(\mathcal{E}(\rho))$ and on the $dim(ker(\mathcal{E}(\lambda_0)))$. The information regarding the Jordan chains and the eigen vectors of $\mathcal{E}(\rho)$ is called the *spectral data* of $\mathcal{E}(\rho)$. The vector $\mathbf{x}_0$ is called *eigen vector* corresponding to the eigen value λ_0, and the vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{k_0-1}$ are called generalized eigen vectors. These may not be linearly independent. If the above vectors in the Jordan chain satisfy equation (3.3), then $\Psi(t) = [\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k_0-1}]e^{\mathbf{J}_{\lambda_0}t}$ satisfies equation (3.1), where $\mathbf{J}_{\lambda_0}$ is the $k_0 \times k_0$ Jordan block, corresponding to the eigen value λ_0. If $dim(ker(\mathcal{E}(\lambda_0))) = r_0$ then there exits r_0 eigen vectors, and there may exist generalized eigen vectors for each eigen vector. If the multiplicity of λ_0 in $det(\mathcal{E}(\lambda_0))$ is k_0 then there exists r_0 Jordan blocks of sizes $k_0^1, k_0^2, \cdots, k_0^{r_0}$ such that $\sum_{j=1}^{r_0} k_0^j = k_0$. Corresponding to eigen value λ_0 there exists vectors

$$\mathbf{X}(\lambda_0) = [\mathbf{x}_{\lambda_0}{}_1^0, \cdots, \mathbf{x}_{\lambda_0}{}_1^{k_0^1-1}, \quad \mathbf{x}_{\lambda_0}{}_2^0, \cdots, \mathbf{x}_{\lambda_0}{}_2^{k_0^2-1}, \quad \cdots, \quad \mathbf{x}_{\lambda_0}{}_{r_0}^0, \cdots, \mathbf{x}_{\lambda_0}{}_{r_0}^{k_0^{r_0}-1}]$$

and the matrix $\mathbf{J}_{\lambda_0} = diag(\mathbf{J}_{\lambda_0}^1, \cdots, \mathbf{J}_{\lambda_0}^{r_0})$. Here the vectors $\mathbf{x}_{\lambda_0}{}_1^0, \mathbf{x}_{\lambda_0}{}_2^0, \cdots, \mathbf{x}_{\lambda_0}{}_{r_0}^0$ are the eigen vectors and are linearly independent. These chains of vectors which satisfy equation (3.3) are called the canonical set of Jordan chains.

Here $\mathbf{X}(\lambda_0)$ is an $M \times r_0$ matrix and $\mathbf{J}_{\lambda_0}$ is a $r_0 \times r_0$ matrix. The pair of matrices $(\mathbf{X}(\lambda_0), \mathbf{J}_{\lambda_0})$ is called the *Jordan pair* of $\mathcal{E}(\rho)$ corresponding to the eigen value λ_0. The following statement [58] [48] holds good for the Jordan pair.

Let $(\mathbf{X}, \mathbf{J})$ be a pair of matrices, where $\mathbf{X}$ is an $M \times \mu$ matrix and $\mathbf{J}$ is an $\mu \times \mu$ matrix with unique eigen value λ_0. Then the following conditions are necessary and sufficient in order that $(\mathbf{X}, \mathbf{J})$ be a Jordan pair of $\mathcal{E}(\rho)$ corresponding to λ_0.

- $det(\mathcal{E}(\rho))$ has a zero of multiplicity μ
- $rank\ col(\mathbf{X}\mathbf{J}^j)_{j=0}^{l-1} = \mu$
- $\mathbf{E}_l\mathbf{X}\mathbf{J}^l + \mathbf{E}_{l-1}\mathbf{X}\mathbf{J}^{l-1} + \cdots + \mathbf{E}_0\mathbf{X} = 0$

Taking the Jordan pair $(\mathbf{X}(\lambda_i), \mathbf{J}_{\lambda_i})$ for every eigen value λ_i of $\mathcal{E}(\rho)$, and defining a finite Jordan pair $(\mathbf{X}_f, \mathcal{J}_f)$ of $\mathcal{E}(\rho)$ as $\mathbf{X}_f = [\mathbf{X}(\lambda_0), \mathbf{X}(\lambda_1), \cdots, \mathbf{X}(\lambda_p)]$, $\mathcal{J}_f = diag(\mathbf{J}_{\lambda_0}, \mathbf{J}_{\lambda_1}, \cdots, \mathbf{J}_{\lambda_p})$, it is found out that $\mathbf{X}_f$ is $M \times \mu_f$ and $\mathcal{J}_f$ is $\mu_f \times \mu_f$ matrices, where μ_f is the degree of $det(\mathcal{E}(\rho))$ (Chrystal's theorem [54]). Having defined the finite Jordan pair, the $M \times \mu_f$ matrix valued function $\Psi(t) = \mathbf{X}_f e^{\mathcal{J}_f t}$ is called a basis matrix for the solution space of the homogeneous matrix differential equation (3.1) [58]. Here the finite Jordan pair satisfies the conditions given for a Jordan pair for λ_0. The solution space obtained by the finite Jordan pair is called the smooth solution space. It was showed in [58] and [48] that the smooth solution space of equation (3.1) is invariant under the unimodular modular transformations of the matrix polynomial $\mathcal{E}(\rho)$. Since a unimodular modular matrix polynomial (say $\mathcal{U}(\rho)$) has constant determinant, smooth solution space due to $\mathcal{U}(\rho)\mathcal{E}(\rho)$ and $\mathcal{E}(\rho)$ are same. In order to find out the unique solution space by $\mathcal{E}(\rho)$ the concept of spectrum at infinity is introduced. Literature on the structure of spectrum at infinity of a matrix polynomial can be found in [51] [58]. It was shown in [58] that spectrum at infinity affects the behavior of impulsive solution space of a matrix polynomial.

In order to find the infinite spectrum, the dual polynomial matrix $\tilde{\mathcal{E}}(\rho) = \rho^l\mathcal{E}(1/\rho)$ is defined. By definition [48], a Jordan chain of matrix valued function $\mathcal{E}(\rho)$ at infinity is just a Jordan chain of the matrix valued function $\mathcal{E}(\rho^{-1})$ at zero. Then an additional Jordan pair $(\mathbf{X}_\infty, \mathcal{J}_\infty)$ of $\mathcal{E}(\rho)$ is defined as

$$\begin{aligned}\mathbf{X}_\infty &= [\mathbf{y}_0^{(1)}, \cdots, \mathbf{y}_{s_1-1}^{(1)}, \mathbf{y}_0^{(2)}, \cdots, \mathbf{y}_{s_2-1}^{(2)} \quad \cdots \quad \mathbf{y}_0^{(q)}, \cdots, \mathbf{y}_{s_q-1}^{(q)}] \\ \mathcal{J}_\infty &= diag[\mathbf{J}_{\infty 1}, \mathbf{J}_{\infty 2}, \cdots, \mathbf{J}_{\infty q}]\end{aligned}$$

where $\mathcal{J}_{\infty j}$ is the Jordan block of size s_j with eigenvalues zero and $\mathbf{y}_j^{(i)}$ are the eigenvectors and generalized eigen vectors of $\tilde{\mathcal{E}}(\rho)$ for zero eigen value. This Jordan pair is called *infinite Jordan pair*. It should be noted that $(\mathbf{X}_\infty, \mathcal{J}_\infty)$ is the Jordan pair of the matrix polynomial

$\tilde{\mathcal{E}}(\rho) = \rho^l \mathcal{E}(1/\rho)$ corresponding to the eigenvalue $\lambda = 0$. The following conditions must be satisfied by a pair $(\hat{\mathbf{X}}, \hat{\mathbf{J}})$ to be an infinite Jordan pair of $\mathcal{E}(\rho)$.

- $\det(\rho^l \mathcal{E}(1/\rho))$ has a a zero at $\lambda_0 = 0$ of multiplicity μ_∞
- rank $\mathrm{col}(\hat{\mathbf{X}}\hat{\mathbf{J}}^j)_{j=0}^{l-1} = \mu_\infty$
- $\mathbf{E}_0\hat{\mathbf{X}}\hat{\mathbf{J}}^l + \mathbf{E}_1\hat{\mathbf{X}}\hat{\mathbf{J}}^{l-1} + \cdots + \mathbf{E}_l = 0$

Having defined the concepts of finite and infinite Jordan pairs, linearization of matrix polynomials is discussed in the next section.

3.1.3 Linearizations and decomposable pairs

If $\mathcal{E}(\rho)$ is a regular $M \times M$ matrix polynomial, an $Ml \times Ml$ linear matrix polynomial $\mathbf{S}_0 + \mathbf{S}_1\rho$ is called the linearization of $\mathcal{E}(\rho)$ if

$$\begin{bmatrix} \mathcal{E}(\rho) & 0 \\ 0 & \mathbf{I}_{M(l-1)} \end{bmatrix} = \mathcal{U}(\rho)(\mathbf{S}_0 + \mathbf{S}_1\rho)\mathcal{V}(\rho).$$

The structures of $\mathcal{U}(\rho)$ and $\mathcal{V}(\rho)$ are given in [48] [58]. If block companion polynomial $\mathcal{C}_{\mathcal{E}}(\rho)$ of $\mathcal{E}(\rho)$ can be taken as the linearization, then

$$\mathcal{C}_{\mathcal{E}}(\rho) = \begin{bmatrix} \mathbf{I} & 0 & \cdots & & 0 \\ 0 & \mathbf{I} & \cdots & & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{I} & 0 \\ 0 & 0 & \cdots & 0 & \mathbf{E}_l \end{bmatrix} \rho + \begin{bmatrix} 0 & -\mathbf{I} & 0 & \cdots & 0 \\ 0 & 0 & -\mathbf{I} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & \cdots & -\mathbf{I} \\ \mathbf{E}_0 & \mathbf{E}_1 & & \cdots & \mathbf{E}_{l-1} \end{bmatrix}$$

The matrix $\mathcal{C}_{\mathcal{E}}(\rho)$ is a matrix pencil. The main idea of linearization is to find the equivalence of $\mathcal{E}(\rho)$ to a linear matrix polynomial $\mathcal{C}_{\mathcal{E}}(\rho)$. Since matrices $\mathcal{U}(\rho)$ and $\mathcal{V}(\rho)$ are unimodular, $det(\mathcal{E}(\rho)) = det(\mathcal{C}_{\mathcal{E}}(\rho))$, i.e. spectrum is preserved.

Next the concept of decomposable pairs, which generalizes the concept of Jordan pairs, is discussed. A pair of matrices $(\mathbf{X}, \mathbf{T})$ is called *admissible of order* p if $\mathbf{X}$ is $n \times p$ and $\mathbf{T}$

is $p \times p$. The admissible pair $(\mathbf{X}, \mathbf{T})$ of order Ml is called a *decomposable pair* (of degree l) if the following conditions are satisfied:

- $\mathbf{X} = [\mathbf{X}_1 \ \mathbf{X}_2], \ \mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & 0 \\ 0 & \mathbf{T}_2 \end{bmatrix}$ where $\mathbf{X}_1$ is an $M \times \mu$ matrix, and $\mathbf{T}_1$ is a $\mu \times \mu$ matrix, for some μ, $0 \leq \mu \leq Ml$, $\mathbf{X}_2$ and $\mathbf{T}_2$ are of sizes $M \times (Ml - \mu)$ and $(Ml - \mu) \times (Ml - \mu)$ respectively.

- The matrix
$$\mathbf{S}_{l-1} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2\mathbf{T}_2^{l-1} \\ \mathbf{X}_1\mathbf{T}_1 & \mathbf{X}_2\mathbf{T}_2^{l-2} \\ \vdots & \vdots \\ \mathbf{X}_1\mathbf{T}_1^{l-1} & \mathbf{X}_2 \end{bmatrix} \tag{3.4}$$
is nonsingular.

The pair $(\mathbf{X}, \mathbf{T})$ is called the decomposable pair of the regular matrix polynomial $\mathcal{E}(\rho) = \sum_{i=0}^{l} \mathbf{E}_i\rho^i$, if in addition to above conditions the following are satisfied.

- $\sum_{i=0}^{l} \mathbf{E}_i\mathbf{X}_1\mathbf{T}_1^i = 0, \quad \sum_{i=0}^{l} \mathbf{E}_i\mathbf{X}_2\mathbf{T}_2^{l-i} = 0$

It is shown in [48] that if $\mathcal{E}(\rho)$ is a regular matrix polynomial with $(\mathbf{X}_f, \mathcal{J}_f)$ and $(\mathbf{X}_\infty, \mathcal{J}_\infty)$ as the finite and infinite Jordan pairs, respectively, then $([\mathbf{X}_f \ \ \mathbf{X}_\infty], \mathcal{J}_f \oplus \mathcal{J}_\infty)$ is a decomposable pair for $\mathcal{E}(\rho)$. Detailed proof is given in [48]. If $([\mathbf{X}_1 \ \mathbf{X}_2], \mathbf{T}_1 \oplus \mathbf{T}_2)$ be a decomposable pair, let $\mathbf{S}_{l-1}$ is as in (3.4) and let $\mathbf{S}_{l-2}$ =col$(\mathbf{X}_1\mathbf{T}_1^i, \mathbf{X}_2\mathbf{T}_2^{l-2-i})_{i=0}^{l-2}$, then the following conditions are true.

- $\mathbf{S}_{l-2}$ has full rank, with rank $M(l-1)$.

- The $Ml \times Ml$ matrix $\mathbf{P} = (\mathbf{I} \oplus \mathbf{T}_2)\mathbf{S}_{l-1}^{-1} \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix} \mathbf{S}_{l-2}$ is a projector matrix with $ker(\mathbf{P}) = ker(\mathbf{S}_{l-2})$.

