A Skew-Normal Canonical Model for Statistical Static Timing Analysis

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***Abstract*—The use of quadratic gate delay models and arrival times result in improved accuracies for parameterized block based statistical static timing analysis (SSTA). However, the computational complexity is significantly higher. As an alternate to this, we propose a canonical model based on skew-normal random variables (SN model). This model is derived from the quadratic canonical models and can take into account the skewness in the gate delay distribution as well as the nonlinearity of the MAX operation. Based on conditional expectations, we derive analytical expressions for the moments of the MAX operator and the tightness probability that can be used along with the SN canonical models. The computational complexity for both timing and criticality analysis is comparable to SSTA using linear models. There is a two to three orders of magnitude improvement in the run-time as compared to the quadratic models. Results on ISCAS benchmarks show that the SN models have lower variance error than the quadratic model, but the error in the third moment is comparable to that of the semi-quadratic model.**

I. Introduction

Integrated circuits today have significant process parameter variations that affect gate delays and hence operating frequency of the chip. A timing analysis technique that takes these variations into account is statistical static timing analysis (SSTA). A computationally efficient method for SSTA is the parameterized block-based SSTA, which involves a PERT-like traversal of the circuit graph [1], [2]. A canonical delay model is used to express both the gate delays and the arrival times (ATs). In its simplest form, the probability distribution function (PDF) of the process parameters is assumed to be Gaussian and the gate delays and ATs are assumed to be linear functions of these process parameters.

However, gate delays are not always well modelled as linear functions of process parameter variations. In addition, the MAX operation that is used to propagate the arrival times is inherently non-linear. One way to account for both these nonlinearities is to use quadratic canonical models. These models match the skewness in addition to the mean, standard deviation and correlations. While the overall errors do reduce, it is at the cost of significant additional computational complexity [6], [7], [8], [9]. Other models that take into account the skewness in the arrival times due to the MAX operator have been proposed in [4] and [5]. In [4], the authors attempt to reduce the error in the standard deviation of the circuit delay (and hence the yield) by assuming that the arrival times have a joint skew normal distribution [4]. They obtain the tightness probability and the moments of this distribution using a procedure similar to the one used by Clark. These are then used to obtain the coefficients of the linear canonical model. The method proposed in [5] is a mixed parameterized SSTA and Monte Carlo technque, wherein limited Monte Carlo sampling is used when the skewness in the arrival times exceeds a threshold value. Both these models do not take into account the non-linearities in the gate delay.

Essentially we require a canonical form that can take both the non-linearities into account, while maintaining the computational tractability of the linear model. This is the motivation for skew-canonical model proposed by us in [18]. This canonical form is a merger of the standard linear canonical model for the delay and the standard skew-normal representation proposed in [10]. It is a linear function of the principal components and contains the modulus of another standard normal random variable that is common to all gates in the circuit. Like the quadratic model, it is possible to match the mean, standard deviations and skewness. It is a promising model as the run-times required for SSTA are only marginally higher than the linear model.

However, a natural extension of the linear gate delay model is the quadratic delay model that is obtained using a Taylor series expansion. The simplest method to obtain a skew-canonical model for the delay would be to derive it from the quadratic delay model by matching moments. Therefore, ideally we require a skew-canonical model that has sufficient degrees of freedom to match the first three moments and also preserve the correlations with each principal component. While it is relatively simple to get a model that matches the first three moments, it becomes difficult (if not impossible) to also preserve all the correlations. A similar problem arises when the result of the MAX operator is to be expressed in the canonical form. The sensitivities are scaled or an additional independent component is added to match the variance, both of which effectively result in an error in the correlation between gates. Following a similar approach, we derive an equivalent skew-canonical model that matches the first three moments of the quadratic delay model exactly, while admitting an error in the correlation between gates. We show that this error is marginal for values of process parameter variations typically seen today. Based on conditional expectations, we derive analytical expressions for the moments of the MAX operator and the tightness probability that can be used along with the SN canonical models. This turns out to be a much simpler method than the method used in [4]. As a result, the computational complexity of using this model is only marginally higher than the linear model based on Clark's approximations, thus allowing for efficient computation of node criticalities. Typical run-time improvements are around two to three orders of magnitude as compared to quadratic delay models. This makes it more suitable for use in timing optimizations than the quadratic delay models.

The paper is organized as follows. Section II contains a brief review of the quadratic canonical models. Section III introduces skew normal representations and contains some properties that are used for SSTA. In section IV we derive equivalent skew-canonical models for the quadratic and semi-quadratic delay models. In section V, we derive analytical expressions for the tightness probability and the moments of the MAX of two skew normal variables. We also describe the moment matching procedure to obtain the coefficients of the canonical form. Section VI describes the procedure to evaluate the node criticalities using the quadratic and the equivalent skew canonical models. Section VII contains the results and section VIII, the conclusions.

II. Quadratic Canonical models

The quadratic canonical model models both the gate delay and the ATs as a quadratic function of the process paramter variations. If these variations are represented in terms of the principal components, the arrival time at node i can be written as [6], [7], [8], [14]

$$A_i = a_{i0} + \mathbf{A}\xi + \xi^T \mathbf{B}\xi + a_{\eta_i}\eta_i \tag{1}$$

Here ξ is a vector containing the principal components, $\mathbf{A}$ and $\mathbf{B}$ are matrices containing the sensitivities to the linear and quadratic components, η_i is the random variable representing an independent source of variation associated with A_i and a_{i0} is the mean AT (delay). The SUM operation continues to be straightforward since each individual component can be added. The MAX operation is more involved since we now need to compute the moments of the MAX of two quadratic functions of random