3.1.4 Decomposable linearization and a resolvent form

After discussing the concepts of Linearizations and decomposable pairs, their role in finding the structure of inverse of a regular matrix polynomial is discussed with the following propositions.

Proposition 3 *Let $\mathcal{E}(\rho)$ be a regular matrix polynomial, and let $([\mathbf{X}_1\mathbf{X}_2], \mathbf{T}_1 \oplus \mathbf{T}_2)$ be its decomposable pair. Then $\mathcal{T}(\rho) = (\mathbf{I}\rho - \mathbf{T}_1) \oplus (\mathbf{T}_2\rho - \mathbf{I})$ is a linearization of $\mathcal{E}(\rho)$ and*

$$\mathcal{C}_{\mathcal{E}}(\rho)\mathbf{S}_{l-1} = \begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{V} \end{bmatrix} \mathcal{T}(\rho) \tag{3.5}$$

where $\mathbf{V} = [\mathbf{E}_1\mathbf{X}_1\mathbf{T}_1^{l-1}, \ -\sum_{i=0}^{l-1}\mathbf{E}_i\mathbf{X}_2\mathbf{T}_2^{l-1-i}]$ and $\mathcal{C}_{\mathcal{E}}(\rho)$ is the companion polynomial discussed earlier.

Using the above decomposable linearization theorem, resolvent form of the inverse for a regular matrix polynomial is given by the following proposition.

Proposition 4 *Let $\mathcal{E}(\rho)$ be a matrix polynomial with decomposable pair $([\mathbf{X}_1 \ \mathbf{X}_2], \mathbf{T}_1 \oplus \mathbf{T}_2)$ and corresponding decomposable linearization $\mathcal{T}(\rho)$. If $\mathbf{V} = [\mathbf{E}_1\mathbf{X}_1\mathbf{T}_1^{l-1}, \ -\sum_{i=0}^{l-1}\mathbf{E}_i\mathbf{X}_2\mathbf{T}_2^{l-1-i}]$, $\mathbf{S}_{l-2} = col(\mathbf{X}_1\mathbf{T}_1^i, \mathbf{X}_2\mathbf{T}_2^{l-2-i})_{i=0}^{l-2}$ and $Z = [\mathbf{I} \oplus \mathbf{T}_2^{l-1}] \begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{V} \end{bmatrix}^{-1} [0 \ \cdots \ 0 \ \mathbf{I}]^T$, then inverse of $\mathcal{E}(\rho)$ is given by,*

$$\mathcal{E}^{-1}(\rho) = [\mathbf{X}_1 \ \mathbf{X}_2]\mathcal{T}^{-1}(\rho)Z. \tag{3.6}$$

Proofs of above propositions are given in [48], and the second proof can also be found in [58]. The decomposable linearization given in the above proposition is the generalization of *Kronecker canonical form or Weierstrass canonical form of regular matrix pencils* [59] [51].

Proposition 5 *Every regular pencil $(\rho\mathbf{E} - \mathbf{A})_{N\times N}$ can be reduced to a canonical quasi-diagonal form*

$$(\rho\mathbf{E} - \mathbf{A}) = \mathbf{P}_K[(\rho\mathbf{I} - \mathcal{N}_f) \oplus (\mathbf{I} - \mathcal{N}_\infty\rho)]\mathbf{Q}_K \tag{3.7}$$

where $(N-K)\times(N-K)$ matrix $\mathcal{N}_\infty$ corresponds to infinite elementary divisors and $K \times K$ matrix $\mathcal{N}_f$ is uniquely determined by the finite elementary divisors of the given pencil.

$\mathbf{P}_K$ and $\mathbf{Q}_K$ are matrices with constant entries. Detailed information regarding elementary divisors is given in [59]. In case of Kronecker canonical form the matrices $\mathcal{N}_f$ and $\mathcal{N}_\infty$ are the matrices corresponding to finite and infinite spectral data, i.e. $\mathcal{J}_f$ and $\mathcal{J}_\infty$ respectively.

3.1.5 "Inverse problem"— construction from decomposable pair

Given a regular matrix polynomial $\mathcal{E}(\rho)$, the spectral data, i.e. the Jordan pairs can be found out and the inversion formula is given in equation (3.6). But the inverse problem, i.e. given the spectral data, construction of the a regular matrix polynomial is the issue of current interest. If $(\mathbf{X}, \mathbf{T})$ is the decomposable pair (of degree l), then the inverse problem is to find all regular matrix polynomials of degree l with $(\mathbf{X}, \mathbf{T})$ as the decomposable pair. Such regular matrix polynomial is constructed with $(\mathbf{X}, \mathbf{T})$ as the decomposable pair from the following proposition [48].

Proposition 6 *Let* $(\mathbf{X}, \mathbf{T}) = ([\mathbf{X}_1\mathbf{X}_2], \mathbf{T}_1 \oplus \mathbf{T}_2)$ *be a decomposable pair of degree* l*, and let* $\mathbf{S}_{l-2} = col(\mathbf{X}_1{\mathbf{T}_1}^i, \mathbf{X}_2{\mathbf{T}_2}^{l-2-i})_{i=0}^{l-2}$*. Then for every* $M \times Ml$ *matrix* $\mathbf{V}$ *such that* $\begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{V} \end{bmatrix}$ *is nonsingular, the matrix polynomial*

$$\mathcal{E}(\rho) = \mathbf{V}(\mathbf{I} - \mathbf{P})[(\mathbf{I}\rho - \mathbf{T}_1) \oplus (\mathbf{T}_2\rho - \mathbf{I})](\mathbf{U}_0 + \mathbf{U}_1\rho + \cdots + \mathbf{U}_{l-1}\rho^{l-1}) \tag{3.8}$$

has $(\mathbf{X}, \mathbf{T})$ *as the decomposable pair, where* $\mathbf{P} = (\mathbf{I} \oplus \mathbf{T}_2)[col(\mathbf{X}_1{\mathbf{T}_1}^i, \mathbf{X}_2{\mathbf{T}_2}^{l-1-i})_{i=0}^{l-1}]^{-1} \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix} \mathbf{S}_{l-2}$ *and* $[\mathbf{U}_0\ \mathbf{U}_1\ \cdots\ \mathbf{U}_{l-1}] = [col(\mathbf{X}_1{\mathbf{T}_1}^i, \mathbf{X}_2{\mathbf{T}_2}^{l-1-i})_{i=0}^{l-1}]^{-1}$.

Conversely, if $\mathcal{E}(\rho) = \sum_{i=0}^{l} \mathbf{E}_i\rho^i$ *has* $(\mathbf{X}, \mathbf{T})$ *as its decomposable pair, then* $\mathcal{E}(\rho)$ *admits representation (3.8) with*

$$\mathbf{V} = \mathbf{V}(\mathbf{I} - \mathbf{P}) = \left[\mathbf{E}_l\mathbf{X}_1\mathbf{T}_1^{l-1}, \ -\sum_{i=0}^{l-1} \mathbf{E}_i\mathbf{X}_2\mathbf{T}_2^{l-1-i} \right]. \tag{3.9}$$

3.2 Proposed FIRPRFB design

The $M \times M$ analysis polyphase matrix $\mathcal{E}(z)$ of degree l is given by

$$\mathcal{E}(z) = \sum_{i=0}^{l} \mathbf{E}_i z^{-i}.$$

Here ρ, the variable used in the previous sections, is replaced by z^{-1}. For FIRPRFBs with FIR inverse $det(\mathcal{E}(z)) = z^{-k}$, where $\mathcal{E}(z)$ is $M \times M$ polyphase matrix [1]. All the eigenvalues of $\mathcal{E}(z)$ are zero, i.e. the Jordan matrix corresponding to finite spectrum ($\mathcal{J}_f$) is nilpotent. It was shown in the earlier discussion that the finite and infinite Jordan pairs are going to define a matrix polynomial uniquely. The Jordan chains are highly nonunique, but the number of eigen vectors are decided by $dim(ker(\mathcal{E}(0)))$. The multiplicities of the eigen value (zero in this case) will decide the matrix polynomial uniquely. If $\mathcal{E}(z)$ is of degree l, and $det(\mathcal{E}(z)) = z^{-k}$, then $\mathbf{X}_f$ and $\mathcal{J}_f$ are $Ml \times k$ are $k \times k$ matrices. Now it will be shown that $det(\tilde{\mathcal{E}}(z)) = z^{-(Ml-k)}$, where $\tilde{\mathcal{E}}(z)$ is the dual polynomial of $\mathcal{E}(z)$.

Theorem 5 *For FIR $\mathcal{E}(z)$ with FIR inverse, if degree of $det(\mathcal{E}(z)) = k$ then for the dual polynomial $\tilde{\mathcal{E}}(z)$ the degree of $det(\tilde{\mathcal{E}}(z)) = Ml - k$.*

Proof: From the concept of linearization of matrix polynomials discussed in section (3.1.3)

$$det(\mathcal{E}(z)) = det(\mathbf{A} + \mathbf{B}z^{-1})$$

where structures of $\mathbf{A}$ and $\mathbf{B}$ are discussed in section (3.1.3). From the Kronecker canonical form of matrix pencils discussed in proposition (5), the pencil matrix $\mathbf{A} + \mathbf{B}z^{-1}$ can be written as

$$\mathbf{A} + \mathbf{B}z^{-1} = \mathbf{P}_K \begin{bmatrix} \mathbf{I}z^{-1} - \mathcal{N}_f & 0 \\ 0 & \mathbf{I} - z^{-1}\mathcal{N}_\infty \end{bmatrix} \mathbf{Q}_K$$

where $\mathcal{N}_f$ and $\mathcal{N}_\infty$ are Jordan nilpotent matrices because the matrix polynomial is FIR. If $\mathcal{N}$ is a Jordan nilpotent matrix, $det(\mathbf{I} - z^{-1}\mathcal{N})$ is 1 because $\mathcal{N}$ is upper triangular with diagonal elements equal to zero. Therefore,

$$\begin{aligned} det(\mathbf{A}+\mathbf{B}z^{-1}) &= c\,det(\mathbf{I}z^{-1}-\mathcal{N}_f)det(\mathbf{I}-z^{-1}\mathcal{N}_\infty) \\ &= c\,det(\mathbf{I}z^{-1}-\mathcal{N}_f) \\ &= cz^{-k} \end{aligned}$$

where $c = det(\mathbf{P}_K\mathbf{Q}_K)$. For the dual polynomial $\tilde{\mathcal{E}}(z)$,

$$\begin{aligned} det(\tilde{\mathcal{E}}(z)) &= det(z^{-l}\mathcal{E}(z^{-1})) \\ &= det(z^{-l}\mathbf{I}\mathcal{E}(z^{-1})) \\ &= det(z^{-l}\mathbf{I})det(\mathcal{E}(z^{-1})) \\ &= det(z^{-1}\mathbf{I}_{Ml})det(\mathbf{A}+\mathbf{B}z) \\ &= det(z^{-1}(\mathbf{A}+\mathbf{B}z)) \\ &= c\,det(\mathbf{I}z^{-1}-\mathcal{N}_\infty) \\ &= cz^{-(Ml-k)}.\square \end{aligned}$$

Since the finite spectral data of $\tilde{\mathcal{E}}(z)$ is same as the infinite spectral data of $\mathcal{E}(z)$, $\mathcal{J}_\infty$ and $\mathbf{X}_\infty$ are $(Ml-k)\times(Ml-k)$ and $Ml\times(Ml-k)$ matrices. So, given the Jordan matrices $\mathcal{J}_f$ and $\mathcal{J}_\infty$ and if the matrices $\mathbf{X}_f$ and $\mathbf{X}_\infty$ are constructed such that $\mathbf{S}_{l-1}$ is invertible, then pair $(\mathbf{X},\mathcal{J}) = ([\mathbf{X}_f\ \mathbf{X}_\infty], \mathcal{J}_f\oplus\mathcal{J}_\infty)$ satisfies the conditions of a decomposable pair (of degree l). Using the above proposition (6), a regular matrix polynomial can be constructed with $(\mathbf{X},\mathcal{J})$ as the decomposable pair of degree l. Now it will be shown that rows of the matrix $\mathbf{V}$ used in the above proposition (6) span the null space of $\mathbf{S}_{l-2}$.

Theorem 6 *The rows of the matrix* $\mathbf{V}$ *used in the construction of matrix polynomial* $\mathcal{E}(z)$ *in (3.8), span the null space of* $\mathbf{S}_{l-2}$.

Proof: From proposition (6) matrix $\mathbf{V}$ is selected such that the composite matrix $\begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{V} \end{bmatrix}$ is invertible. If $\mathbf{X}_f$ and $\mathbf{X}_\infty$ are taken such that $([\mathbf{X}_f\ \mathbf{X}_\infty], \mathcal{J}_f\oplus\mathcal{J}_\infty)$ is the decomposable pair of a regular matrix polynomial $\mathcal{E}(z)$, i.e. then matrix $\mathbf{S}_{l-1}$ is invertible, then it was shown from the additional properties of a decomposable pair that $M(l-1) \times Ml$ matrix, $\mathbf{S}_{l-2}$ is full rank (with rank $M(l-1)$). So, $\mathbf{V}$ must be an $M \times Ml$ full rank matrix for the composite matrix to be invertible. If the rows of matrix $\mathbf{S}_{l-2}$ span a $M(l-1)$ dimensional space, say $\mathbb{W}_1$, rows of $\mathbf{S}_{l-2}$ are taken as the basis vectors of $\mathbb{W}_1$. Since each row of $\mathbf{S}_{l-2}$ is $1 \times Ml$, $\mathbb{W}_1$ can be taken as the $M(l-1)$ dimensional subspace of the Ml dimensional vector space $\mathbb{V}$. The remaining M basis vectors of the space $\mathbb{V}$ are taken from $ker(\mathbf{S}_{l-2})$, which is M dimensional and orthogonal complement to $\mathbb{W}_1$. If $\mathbf{N}_{\mathbf{S}_{l-2}}$ represents the matrix whose rows are the basis of $ker(\mathbf{S}_{l-2})$, rows spanning $\mathbf{V}$ may be taken as linear combinations of the basis of $\mathbb{W}_1$ and $ker(\mathbf{S}_{l-2})$, i.e.

$$\mathbf{V} = [\mathbf{A}_1 \ \mathbf{A}_2] \begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{N}_{\mathbf{S}_{l-2}} \end{bmatrix}. \tag{3.10}$$

From the concept of rank of product of matrices [59],

$$rank(\mathbf{AB}) = rank(\mathbf{B}) - dim(ker(\mathbf{A}) \cap range(\mathbf{B})). \tag{3.11}$$

So, applying the above rank equality to (3.10) we have

$$rank(\mathbf{V}) = rank\left(\begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{N}_{\mathbf{S}_{l-2}} \end{bmatrix}\right) - dim\left(ker([\mathbf{A}_1 \ \mathbf{A}_2]) \cap range\left(\begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{N}_{\mathbf{S}_{l-2}} \end{bmatrix}\right)\right).$$

Since $rank\left(\begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{N}_{\mathbf{S}_{l-2}} \end{bmatrix}\right) = Ml$, $range\left(\begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{N}_{\mathbf{S}_{l-2}} \end{bmatrix}\right) = \mathbb{V}$, and $\mathbf{V}$ being full rank we have,

$$dim(ker([\mathbf{A}_1 \ \mathbf{A}_2])) \cap \mathbb{V}) = Ml - M$$

which happens only when $[\mathbf{A}_1 \ \mathbf{A}_2]$ is full rank.

In the expression for the construction of matrix polynomial from equation (3.8), the term $\mathbf{V}(\mathbf{I}-\mathbf{P})$ can be written as:

$$\begin{aligned}
\mathbf{V}(\mathbf{I}-\mathbf{P}) &= (\mathbf{A}_1\mathbf{S}_{l-2}+\mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}})(\mathbf{I}-\mathbf{P}) \\
&= \mathbf{A}_1\mathbf{S}_{l-2}(\mathbf{I}-\mathbf{P})+\mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}(\mathbf{I}-\mathbf{P}) \\
&= \mathbf{A}_1\mathbf{S}_{l-2}\left(\mathbf{I}-(\mathbf{I}\oplus\mathcal{J}_\infty)\mathbf{S}_{l-1}^{-1}\begin{bmatrix}\mathbf{I}\\0\end{bmatrix}\mathbf{S}_{l-2}\right)+\mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}(\mathbf{I}-\mathbf{P}) \\
&= \mathbf{A}_1\left(\mathbf{S}_{l-2}-\mathbf{S}_{l-2}(\mathbf{I}\oplus\mathcal{J}_\infty)\mathbf{S}_{l-1}^{-1}\begin{bmatrix}\mathbf{I}\\0\end{bmatrix}\mathbf{S}_{l-2}\right)+\mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}(\mathbf{I}-\mathbf{P}) \qquad (3.12)
\end{aligned}$$

The term $\mathbf{S}_{l-2}(\mathbf{I}\oplus\mathcal{J}_\infty)\mathbf{S}_{l-1}^{-1}$ can be written as:

$$\begin{aligned}
\mathbf{S}_{l-2}(\mathbf{I}\oplus\mathcal{J}_\infty)\mathbf{S}_{l-1}^{-1} &= \begin{bmatrix}\mathbf{X}_f & \mathbf{X}_\infty{\mathcal{J}_\infty}^{l-2}\\ \mathbf{X}_f\mathcal{J}_f & \mathbf{X}_\infty{\mathcal{J}_\infty}^{l-3}\\ \vdots & \vdots \\ \mathbf{X}_f{\mathcal{J}_f}^{l-2} & \mathbf{X}_\infty\mathcal{J}_\infty\end{bmatrix}\begin{bmatrix}\mathbf{I} & 0\\ 0 & \mathcal{J}_\infty\end{bmatrix}\mathbf{S}_{l-1}^{-1} \\
&= \begin{bmatrix}\mathbf{I}_{M(l-1)} & 0\end{bmatrix}\mathbf{S}_{l-1}\mathbf{S}_{l-1}^{-1} \\
&= \begin{bmatrix}\mathbf{I}_{M(l-1)} & 0\end{bmatrix} \qquad (3.13)
\end{aligned}$$

Now using (3.13) in (3.12), $\mathbf{V}(\mathbf{I}-\mathbf{P})$ is written as:

$$\begin{aligned}
\mathbf{V}(\mathbf{I}-\mathbf{P}) &= \mathbf{A}_1\left(\mathbf{S}_{l-2}-\begin{bmatrix}\mathbf{I}_{M(l-1)} & 0\end{bmatrix}\begin{bmatrix}\mathbf{I}_{M(l-1)}\\0\end{bmatrix}\mathbf{S}_{l-2}\right)+\mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}(\mathbf{I}-\mathbf{P}) \\
&= \mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}(\mathbf{I}-\mathbf{P})
\end{aligned}$$

Form the above it is obvious that $\mathbf{V}=\mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}$, thus the rows of matrix $\mathbf{V}$ span the null space of $\mathbf{S}_{l-2}$.□

Matrix $\mathbf{A}_2$ is full rank for $\mathbf{V}$ to be full rank. Having obtained $\mathbf{V}$, the explicit representation for $\mathcal{E}(z)$ and $\mathcal{R}(z)$ is given by:

$$\mathcal{E}(z)=\mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}(\mathbf{I}-\mathbf{P})[(z^{-1}\mathbf{I}-\mathcal{J}_f)\oplus(z^{-1}\mathcal{J}_\infty-\mathbf{I})]\mathbf{S}_{l-1}^{-1}col(z^{-i}\mathbf{I})_{i=0}^{l} \qquad (3.14)$$

$$\begin{aligned}\mathcal{R}(z) &= z^{-l_0}\mathcal{E}^{-1}(z) \\ &= z^{-l_0}[\mathbf{X}_f\ \mathbf{X}_\infty]\begin{bmatrix}(z^{-1}\mathbf{I}-\mathcal{J}_f) & 0 \\ 0 & (z^{-1}\mathcal{J}_\infty-\mathbf{I})\end{bmatrix}^{-1}[\mathbf{I}\oplus\mathcal{J}_\infty^{l-1}]\begin{bmatrix}\mathbf{S}_{l-2} \\ \mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}\end{bmatrix}^{-1}[0\ \cdots\ 0\ \mathbf{I}]^T\end{aligned} \tag{3.15}$$

3.2.1 Length of synthesis filters

In this section it will be shown that for FIR $\mathcal{E}(z)$ with FIR inverse, the reconstruction delay introduced by the cascade of $\mathcal{E}(z)$ and $\mathcal{R}(z)$ is equal to the number of finite spectral points.

Theorem 7 *The minimum delay introduced for synthesis section to be causal is equal to $z^{-\mu_f}$, where μ_f is the index of nil potency of $\mathcal{J}_f$.*

Proof: Recall that the matrix $\mathcal{R}(z) = z^{-l_0}\mathcal{E}^{-1}(z)$, where the delay z^{-l_0} is introduced to make the synthesis section causal. Using the explicit expression of $\mathcal{E}^{-1}(z)$,

$$\begin{aligned}\mathcal{R}(z) &= z^{-l_0}[\mathbf{X}_f\ \mathbf{X}_\infty]\begin{bmatrix}(\mathbf{I}z^{-1}-\mathcal{J}_f)^{-1} & 0 \\ 0 & (z^{-1}\mathcal{J}_\infty-\mathbf{I})^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{I} & 0 \\ 0 & \mathcal{J}_\infty^{l-1}\end{bmatrix}\begin{bmatrix}\mathbf{S}_{l-2} \\ \mathbf{V}\end{bmatrix}^{-1}\begin{bmatrix}0 \\ \vdots \\ 0 \\ \mathbf{I}\end{bmatrix} \\ &= z^{-l_0}[\mathbf{X}_f\ \mathbf{X}_\infty]\begin{bmatrix}z\mathbf{I}+z^2\mathcal{J}_f+\cdots+z^{\mu_f}\mathcal{J}_f^{\mu_f-1} & 0 \\ 0 & -(\mathcal{J}_\infty^{l-1}+z^{-1}\mathcal{J}_\infty^{l}+\cdots+z^{-(\mu_\infty-l)}\mathcal{J}_\infty^{\mu_\infty-1})\end{bmatrix} \\ &\qquad\begin{bmatrix}\mathbf{S}_{l-2} \\ \mathbf{V}\end{bmatrix}^{-1}\begin{bmatrix}0 \\ \vdots \\ 0 \\ \mathbf{I}\end{bmatrix}\end{aligned}$$

The above matrix is causal when $\mu_f - l_0 = 0$, thus the minimum delay introduced is equal to $z^{-\mu_f}$.□

While the synthesis polyphase matrix $\mathcal{R}(z) = z^{-\mu_f}\mathcal{E}(z)$, the degree of the synthesis polyphase depends upon μ_∞.

- If $\mu_\infty < l$, then the term $(\mathcal{J}_\infty z^{-1} - \mathbf{I})^{-1}\mathcal{J}_\infty{}^{l-1} = 0$ in the expression of $\mathcal{E}^{-1}(z)$, so the degree of the synthesis polyphase matrix is $l_0 - 1$ and the length of the synthesis filters is Ml_0.

- If $\mu_\infty \geq l$, then the degree of the synthesis polyphase matrix is $l_0 + \mu_\infty - l$ and the length of the synthesis filters is $M(l_0 + \mu_\infty - l + 1)$.

Free variables: Since the matrix $\mathbf{S}_{l-1}$ must be full rank given $\mathcal{J}_f$ and $\mathcal{J}_\infty$, characterization of $\mathbf{X}_f$ and $\mathbf{X}_\infty$ such that $\mathbf{S}_{l-1}$ is full rank is very difficult. It was observed in simulations that in most of the cases the matrix $\mathbf{S}_{l-1}$ becomes invertible for arbitrary selection of $\mathbf{X}_f$ and $\mathbf{X}_\infty$, which requires Mk and $M(Ml-k)$ free variables. The full rank matrix used in the construction of matrix $\mathbf{V}$, given by $\mathbf{V} = \mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}$, requires M^2 free variables for $\mathbf{A}_2$. So, for characterizing an M channel degree l analysis polyphase matrix, we require to optimize $M^2(l+1)$ free variables.

3.3 Linear phase filter bank design

The conditions for $\mathcal{E}(z)$ to be linear phase is given by

$$\mathcal{E}(z) = z^{-l}\mathbf{D}\mathcal{E}(z^{-1})\mathbf{J} \tag{3.16}$$

where $\mathbf{D}$ is an $M \times M$ diagonal matrix with diagonal elements as $+1$ and -1 depending upon the number of symmetric and antisymmetric filters, $\mathbf{J}$ is an $M \times M$ exchange matrix. From the reconstruction theorem the structure of $\mathcal{E}(z)$ is given by equation (3.14). Now using this structure the conditions on finite and infinite spectra and Jordan chains are derived.

Theorem 8 *For an M channel linear phase FIR filter bank with FIR inverse, following conditions are true:*

1. *The product Ml is always even.*

2. *$deg(det(\mathcal{E}(z))) = Ml/2$.*

3. *The number of points in the finite and infinite spectrum are equal to* $Ml/2$.

Proof: From the linear phase condition given in the equation (3.16),

$$\mathcal{E}(z) = \mathbf{D}\tilde{\mathcal{E}}(z)\mathbf{J} \tag{3.17}$$

where $\tilde{\mathcal{E}}(z)$ is the dual polynomial of $\mathcal{E}(z)$. Taking the determinant on both sides of equation (3.17), and since $\mathbf{D}$ and $\mathbf{J}$ are have constant determinants,

$$deg(det(\mathcal{E}(z))) = deg(det(\tilde{\mathcal{E}}(z))) \tag{3.18}$$

If $deg(det(\mathcal{E}(z)))$ is k, then from theorem (5) $deg(det(\tilde{\mathcal{E}}(z)))$ is $Ml-k$. Then from equation (3.18) $k = Ml - k$ implies, $k = Ml/2$. Since $k \in \mathbb{Z}$, Ml must be even. k is the size of finite spectrum and is equal to $Ml/2$. So, both finite and infinite spectrum have $Ml/2$ points. □

Theorem 9 *For an* M *channel FIR filter bank with FIR inverse with* $\mathcal{J}_f = \mathcal{J}_\infty$ *to be linear phase the following conditions must be satisfied.*

1. $\mathbf{X}_f = \mathbf{J}\mathbf{X}_\infty$
2. $\mathbf{V}(\mathbf{I}-\mathbf{P}) = -\mathbf{D}\mathbf{V}(\mathbf{I}-\mathbf{P})\mathbf{J}_2$, *where* $\mathbf{J}_2 = \begin{bmatrix} 0 & \mathbf{I}_{Ml/2} \\ \mathbf{I}_{Ml/2} & 0 \end{bmatrix}$.

Proof: From the structure of $\mathcal{E}(z)$ from (3.14) and linear phase condition of equation (3.16), assuming $\mathcal{Q}(z) = col(z^{-i}\mathbf{I})_{i=0}^{l-1}$ we have,

$$\mathbf{V}(\mathbf{I}-\mathbf{P})\mathcal{T}(z)\mathbf{S}_{l-1}^{-1}\mathcal{Q}(z) = \mathbf{D}\mathbf{V}(\mathbf{I}-\mathbf{P})z^{-1}\mathcal{T}(z^{-1})\mathbf{S}_{l-1}^{-1}z^{-(l-1)}\mathcal{Q}(z^{-1})\mathbf{J} \tag{3.19}$$

where $\mathcal{T}(z) = [(z^{-1}\mathbf{I} - \mathcal{J}_f) \oplus (z^{-1}\mathcal{J}_\infty - \mathbf{I})]$. The term $z^{-1}\mathcal{T}(z^{-1})$ is analyzed as follows.

$$z^{-1}\mathcal{T}(z^{-1}) = z^{-1}\begin{bmatrix} \mathbf{I}z - \mathcal{J}_f & 0 \\ 0 & z\mathcal{J}_\infty - \mathbf{I} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} - z^{-1}\mathcal{J}_f & 0 \\ 0 & \mathcal{J}_\infty - z^{-1}\mathbf{I} \end{bmatrix}$$
$$= -\mathbf{J}_2 \begin{bmatrix} z^{-1}\mathbf{I} - \mathcal{J}_\infty & 0 \\ 0 & z^{-1}\mathcal{J}_f - \mathbf{I} \end{bmatrix} \mathbf{J}_2$$

Taking $\tilde{\mathbf{J}}_{Ml}$ to be an $Ml \times Ml$ exchange matrix, the term $\mathbf{S}_{l-1}^{-1} z^{-(l-1)} \mathcal{Q}(z^{-1})\mathbf{J}$ is analyzed as follows.

$$\begin{aligned} \mathbf{S}_{l-1}^{-1} z^{-(l-1)} \mathcal{Q}(z^{-1})\mathbf{J} &= \mathbf{J}_2(\tilde{\mathbf{J}}_{Ml}\mathbf{S}_{l-1}\mathbf{J}_2)^{-1}\tilde{\mathbf{J}}_{Ml} z^{-(l-1)}\mathcal{Q}(z^{-1})\mathbf{J} \\ &= \mathbf{J}_2(\tilde{\mathbf{J}}_{Ml}\mathbf{S}_{l-1}\mathbf{J}_2)^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & \mathbf{J} \\ 0 & 0 & \cdots & \mathbf{J} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \mathbf{J} & \cdots & 0 & 0 \\ \mathbf{J} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} z^{-(l-1)}\mathbf{I} \\ z^{-(l-2)}\mathbf{I} \\ \cdots \\ z^{-1}\mathbf{I} \\ \mathbf{I} \end{bmatrix} \mathbf{J} \\ &= \mathbf{J}_2(\tilde{\mathbf{J}}_{Ml}\mathbf{S}_{l-1}\mathbf{J}_2)^{-1}\mathcal{Q}(z) \end{aligned}$$

Using the above equalities in the linear phase condition and cancelling the common terms we have

$$\begin{aligned} &\mathbf{V}(\mathbf{I}-\mathbf{P}) \begin{bmatrix} z^{-1}\mathbf{I} - \mathcal{J}_f & 0 \\ 0 & z^{-1}\mathcal{J}_\infty - \mathbf{I} \end{bmatrix} \mathbf{S}_{l-1}^{-1} \\ &= -\mathbf{D}\mathbf{V}(\mathbf{I}-\mathbf{P})\mathbf{J}_2 \begin{bmatrix} z^{-1}\mathbf{I} - \mathcal{J}_\infty & 0 \\ 0 & z^{-1}\mathcal{J}_f - \mathbf{I} \end{bmatrix} (\tilde{\mathbf{J}}_{Ml}\mathbf{S}_{l-1}\mathbf{J}_2)^{-1}. \end{aligned}$$

For the above equation one possible solution is obtained by assuming $\mathcal{J}_f = \mathcal{J}_\infty$. This assumption is valid because in this case the Jordan forms of finite and infinite spectra are assumed to be same, which is a special case of a more general condition — determinant equality— that does not give information regarding the Jordan structure of finite and infinite spectra. With the above assumption, the above equation leads to following equalities.

$$\mathbf{V}(\mathbf{I}-\mathbf{P}) = -\mathbf{D}\mathbf{V}(\mathbf{I}-\mathbf{P})\mathbf{J}_2 \tag{3.20}$$

$$(\mathbf{S}_{l-1})^{-1} = (\tilde{\mathbf{J}}_{Ml}\mathbf{S}_{l-1}\mathbf{J}_2)^{-1} \tag{3.21}$$

The last equation can be simplified as follows.

$$\mathbf{S}_{l-1} = \tilde{\mathbf{J}}_{Ml}\mathbf{S}_{l-1}\mathbf{J}_2$$

$$\begin{bmatrix} \mathbf{X}_f & \mathbf{X}_\infty \mathcal{J}_\infty^{l-1} \\ \mathbf{X}_f \mathcal{J}_f & \mathbf{X}_\infty \mathcal{J}_\infty^{l-2} \\ \vdots & \vdots \\ \mathbf{X}_f \mathcal{J}_f^{l-2} & \mathbf{X}_\infty \mathcal{J}_\infty \\ \mathbf{X}_f \mathcal{J}_f^{l-1} & \mathbf{X}_\infty \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \mathbf{J} \\ 0 & 0 & \cdots & \mathbf{J} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \mathbf{J} & \cdots & 0 & 0 \\ \mathbf{J} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_f & \mathbf{X}_\infty \mathcal{J}_\infty^{l-1} \\ \mathbf{X}_f \mathcal{J}_f & \mathbf{X}_\infty \mathcal{J}_\infty^{l-2} \\ \vdots & \vdots \\ \mathbf{X}_f \mathcal{J}_f^{l-2} & \mathbf{X}_\infty \mathcal{J}_\infty \\ \mathbf{X}_f \mathcal{J}_f^{l-1} & \mathbf{X}_\infty \end{bmatrix} \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{J}\mathbf{X}_\infty & \mathbf{J}\mathbf{X}_f \mathcal{J}_f^{l-1} \\ \mathbf{J}\mathbf{X}_\infty \mathcal{J}_\infty & \mathbf{J}\mathbf{X}_f \mathcal{J}_f^{l-2} \\ \vdots & \vdots \\ \mathbf{J}\mathbf{X}_\infty \mathcal{J}_\infty^{l-2} & \mathbf{J}\mathbf{X}_f \mathcal{J}_f \\ \mathbf{J}\mathbf{X}_\infty \mathcal{J}_\infty^{l-1} & \mathbf{J}\mathbf{X}_f \end{bmatrix}$$

From the above matrix equality $\mathbf{X}_f = \mathbf{J}\mathbf{X}_\infty$ and $\mathbf{X}_\infty = \mathbf{J}\mathbf{X}_f$ which are equivalent. □

3.3.1 Length of synthesis filters

From section (3.2.1), the degree of the synthesis polyphase matrix is $\mu_f + \mu_\infty - l$. Since we assumed $\mu_f = \mu_\infty$, the degree of the synthesis polyphase matrix is $2\mu_f - l$, so the length of the synthesis filters is $M(2\mu_f - l + 1)$.

3.3.2 Near linear phase filter banks

Characterizing $\mathbf{V}$ satisfying the conditions $\mathbf{V}(\mathbf{I}-\mathbf{P}) = -\mathbf{D}\mathbf{V}(\mathbf{I}-\mathbf{P})\mathbf{J}_2$ and full rank $\begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{V} \end{bmatrix}$ is difficult because there is no easy way of solving the above equality for $\mathbf{V}$. As already discussed in theorem (6), $\mathbf{V}$ is the span of null space of $\mathbf{S}_{l-2}$. So the norm of the error matrix $\mathbf{eV} = \mathbf{V}(\mathbf{I}-\mathbf{P}) + \mathbf{D}\mathbf{V}(\mathbf{I}-\mathbf{P})\mathbf{J}_2$, $\|\mathbf{eV}\|$, is reduced in the optimization while designing the filter bank. $\|\mathbf{eV}\| \to 0$ if all the elements in the matrix $\mathbf{eV}$ are zero. This leads to the minimization of M^2l elements of the matrix $\mathbf{eV}$ to zero. Instead if SVD of $\mathbf{eV}$ is considered i.e. $\mathbf{eV} = \mathbf{P}\mathbf{D}_{\mathbf{eV}}\mathbf{Q}$, where $\mathbf{P}$ and $\mathbf{Q}$ are $M \times M$ and $M^2l \times M^2l$ orthogonal matrices whose norms are unity, and $\mathbf{D}_{\mathbf{eV}}$ is a diagonal $M \times M^2l$ matrix with M diagonal elements. As $\|\mathbf{eV}\| = \|\mathbf{P}\mathbf{D}_{\mathbf{eV}}\mathbf{Q}\| = \|\mathbf{D}_{\mathbf{eV}}\|$, by taking SVD of $\mathbf{eV}$, minimization of M elements is enough, and this is irrespective of the degree of analysis polyphase matrix.

Free variables: The construction of linear phase $\mathcal{E}(z)$ involves the inversion of matrix $\mathbf{S}_{l-1}$, which is formed from the matrices $\mathbf{X}_f$ and $\mathcal{J}_f$. Since $\mathcal{J}_f = \mathcal{J}_\infty$ and $\mathbf{X}_\infty = \mathbf{J}\mathbf{X}_f$, $\mathbf{X}_\infty$ and $\mathcal{J}_\infty$ are known once $\mathbf{X}_f$ and $\mathcal{J}_f$ are fixed. So, $\mathcal{E}(z)$ is a regular matrix polynomial with $([\mathbf{X}_f \ \mathbf{J}\mathbf{X}_f], \mathcal{J}_f \oplus \mathcal{J}_f)$ as the decomposable pair. Different $\mathcal{E}(z)$ are possible for different $\mathcal{J}_f$ for a given M and l. Constraining $\mathbf{X}_f$, given $\mathcal{J}_f$, such that $\mathbf{S}_{l-1}$ is full rank is very difficult. Depending upon the Jordan structure of $\mathcal{J}_f$, columns of $\mathbf{X}_f$ are taken to be linearly independent for the canonical Jordan chain condition to be satisfied. In the simulations, all the elements in $\mathbf{X}_f$ are assumed to be arbitrary and in most cases the matrix $\mathbf{S}_{l-1}$ is seen to be invertible. So, M^2 free variables for matrix $\mathbf{V}$ and $M^2(l/2)$ free variables for $\mathbf{X}_f$ are required to be optimized.

3.4 Low delay filter bank design

3.4.1 Matrix polynomials with invertible $\mathcal{J}_f$

In this chapter a class of matrix polynomials called comonic matrix polynomials with nonzero finite spectrum is discussed. This leads to the construction of unimodular matrix polynomials. If $\mathcal{J}_f$ is invertible, its eigenvalues take nonzero values. Then $det(\mathcal{E}(z))$ is a polynomial with nonzero roots, i.e. $\mathcal{E}(0) = \mathbf{E}_0$ is invertible. Then taking $\mathcal{E}_0 = \mathbf{I}$, $\mathcal{E}(z)$ is called *comonic.*

Let$(\mathbf{X}_f, \mathcal{J}_f)$ and $(\mathbf{X}_\infty, \mathcal{J}_\infty)$ be the finite and infinite Jordan pairs of the *comonic* polynomial $\mathcal{E}(z)$. Since $\mathbf{E}_0 = \mathbf{I}$ and $\mathcal{J}_f$ is invertible, taking $\mathcal{J} = \mathcal{J}_f^{-1} \oplus \mathcal{J}_\infty$ and $\mathbf{X} = [\mathbf{X}_f \;\; \mathbf{X}_\infty]$, the pair $(\mathbf{X}, \mathcal{J})$ can be taken as *comonic Jordan pair*, which is a convenient representation of comonic matrix polynomial. Linearization of the comonic matrix polynomial is defined as

$$\mathbf{I} - \mathbf{R}z^{-1} \;\; = \;\; \mathcal{B}(z) \; diag[\mathcal{E}(z), \mathbf{I}, \cdots, \mathbf{I}] \; \mathcal{D}(z)$$

where the structures of $\mathcal{B}(z)$ and $\mathcal{D}(z)$ are given in [48], and $\mathbf{R}$ is called the comonic companion matrix whose structure is given by

$$\mathbf{R} = \begin{bmatrix} 0 & \mathbf{I} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{I} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{I} \\ -\mathbf{E}_l & -\mathbf{E}_{l-1} & -\mathbf{E}_{l-2} & \cdots & -\mathbf{E}_1 \end{bmatrix}.$$

The matrix pencil $\mathbf{I} - \mathbf{R}z^{-1}$ has a canonical form [59] [48] given by

$$col(\mathbf{X}\mathcal{J}^i)_{i=0}^{l-1} \; \mathcal{J} = \mathbf{R} \; col(\mathbf{X}\mathcal{J}^i)_{i=0}^{l-1}.$$

and the above equation is equivalent to

$$\mathbf{X}\mathcal{J}^l + \mathbf{E}_l\mathbf{X} + \mathbf{E}_{l-1}\mathbf{X}\mathcal{J} + \cdots + \mathbf{E}_1\mathbf{X}\mathcal{J}^{l-1} = 0$$

The pair $(\mathbf{X}, \mathcal{J})$ is taken as the decomposable pair if the matrix $col(\mathbf{X}\mathcal{J}^i)_{i=0}^{l-1}$ is nonsingular. The above equation leads to the construction of a comonic matrix polynomial given the spectral data $(\mathbf{X}, \mathcal{J})$. The matrix polynomial $\mathcal{E}(z)$ is given by

$$\mathcal{E}(z) = \mathbf{I} - \mathbf{X}\mathcal{J}^l(\mathbf{V}_1 z^{-l} + \mathbf{V}_2 z^{-(l-1)} + \cdots + \mathbf{V}_l z^{-1})$$

where $[\mathbf{V}_1 \ \cdots \ \mathbf{V}_l] = [col(\mathbf{X}\mathcal{J}^i)_{i=0}^{l-1}]^{-1}$. If $(\mathbf{X}, \mathcal{J})$ is the comonic Jordan pair and if $\mathcal{E}(z)$ has the structure given above then the inverse of the matrix polynomial is given by

$$\mathcal{E}^{-1}(z) = \mathbf{X}\mathcal{J}^{l-1}(\mathbf{I} - \mathcal{J}z^{-1})^{-1}[col(\mathbf{X}\mathcal{J}^i)_{i=0}^{l-1})]^{-1}[0 \ 0 \ \cdots \ \mathbf{I}]^T. \quad (3.22)$$

3.4.2 Construction of unimodular matrix polynomials

If the analysis polyphase matrix is unimodular then delay due to analysis and synthesis sections is zero, and the filter bank delay is due to the delay chain only. It was shown previously that for an unimodular matrix polynomial, finite spectrum is absent and has spectrum at infinity only, i.e. $\mathcal{J}$ is Jordan nilpotent [62]. So, $\mathcal{J} = \mathcal{J}_\infty$. Assuming $\mathcal{J}_\infty = diag(\mathcal{J}_{\infty 1}, \cdots, \mathcal{J}_{\infty k})$, with Jordan blocks having sizes $m_1, \cdots, m_k$ such that $m_1 \geq m_2 \geq \cdots \geq m_k$ and $\sum_{i=1}^k m_i = Ml$, the $M \times Ml$ matrix $\mathbf{X}$ has the structure

$$\mathbf{X} = [\mathbf{x}_1^0, \cdots, \mathbf{x}_1^{m_1-1}, \mathbf{x}_2^0, \cdots, \mathbf{x}_2^{m_2-1} \ \cdots \ \mathbf{x}_k^0, \cdots, \mathbf{x}_k^{m_k-1}]$$

and the eigen vectors $\mathbf{x}_1^0, \mathbf{x}_2^0. \cdots, \mathbf{x}_k^0$ are linearly independent. Now the requirement for the construction of a unimodular matrix polynomial is, $(\mathbf{X}, \mathcal{J})$ must be a decomposable pair, or matrix $col(\mathbf{X}\mathcal{J}^i)_{i=0}^{l-1}$ must be nonsingular. Now the inverse problem reduces to finding $\mathbf{X}$, given $\mathcal{J}$, such that $col(\mathbf{X}\mathcal{J}^i)_{i=0}^{l-1}$ is nonsingular. Closest possible solutions are given in [61], in the context of finding $\mathbf{Y}$ for the pair $(\mathbf{Y}, \mathbf{T})$ to be controllable. The controllability matrix $\mathbf{C}_k = [\mathbf{Y} \ \mathbf{TY} \ \mathbf{T}^2\mathbf{Y} \cdots \mathbf{T}^{k-1}\mathbf{Y}]$ is assumed, where $\mathbf{Y}$ is $Ml \times M$ matrix and $\mathbf{T}$ is $Ml \times Ml$ matrix. Structures of the matrix $\mathbf{Y}$ are given in [61] if the controllability index k [61] lies in the bounds (l, t_∞). t_∞ is the nilpotent index of $\mathbf{T}$. If $col(\mathbf{X}\mathcal{J}^i)_{i=0}^{l-1}$ is invertible, then it's transpose $[\mathbf{X}^T \ \mathbf{X}^T\mathcal{J}^T \cdots \mathbf{X}^T(\mathcal{J}^T)^{l-1}]$ is nonsingular. Taking $\mathbf{Y} = \mathbf{X}^T$ and $\mathbf{T} = \mathcal{J}^T$, the pair $(\mathbf{X}^T, \mathcal{J}^T)$ must be controllable with the controllability index always achieving the lower bound l. The structure for $\mathbf{Y}$ given in [61] is valid if $l \leq k \leq \mu_\infty$, but the inverse problem demands $k = l$. So, for simulations $\mathbf{X}$ is taken arbitrarily with eigen vectors linearly independent and it was seen in simulations that in most of the cases the matrix $\mathbf{C}_l$ becomes nonsingular.

3.4.2.1 Degree of the inverse polynomial

If μ_∞ is the nilpotent index of $\mathcal{J}$ then the term $\mathcal{J}^{l-1}(\mathbf{I} - \mathcal{J}z^{-1})^{-1}$ in equation (3.22) is given by

$$\mathcal{J}^{l-1}(\mathbf{I} - \mathcal{J}z^{-1})^{-1} = \mathbf{J}^{l-1}(\mathbf{I} + \mathcal{J}z^{-1} + \cdots + \mathcal{J}^k z^{-k})$$

where k is the degree of the inverse polynomial. The highest degree of the above polynomial is $l + k - 1$, which must be equal to $\mu_\infty - 1$, which implies $k = \mu_\infty - l$, so the degree of the inverse polynomial is $\mu_\infty - l$.

3.5 Simulation results

3.5.1 Simulation results for FIR design

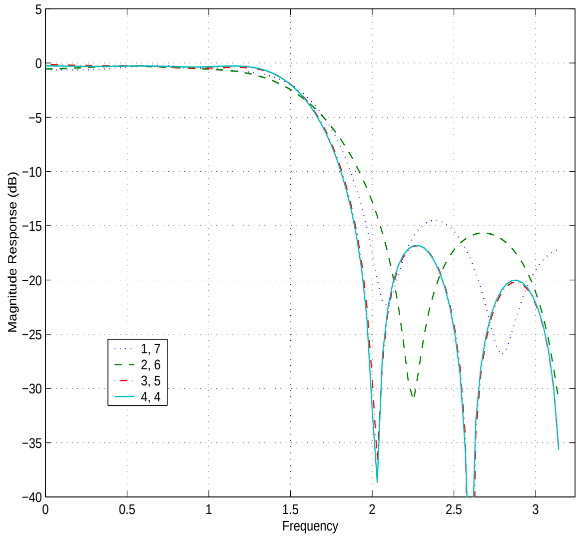
The design procedure is briefly recalled below. Given number of channels M and analysis polyphase order l, some μ_f and some corresponding possible case of $\mathcal{J}_f$ and $\mathcal{J}_\infty$ are assumed. The finite and infinite Jordan chains $\mathbf{X}_f$ and $\mathbf{X}_\infty$ are taken arbitrarily as free variables to be optimized, because constructing them with the constraint that $\mathbf{S}_{l-1}$ is full rank is difficult. In simulations it was seen that $\mathbf{S}_{l-1}$ is invertible for most of the cases. The regular matrix polynomial $\mathcal{E}(z)$ with $([\mathbf{X}_f\ \mathbf{X}_\infty], diag\{\mathcal{J}_f, \mathcal{J}_\infty\})$ as the decomposable pair is constructed from proposition (6). $\mathbf{V} = \mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}$ is optimized. The full rank $\mathbf{A}_2$ is obtained from SVD method, $\mathbf{A}_2 = \mathbf{UDV}^T$, where $\mathbf{U}$ and $\mathbf{V}$ are the real orthogonal matrices and $\mathbf{D}$ is a diagonal matrix. The orthogonal matrices are obtained form Givens rotations [1]. Optimization may be run for any appropriate cost function, such as frequency selectivity. Once $\mathcal{E}(z)$ is obtained from optimization, $\mathcal{R}(z)$ is found as mentioned with l_0, the delay, taken such that the synthesis filters are causal. Simulations are done 2 channel and 4 channel cases. Matlab unconstrained optimization routine *fminunc* is used for optimizing the filters.

For 2 channel case the analysis polyphase matrix is considered to be of order 4. Some possible cases of $\mathcal{J}_f$ and $\mathcal{J}_\infty$ are assumed. Design results are tabulated in table (3.1). The notations used are, l_a and l_s correspond to length of analysis and synthesis filters, LA_{SB} and HA_{SB} correspond to the low pass and high pass stop band attenuation (in dB, taken for first side lobe), NP_{LP} and NP_{HP} represent the first null position of low pass and high pass filters. As the degree of the determinant of $\mathcal{E}(z)$ increases from 1 to 4, the filter responses improve, especially the stop band attenuation is improved. But for further values of the degree of the determinant, the results are not consistent. For example, for degree 5 the stop band attenuation is better compared to degree 4, but transition band is more when compared to former one. For degree 6, two cases are possible. First one is having a finite Jordan structure $[5, 1]$, which leads to $l_s = 10$ and better stopband attenuation at the cost of transition band performance when compared to degree 5 case. For the latter case (having a finite Jordan structure 6) $l_s = 12$, and the transition band performance is good compared to $[5, 1]$ case. The reason for the performance of the filters becoming bad after degree 5, could be because the term $(\mathcal{J}_\infty z^{-1} - \mathbf{I})^{-1} {\mathcal{J}_\infty}^{l-1}$ becomes zero in the synthesis polyphase expression for $\mu_\infty < l$, so this leads to zeroing of coefficients in synthesis filters. Since the FB is PR, the effects on synthesis section can be seen in the analysis. The zeroing of coefficients in synthesis filters has been clearly observed. The responses for different Jordan structures are shown in figure (3.1).

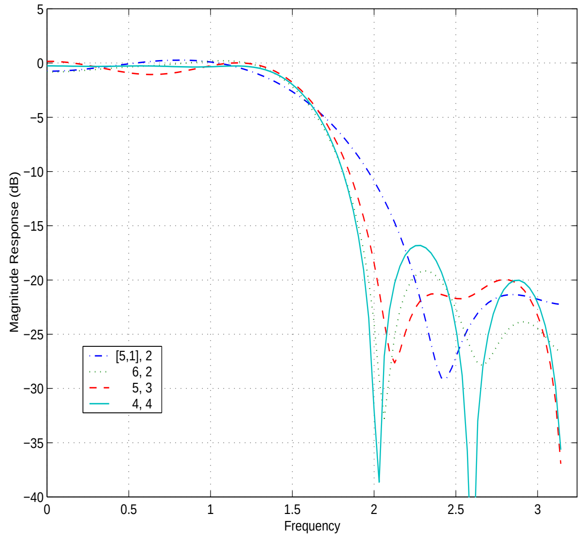
For 4 channel case, simulations are done for $l = 2$. Regarding the spectral data, following cases are taken.

- $\mathcal{J}_f$ and $\mathcal{J}_\infty$ have same Jordan structure, i.e. [2,2], such that $l_a = l_s = 12$.
- $\mathcal{J}_f$ and $\mathcal{J}_\infty$ having $[3, 1]$ and $[2, 2]$ as Jordan structures, such that $l_a = 12$ and $l_s = 16$.

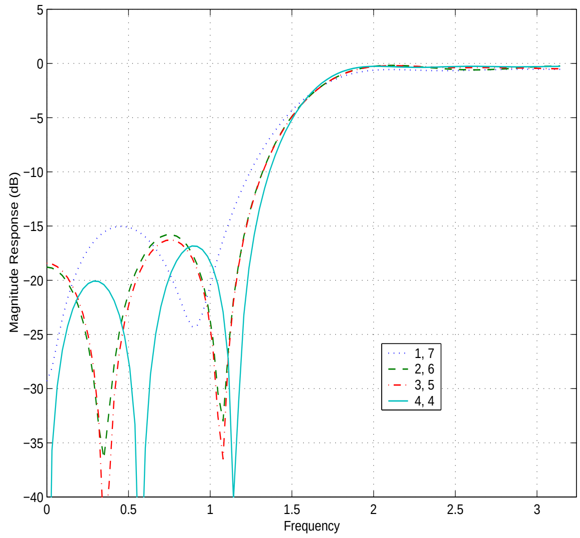
For the former case, analysis filter coefficients are shown in table (3.2). It can be seen that the fourth filter is approximately a flipped and modulated version of the first one,



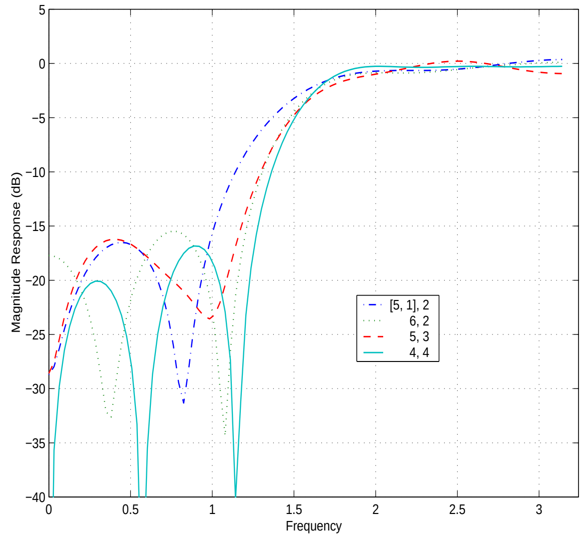
(a) Low pass filter responses



(b) Low pass filter responses



(c) High pass filter responses



(d) High pass filter responses

Figure 3.1: 2 channel analysis filter responses for different $(\mathcal{J}_f, \mathcal{J}_\infty)$

$\mathcal{J}_f$	$\mathcal{J}_\infty$	l_a	l_s	LA_{SB} (dB)	HA_{SB} (dB)	NP_{LP} $max.\pi$	NP_{HP} $max.\pi$
1	7	10	10	-14.5000	-15.0234	2.9044	0.8875
2	6	10	10	-15.6700	-15.7938	2.2532	1.0788
3	5	10	10	-16.8147	-16.2723	2.0310	1.0788
4	4	10	10	-16.8147	-16.8381	2.0310	1.1423
5	3	10	10	-19.1705	-15.4600	2.0628	1.0788
[5, 1]	2	10	10	-21.3450	-16.5097	2.4231	0.8250
6	2	10	12	-21.2583	-16.2208	2.1262	0.9871

Table 3.1: Comparison of 2 channel filters for different delays.

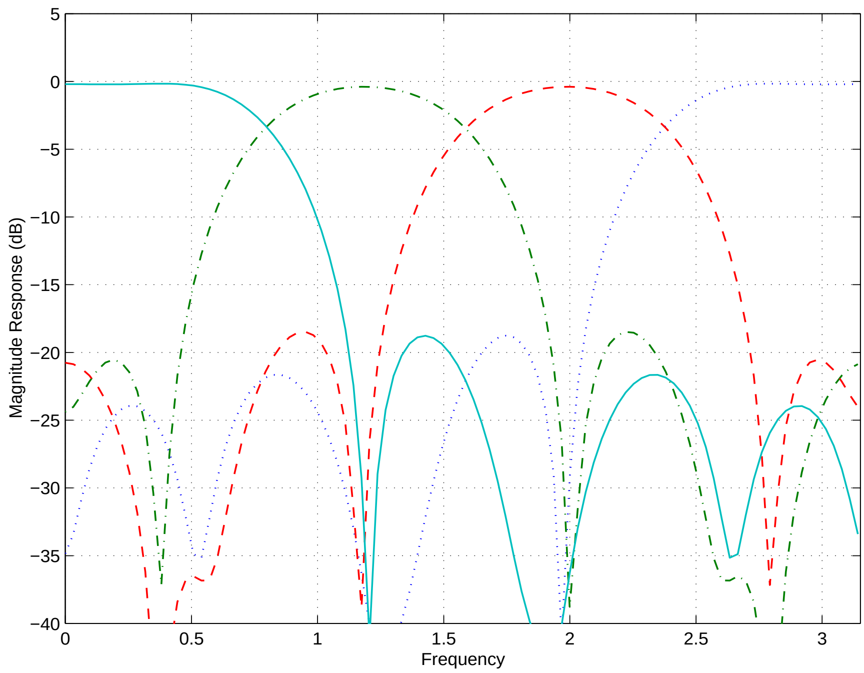


Figure 3.2: Magnitude responses of analysis filters of 4 channel filter bank

$h_0(n)$	$h_1(n)$	$h_2(n)$	$h_3(n)$
0.11213728509219	-0.04298586041109	0.12665769631184	0.03123787884481
0.19013060467551	-0.01754633683769	-0.10383208113384	-0.01029271357353
0.26296147394038	-0.02236899046822	-0.15453616756111	-0.01640574304014
0.25887231104063	-0.09708148443751	0.27999987670062	0.06965281023631
0.18336772540933	-0.05005092133824	-0.03577015776387	-0.08202426091665
0.09807788074062	0.19390668366595	-0.21557421007789	0.00110570000830
-0.00109759645188	0.21558088446386	0.19390402404537	0.09806704131317
-0.08202164924866	-0.03576214764423	0.05004380212950	-0.18336249719728
-0.06965350844238	-0.27999832846542	-0.09707979680717	0.25887323876379
-0.01640815624907	-0.15454142398085	0.02236454691044	-0.26295978785254
0.01029010177065	0.10381854905884	-0.01754917748475	0.19014549193729
0.03123653503479	0.12664762153764	0.04298418248956	-0.11214125172197

Table 3.2: Analysis filter coefficients of 4 channel filter bank.

and similar behavior is displayed between the third filter and the second. This behavior is seen in synthesis case also, which are shown in table (3.3). This behavior is expected because same weightage is given for all the filters in the optimization. The analysis filters are shown in figure (3.2). If M is even, the condition

$$h_{M-i-1}(n) = (-1)^n h_i(N-n-1) \quad \forall\; i = 0, \cdots, M-1 \qquad n = 0, \cdots, N-1.$$

is satisfied if $\mathcal{E}(z)$ satisfies the condition

$$\mathbf{J}\mathcal{E}(z) = \tilde{\mathcal{E}}(z)\mathbf{J}' \tag{3.23}$$

where $\mathbf{J}'$ is an $M \times M$ exchange matrix with alternate sign change and $\mathbf{J}'_{1,M} = 1$. Taking the determinant on both sides of the equation (3.23)

$$\begin{aligned} deg(det(\mathbf{J}\mathcal{E}(z))) &= deg(det(\tilde{\mathcal{E}}(z)\mathbf{J}')) \\ k &= Ml - k. \end{aligned}$$

Form the above expressions it is evident that both finite and infinite spectrum have same

$f_0(n)$	$f_1(n)$	$f_2(n)$	$f_3(n)$
0.12868974904658	0.55517631406401	0.18709076368596	-0.45658262322867
0.04462934270879	0.46558503335321	-0.07454670831303	0.78273158015719
-0.06826233784958	-0.68150180727814	0.09838196825780	-1.09165612507282
-0.28599988740900	-1.21793880825980	-0.42390119530081	1.06945328574356
-0.34863865716744	-0.16890381065029	0.23154810513185	-0.77026610744241
-0.01997052794912	0.93721477087846	0.83868294043326	0.41477161936311
0.41481125927659	0.83869225900633	-0.93720376456514	0.01998655851073
0.77029298024320	-0.23158667290195	-0.16896434558828	-0.34863407842791
1.06949077039530	-0.42392092121605	1.21795246837691	0.28599021500691
1.09165358937507	-0.09838880675805	-0.68147744961239	-0.06825306219248
0.78269817601883	-0.07453897862460	-0.46560467311364	-0.04463158309743
0.45651366592867	-0.18708611562677	0.55520328738344	0.12868460762566

Table 3.3: Synthesis filter coefficients for 4 channel filter bank.

number of points, implying $\mathcal{J}_f$ and $\mathcal{J}_\infty$ are of same size, i.e. $Ml/2$. For the present case $\mathcal{J}_f$ and $\mathcal{J}_\infty$ are of same size. It is seen that after optimization both the analysis and synthesis filters satisfy the condition (3.23), if $\mathcal{J}_f = \mathcal{J}_\infty$.

For the latter case, the finite Jordan structure is changed from $[2, 2]$ to $[3, 1]$. The analysis filters show the same behavior. Synthesis filter coefficients for the latter case are shifted version of the former by a factor of 4, padded with zeros for first 4 coefficients. The cost function for optimization for both the cases depends on the magnitude response, so the optimized cost value is same for both the cases. It was clearly observed that for synthesis filters phase responses are different, whereas the magnitude responses are same, for the above cases.

When the length of analysis and synthesis is constrained to be the same, the delay value after which the filter responses become worse is equal to $M(l-1)$, because the term $(\mathcal{J}_\infty z^{-1} - \mathbf{I})^{-1} {\mathcal{J}_\infty}^{l-1}$ in the synthesis section goes to zero for $\mu_\infty < l$, leading to zeroing of filter coefficients.

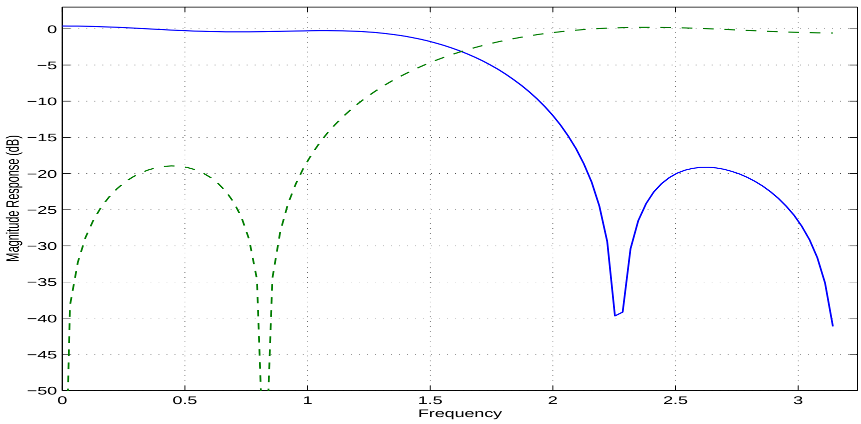


Figure 3.3: Magnitude response of analysis filters of 2 channel near linear phase filter bank

3.5.2 Simulation results for near linear phase design

Simulations are done for 2 channel and 3 channel case and the matlab unconstrained routine *fminunc* is used for optimizing the free variables. For 2 channel case the analysis polyphase matrix with order 4 ($l = 4$) is taken, and the length of the synthesis filters are determined by the Jordan structure of $\mathcal{J}_f$. For simulations all the free variables are taken unconstrained and in optimization it is observed that the matrix $\mathbf{S}_{l-1}$ is invertible in most cases. Norm of SVD of the error matrix is reduced for near linear phase condition. For 2 channel case, the analysis and synthesis coefficients are given in table (3.4) for a choice of $\mathcal{J}_f = 4$. It can be seen from the filter coefficients that the filter bank is indeed near linear phase. The responses are shown in figure (3.3).

For 3 channel case, order 2 analysis polyphase matrix is considered. For reducing the number of free variables, the finite Jordan chain matrix, which becomes a square matrix in this case, is considered to be diagonal. Two possible Jordan structures are taken, i.e. [2,1] and 3. For the former case the analysis and synthesis are of same length, i.e. $l_a = l_s = 9$, and for the latter case, $l_a = 9$ and $l_s = 12$. Filter coefficients for the two cases are tabulated in tables (3.5) and (3.6) respectively. In figure (3.4) it is clearly seen that analysis filters are better if the index of nil potency of $\mathcal{J}_f$ is more, i.e. greater length synthesis filters.

$h_0(n)$	$h_1(n)$	$f_0(n)$	$f_1(n)$
-0.02604587183775	-0.00288480790935	0.00589584654412	-0.05323143456644
-0.02064955544161	-0.00228711871246	-0.00467431502568	0.04220267481018
0.10555274673549	-0.10235252183210	0.20918368955188	0.21572417180581
-0.10576241242400	-0.10212942752748	-0.20872773899080	0.21615267753786
-0.47591109009956	0.46880725795212	-0.95812814527420	-0.97264665240974
-0.47581352281502	-0.46873436568339	-0.95797917118504	0.97244724858269
-0.10578367074698	0.10214562977100	-0.20876085244045	-0.21619612438560
0.10553144534750	0.10233635496466	0.20915064840234	-0.21568063694362
-0.02065003179864	0.00229028183399	-0.00468077967765	-0.04220364836822
-0.02604647268042	0.00288879764453	0.00590400059359	0.05323266254290

Table 3.4: Coefficients for 2 channel near linear phase filter bank with $\mathcal{J}_f$ having a Jordan structure 4.

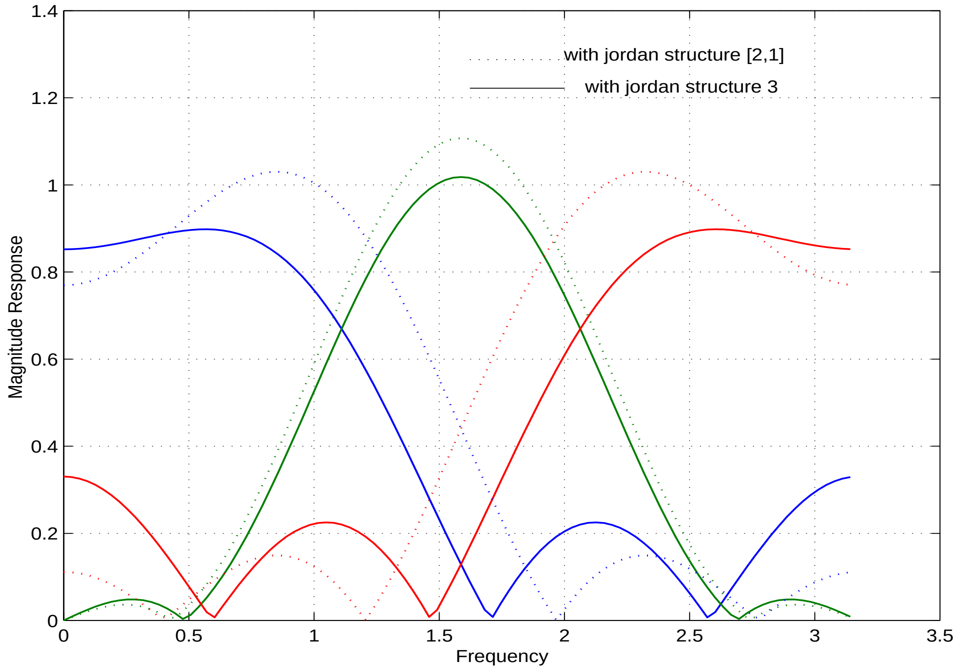


Figure 3.4: Comparison of magnitude response of analysis filters of 3 channel near linear phase filter bank for the cases $a)$ $(\mathcal{J}_f, \mathcal{J}_\infty)$ having a Jordan structure $(3,3)$ $b)$ $(\mathcal{J}_f, \mathcal{J}_\infty)$ having a Jordan structure $([2,1],[2,1])$

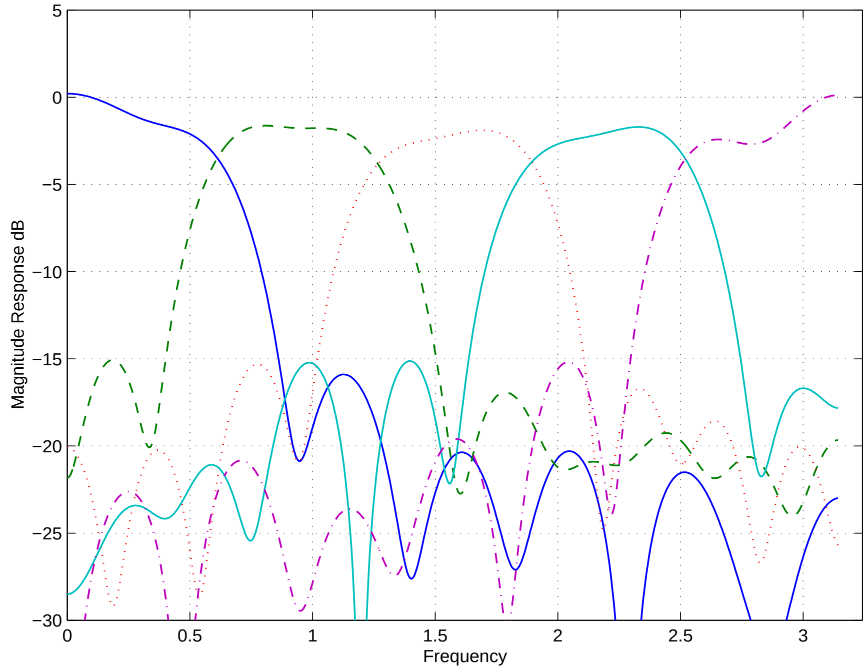


Figure 3.5: Magnitude response of analysis filters of 5 channel low delay filter bank with $\mathcal{J}$ having a Jordan structure $[12,3]$

$h_0(n)$	$h_1(n)$	$h_2(n)$
0.00000000000000	0.00000000000000	0.00000000000000
-0.12503034587400	0.16698913300922	0.12505221460313
0.00000000000000	0.00000000000000	0.00000000000000
0.28955860130023	-0.38673123271382	-0.28960924723400
0.44045166694806	-0.00000591123829	0.44040152046106
0.28955006260033	0.38672914765297	-0.28960127718868
0.00000000000000	0.00000000000000	0.00000000000000
-0.12502665889459	-0.16698823268757	0.12504877316668
0.00000000000000	0.00000000000000	0.00000000000000

Table 3.5: Coefficients for 3 channel near linear phase analysis filters with $\mathcal{J}_f = \mathcal{J}_\infty$ having a Jordan structure $[2,1]$.

$h_0(n)$	$h_1(n)$	$h_2(n)$
0.00000000000000	0.00000000000000	0.00000000000000
0.11306115575886	0.16141960246122	0.11305906522716
-0.11557835446308	-0.00000001225421	0.11557750020210
-0.24353238388602	-0.34769590253577	-0.24352788090561
-0.36016154098692	-0.00000003818619	0.36015887896642
-0.24353331512708	0.34769736931061	-0.24353066907049
-0.11557835446308	-0.00000001225421	0.11557750020210
0.11306158809229	-0.16142028341894	0.11306035964697
0.00000000000000	0.00000000000000	0.00000000000000

Table 3.6: Coefficients for 3 channel near linear phase analysis filters with $\mathcal{J}_f = \mathcal{J}_\infty$ having a Jordan structure 3.

3.5.3 Simulation results for low delay filter bank design

Based on the section (3.4), simulations are done for 5 channel case with order 3 analysis polyphase with $\mathcal{J}$ having a Jordan structure [12, 3]. Here $l_a = 20$ and $l_s = 50$, but as shown in figure (3.5) the filters have a stop band attenuation at $-15dB$. This result justifies the observation done by [32], that low reconstruction delay leads to poor filter responses.

To summerize, as the Jordan chains are taken to be arbitrary (because of difficulty in characterizing $\mathbf{X}_f$ and $\mathbf{X}_\infty$ for $\mathbf{S}_{l-1}$ to be full rank), the free variable space increases. The optimization time is more and very sensitive to initialization. Especially for the near linear phase case, optimization time is more for minimizing the norm of SVD error matrix $\mathbf{D}_{\mathrm{e}\mathbf{V}}$.

Chapter 4

Conclusion and future work

In this dissertation, first the problem of design of M channel causal stable IIR perfect reconstruction filter banks in terms of polyphase matrix was considered. The following approaches were proposed.

- Function of a matrix approach where $\mathbf{A}^*$ is obtained as $f(\mathbf{A})$
- Similarity transformation approach where $\mathbf{A}^*$ is obtained as $\mathbf{T}^{-1}\mathbf{AT}$
- Factorization based approach where $\mathcal{E}(z)$ is expressed as the product of degree 1 factors, with the diagonal matrix case where one of $\mathbf{A}$ and $\mathbf{A}^*$ is diagonal and the other is triangular
- Factorization based approach with the triangular matrix case where $\mathbf{A}$ is upper triangular and $\mathbf{A}^*$ is lower triangular
- Factorization based approach with triangular matrix case and a sequential design method where each degree 1 factor is sequentially optimized.

We derived some theorems, which play crucial role in the design. In the factorization structure, where simple change of sign in the analysis side gives the synthesis was discussed. For this case the poles of analysis and synthesis polyphase matrices are same. The sequential design method is introduced to reduce the optimization burden. The design examples show better results than existing designs.

In function of a matrix approach, there may exist more than one function for the rank criterion to be satisfied. Better filters are obtained when good functions are taken, therefore work may be done in exploring a class of these interpolating functions. In similarity transformation technique, there exists more than one pair of matrices $\{\mathbf{A}, \mathbf{A}\}$ for the rank criterion to be satisfied. Work may be done to characterize these pairs, which demand sound linear algebra concepts.

In the second part, design of M channel FIR perfect reconstruction filter banks in terms of polyphase matrix was considered. The analysis polyphase matrix must be invertible with its determinant being monomial. As the determinant information was known, spectral theory of matrix polynomials was used, and the design problem was viewed as the inverse problem of construction of a regular (invertible) matrix polynomial when spectral data is provided. The following design approaches were proposed.

- **General M channel FIRPRFB design.** It was proved that the delay needed to make the synthesis filters causal is the index of the nil potency of matrix corresponding to the finite spectrum of the matrix polynomial. The length of the synthesis filters can be varied in the design.

- **Near linear phase FIRPRFB design.** Conditions on the spectrum of the matrix polynomial were given, so that it becomes the analysis polyphase matrix of a linear phase filter bank. Near linear phase filters were designed by reducing the error corresponding to one of the linear phase conditions.

- **Low delay FIRPRFB design.** Filter banks that offer minimum delay were designed using construction theorems of unimodular matrix polynomials.

For a given degree of the determinant of analysis polyphase and order of synthesis polyphase, there exists lot of PR structures because more than one Jordan structure may exist. To find a better PR structure among this, simulations have to be done for all cases. Moreover, there is no simple way to initialize the free variables for optimization. Work may be done to reduce the free variables by restricting to some structures.

Appendix-A

If $f(.)$ is any analytic function, and $\mathbf{A}$ is an $m \times m$ matrix then from matrix theory [59], if spectrum of $\mathbf{A}$ is completely known then the values of $f(\lambda)$ on the spectrum of the matrix $\mathbf{A}$ determines $f(\mathbf{A})$. If $\mathbf{A} = \mathbf{P}_{\mathbf{A}}^{-1}\mathcal{J}_{\mathbf{A}}\mathbf{P}_{\mathbf{A}}$ is the Jordan decomposition of $\mathbf{A}$, then function of a matrix is given by the expression

$$f(\mathbf{A}) = \mathbf{P}^{-1}f(\mathcal{J}_{\mathbf{A}})\mathbf{P} \tag{4.1}$$

Let the Jordan matrix $\mathcal{J}_{\mathbf{A}}$ has the form $diag(\mathbf{J}_1, \mathbf{J}_2, \cdots, \mathbf{J}_k)$, where $\mathbf{J}_i$ is the Jordan matrix corresponding the eigen value λ_i, with the structure $diag(\mathbf{J}_i^{a_1}, \mathbf{J}_i^{a_2}, \cdots, \mathbf{J}_i^{a_m})$, where $\forall a_k \in \mathbb{Z}$, $a_1 \geq a_2 \geq \cdots \geq a_m$. $\mathbf{J}_i^{a_j}$ has the structure

$$\mathbf{J}_i^{a_j} = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}_{a_j \times a_j}$$

Now the function of the matrix is $f(\mathcal{J}_{\mathbf{A}}) = diag(f(\mathbf{J}_1), f(\mathbf{J}_2), \cdots, f(\mathbf{J}_k))$ [59] and each term matrix $f(\mathbf{J}_i) = diag(f(\mathbf{J}_i^{a_1}), f(\mathbf{J}_i^{a_2}), \cdots, f(\mathbf{J}_i^{a_m}))$ where the matrix $f(\mathbf{J}_i^{a_j})$ has the structure

$$f(\mathbf{J}_i^{a_j}) = \begin{bmatrix} f(\lambda_i) & \frac{f^1(\lambda_i)}{1!} & \frac{f^2(\lambda_i)}{2!} & \cdots & \frac{f^{a_j-1}(\lambda_i)}{(a_j-1)!} \\ 0 & f(\lambda_i) & \frac{f^1(\lambda_i)}{1!} & \cdots & \frac{f^{a_j-2}(\lambda_i)}{(a_j-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & f(\lambda_i) & \frac{f^1(\lambda_i)}{1!} \\ 0 & 0 & \cdots & 0 & f(\lambda_i) \end{bmatrix}_{a_j \times a_j}$$

So, the values of the function and its higher order derivatives at the spectra of $\mathbf{A}$ completes the matrix $f(\mathbf{A})$. If $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ are the eigen values of $\mathbf{A}$ then $\{f(\lambda_1), f(\lambda_2), \cdots, f(\lambda_k)\}$ are the eigenvalues of $f(\mathbf{A})$. So, if the eigen values of $\mathbf{A}$ and $f(\mathbf{A})$ are known and $\mathbf{A}$ is known, then $f(\mathbf{A})$ can be determined once the interpolating function is known.

Construction of $f(.)$

If $\mathbf{A}$ and eigen values of $\mathbf{A}$ and $f(\mathbf{A})$ are given, $f(.)$ can be constructed as follows. Since λ_i are the eigen values of $\mathbf{A}$, equate $f(\lambda_i)$ to the i^{th} eigen value of $f(\mathbf{A})$. However, there can exists more than one function which passes through the points $\{f(\lambda_1), f(\lambda_2), \cdots, f(\lambda_k)\}$. Now out of these the one with *minimum degree* is constructed, using the function values at points $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$, by *Lagrange interpolation* technique. $f(.)$ is given as $f(\lambda) = \Sigma_{i=0}^{k} f(\lambda_i) L_i(\lambda)$, where $L_i(\lambda)$ is given by

$$L_i(\lambda) = \frac{(\lambda - \lambda_1)\cdots(\lambda - \lambda_{i-1})(\lambda - \lambda_{i+1})\cdots(\lambda - \lambda_k)}{(\lambda_i - \lambda_1)\cdots(\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1})\cdots(\lambda_i - \lambda_k)}$$

The general solutions for the functions of type $f(.)$, that require information regarding the derivatives of $f(.)$ at the spectral points, are exhaustively discussed in [59].

EXAMPLE - 1 If λ_1 and λ_2 are distinct eigenvalues and $f(.)$ be an analytic function that maps $\lambda_1 \overset{f(.)}{\to} \lambda_1$ and $\lambda_2 \overset{f(.)}{\to} 2\lambda_2 - \lambda_1$, such that $\lambda_2 \neq \lambda_1$ then $f(.)$ can be constructed as follows:

Here derivatives of $f(.)$ at the points λ_1 and λ_2 are not given, so an analytic function with minimum degree can be constructed using Lagrange interpolation.

$$\begin{aligned} f(\lambda) &= f(\lambda_1)\frac{(\lambda - \lambda_2)}{(\lambda_1 - \lambda_2)} + f(\lambda_2)\frac{(\lambda - \lambda_1)}{(\lambda_2 - \lambda_1)} \\ &= \lambda_1 \frac{(\lambda - \lambda_2)}{(\lambda_1 - \lambda_2)} + (2\lambda_2 - \lambda_1)\frac{(\lambda - \lambda_1)}{(\lambda_2 - \lambda_1)} \\ &= 2\lambda - \lambda_1 \end{aligned}$$

Bibliography

[1] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice Hall, Englewood Cliffs, New Jersey, 1992.

[2] M. J. T. Smith and S. L. Eddins, "Analysis/Synthesis Techniques for Subband Image Coding," *IEEE Transactions on Acoustics, Speech and Signal Processing*, Vol.38, No.8, pp. 1446-1456, Aug. 1990.

[3] P. P. Vaidyanathan, "Multirate digital filters, filter banks, polyphase networks and applications: A tutorial," *Proc. IEEE*, Vol. 78, pp. 56-91, Jan. 1990.

[4] Z. Doganata and P. P. Vaidyanathan, "Minimal Structures for the Implementation of Digital Rational Lossless Systems," *IEEE Transactions on Acoustics, Speech and Signal Processing*, Vol. 38, No. 12, pp. 2058-2074, Dec. 1990.

[5] S. Basu, C. H. Chiang and Han Mook Choi, "Wavelets and Perfect Reconstruction Subband Coding with Causal Stable Fiters," *IEEE Transactions on Circuits and Systems II*, Vol. 42, No. 1, pp.24-39.

[6] S. M. Phoong, C. W. Kim and P. P. Vaidyanathan, " A New Class of Two-Channel Bi orthogonal Fiter Banks and Wavelet Bases," *IEEE Transactoins on Signal Processing*, Vol. 43, No. 5, pp. 649-665, Oct. 1995.

[7] M. M. Ekanayake and K. Premaratne, "Two-Channel IIR QMF Banks with Approximately Linear-Phase Analysis and Synthesis Filters," *IEEE Transactions on Signal Processing*, Vol. 43, pp. 2313-2322, Oct-1995.

[8] F. Argenti, V. Cappellini, A. Sciorpes and A. N. Venetsanopolous, " Design of IIR Linear-Phase QMF Banks Based on Complex Allpass Sections," *IEEE Transactions on Signal Processing*, Vol. 44, No. 5, pp. 1262-1267, May 1996.

[9] M. Okuda, T. Fukuoka, M. Ikehara and S. Takahashi, "Design of causal IIR filter banks satisfying perfect reconstruction," *IEEE symposium on circuits and systems* 1997, pp. 2433-2436.

[10] X. Zhang, T. Yoshikawa, "Design of orthonormal IIR wavelet filter banks usign allpass filters," *Signal Processing* 1999, Vol.78, No. 1, pp. 91-100.

[11] A. K. Djedid, "Design of Stable, Causal, Perfect Reconstruction, IIR Uniform DFT Fitler Banks," *IEEE Transactions on Signal Processing*, vol. SP-48, pp. 1110-1119, Apr-2000.

[12] M. J. T. Smith and T. P. Barnwell, III,"Exact Reconstruction Techniques for Tree Structured Subband Coders," *IEEE Transactions on Acoustics, Speech and Signal Processing*, Vol. 34, No. 6, Jun. 1986, pp. 434–441.

[13] P. P. Vaidyanathan, "Theory and Design of M-Channel Maximally Decimated Quadrature Mirror Filters with Arbitrary M having Perfect Reconstruction Property," *IEEE Transaction on Acoustics, Speech and Signal Processing*, Vol. 35, No. 4, pp. 476–492, April. 1987.

[14] M. Vetterli, "Filter Banks Allowing Perfect Reconstruction Property," *Signal Processing*, Vol. 10, No. 3, Apr. 1986, pp. 219–244.

[15] M. Vetterli and D. LeGall, "Perfect Reconstruction FIR Filter Banks: Some Properties and Factorizations," *IEEE Transactions on Acoustics, Speech amd Signal Processing*, vol. 37, no. 7, pp. 1057-1071, July 1989.

[16] G. Schuller and M. J. T. Smith, "A New Algorithm for Efficient Low Delay Filter Bank Design," *IEEE ICASSP*, Detroit, Michigan, May 9-12, 1995.

[17] H. Ochi, M. Ohta and S. Kinjo, “Linear Programming Design of Two-Channel Perfect-Reconstruction Biorthogonal Filter Banks—Linear Phase and Low-delay,” *IEEE ISCAS*, 1997, pp. 2401-2404.

[18] M. R. K. Khansai and E. Dubois, “Pade Table, Continued Fraction Expansion and Perfect Reconstruction Filter Banks,” *IEEE Transactions on Signal Processing*, Vol. 44, No. 8, Aug. 1996, pp. 1955–1963.

[19] J. Huang and G. Gu, “A Direct Approach to the Design of QMF Banks via Frequency Domain Optimization,” *IEEE Transactions on Signal Processing*, Vol. 46, No. 8, pp. 2131-2138, Aug. 1998.

[20] T. Q. Nguyen and P. P. Vaidyanathan, “Two-Channel Perfect Reconstruction FIR QMF Structures which Yield Linear Phase FIR Analysis and Synthesis Filters,” *IEEE Transactions on Acoustics, Speech and Signal Processing*, Vol. 37, No. 5, May 1989, pp. 676–690.

[21] B. R. Horng and A. N. Willson Jr., “The Design of Multiplierless Two-Channel Linear Phase FIR Filter Banks with Application to Image Subband Coding,” *Proc. of IEEE ISCAS*, 1990, Vol. 1, pp. 650-653.

[22] H. Xu, W. S. Lu and A. Antoniou, “Design of Perfect Reconstruction QMF Banks by A Null-Space Projection Method,” *Proc. ISCAS*, 1995, pp. 965-968.

[23] D. Pinchon, “A Complete Canonical Structure for Even Length Linear Phase PR Filter Banks,” *Proc. of IEEE ISCAS*, 1997, Vol. 4, pp. 2385–2388.

[24] D. Pinchon and P. Siohan, “Analysis, Design and Implementation of Two-Channel Linear Phase Filter Banks: A New Approach,” *IEEE Transaction on Signal Processing*, Vol. 46, No. 7, Jul. 1998, pp. 1814–1826.

[25] C. K. Chen, “A Novel design technique for two-channel linear-phase perfect-reconstruction FIR fiter banks,” *Signal Processing*, Vol. 58, pp. 261-271, 1997.

[26] C. K. Goh and Y. C. Lim, “Novel Approach for the Design of Two-Channel Perfect Reconstruction Linear Phase FIR Filter Banks,” *IEEE Transaction on Circuits and Systems-Part II: Analog and Digital Signal Processing*, Vol. 45, No. 8, Aug. 1998, pp. 1141–1146.

[27] A. Muthuvel and A. Makur, “Design of two-channel linear phase FIR PR filter banks with even length filters using convolution matrices,” *IEEE Transactions on Circuits and Systems II*, Vol.47, No. 12, pp. 1413-1418, Dec. 2000.

[28] T. Q. Nguyen and P. P. Vaidyanathan, “Maximally Decimated Perfect Reconstruction FIR Filter Banks with Pairwise Mirror Image Analysis (and Synthesis) Frequency Responses,” *IEEE Transactions on Acoustics, Speech and Signal Processing*, Vol. 36, No. 5, pp. 693–706, May 1988.

[29] Z. Doganata, P. P. Vaidyanathan and T. Q. Nguyen, “General Synthesis Procedures for FIR Lossless Transfer Matrices for Perfect Reconstruction Multirate Filter Bank Applications,” *IEEE Transactions on Acoustics, Speech and Signal Processing*, Vol. 36, No. 10, pp. 1561–1574, Oct. 1988.

[30] P. P. Vaidyanathan, T. Q. Nguyen, Z. Doganata and T. Saramaki, “Improved Techniques for Design of Perfect Reconstruction FIR QMF Banks with Lossless Polyphase Matrices,” *IEEE Transactions on Acoustics, Speech and Signal Processing*, Vol. 37, No. 7, pp. 1042–1056, Jul. 1989.

[31] X. G. Xia and B. W. Suter, “FIR Paraunitary Filter Banks Given Several Analysis Filters: Factorization and Constructions,” *IEEE Transactions on Signal Processing*, Vol. 44, No. 3, pp. 720–723, Mar. 1996.

[32] K. Nayebi, T. P. Barnwall, III and M. J. T. Smith, “Time Domain Filter Bank Analysis: A New Design Theory,” *IEEE Transactions on Signal Processing*, Vol. 40, No. 6, pp. 1412-1429, June 1992.

[33] P. P. Vaidyanathan, “How to Capture All FIR Perfect Reconstruction QMF Banks with Unimodular Matrices ?,” *Proc. of IEEE ISCAS*, 1990, Vol. 3, pp. 2030–2033.

[34] P. P. Vaidyanathan and T. Chen, "The Role of Anticausal Inverses in Multirate Filter Banks—Part I: System Theoretic Fundamentals," *IEEE Transactions on Signal Processing*, Vol. 43, No. 5, pp. 1090–1102, May 1995.

[35] P. P. Vaidyanathan and T. Chen, "The Role of Anticausal Inverses in Multirate Filter Banks—Part II: The FIR Case, Factorizations and Biorthogonal Lapped Transform," *IEEE Transactions on Signal Processing*, Vol. 43, No. 5, pp. 1103–1115, May 1995.

[36] A. Muthuvel and A. Makur, "Eigen structure approach for characterization of FIR PR filter banks with order one polyphase," *IEEE Transactions on Signal Processing*, Vol. 49, No. 10, pp. 2283-2291, Oct. 2001.

[37] E. Kofidis, S. Theodoridis and N. Kalouptsidis, "On the Perfect Reconstruction Problem in N-Band Multirate Maximally Decimated FIR Filter Banks," *IEEE Transactions on Signal Processing*, Vol. 44, No. 10, pp. 2413–2423, Oct. 1996.

[38] A. K. Soman, P. P. Vaidyanathan and T. Q. Nguyen, "Linear Phase Paraunitary Filter Banks: Theory, Factorization and Design," *IEEE Transactions on Signal Processing*, Vol. 41, No. 12, Dec. 1993, pp. 3480–3496.

[39] c. W. Kok, M. Ikehara and T. Q. Nyugen, "Design and factorization of FIR paraunitary filter banks given several analysis fitlers," *IEEE Transactions on Signal Processing*, Vol. 48, No. 7, pp. 2157-2161, Jul. 2000.

[40] T. Nagai, T. Fuchie and M. Ikehara, "Design of Linear Phase M-Channel Perfect Reconstruction FIR Filter Banks," *IEEE Transactions on Signal Processing*, Vol. 45, No. 9, pp. 2380-2387, Sep. 1997.

[41] L. Chen, K. P. Chan and T. Q. Nguyen, "A Generalized Algorithm for Linear Phase Paraunitary Filter Banks," *IEEE Transactions on Signal Processing*, Vol. 46, No. 4, Apr. 1998, pp. 1154–1158.

[42] T. Q. Nguyen and P. P. Vaidyanathan, "Structures for M-Channel Perfect Reconstruction FIR QMF Banks which Yield Linear Phase Analysis Filters," *IEEE Transactions on Acoustics, Speech and Signal Processing*, Vol. 38, No. 3, Mar. 1990, pp. 433-446.

[43] T. D. Tran and T. Q. Nguyen, "On M-Channel Linear-Phase FIR Filter Banks and Applications in Image Compression," *IEEE Transactions on Signal Processing*, Vol. 45, No. 9, Sep. 1997, pp. 2175–2187.

[44] S. Basu and H. M. Choi, "Hermite Like Reduction Method for Design of Perfect Reconstruction Multiband Linear Phase Filter Banks," *Proc. of IEEE ISCAS*, 1995, Vol. 1, pp. 597–600.

[45] S. Basu and H. M. Choi, "Hermite-Like Reduction Method for Linear Phase Perfect Reconstruction Filter Bank Design," *Proc. of IEEE ICASSP*, 1995, Vol. 2, pp. 1512–1515.

[46] G. Schuller and W. Sweldens, "Filter bank design using nilpotent matrices," *Proc. 1999 IEEE Workshop on applications of signal processing to Audio and Acoustics*, 1999, pp. 17-20.

[47] A. Hjrungnes, H. Coward and T. A. Ramstad, "Minimum Mean Square Error FIR Filter Banks with Arbitrary Filter Lengths," *Proc. of the IEEE International Conference on Image Processing*, Kobe, Japan, pp. 25-28, Oct. 1999.

[48] I. Gohberg, L. Rodman and P. Lancaster, *Matrix Polynomials*, Academic Press, New York, 1982.

[49] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, 1996.

[50] I. Gohberg. *Orthogonal Matrix-Valued Polynomials & Applications*, Birkhauer Boston, 1988.

[51] T. Kailath, *Linear Systems*, Prentice Hall, Englewood Cliffs, New Jersey, 1980.

[52] P. Lancaster, *Lambda Matrices and Vibrating Systems*, Oxford, Pergamon Press, 1966.

[53] M. V. Keldysh, "On the Completeness of the Eigenfunctions of some classes of Non-selfadjoint Linear Operators", Russian Mathematical Surveys, vol 24, No. 4, pp. 15-44, 1971.

[54] P. Lancaster, "A Fundamental Theorem of Lambda Matrices with Applications - I,II", Linear Algebra and Applications, vol. 18. pp. 189-211(I), pp. 2313-222(II), 1977.

[55] I. Gohberg, L. Rodman and P. Lancaster, "Spectral Analysis of Matrix Polynomials - I,II," Linear Algebra and Applications, vol. 20, pp. 1-44(I), 1978 and vol. 21, pp. 65-88(II), 1978.

[56] I. Gohberg, L. Rodman and P. Lancaster, *Invariant Subspaces of Matrices with Applications*, New York, John Wiley, 1986.

[57] Liansheng Tan and A. C. Pugh, "A note on the solution of regular PMDs", Preprint 99-11, Department of Mathematical Sciences Preprint Archive, Loughborough University, Loughborough, Leicestershire, U.K.

[58] A. I. G. Vardulakis, *Linear Multivariable Control*, John Wiley & Sons, Chichester, England, 1991.

[59] F. R. Gantmacher, *The Theory of Matrices-I,II*, Chelsea, New York, 1959.

[60] G. C. Verghese, B. C. Lévy and T. Kailath, "A generalized state-space for singular system," *IEEE Transanctions on Automatic Control* , AC-26, pp. 811-831, Aug. 1981.

[61] C. T. Chen, *Linear System Theory and Design*, Oxford University Press, 3rd Edition, 1999.

[62] T. Beelen and P. Van Dooren, "A pencil approach for embedding a polynomial matrix into a unimodular matrix," SIAM J. Matrix Anal. Appl., vol. 9, pp. 77-89, Jan. 1988.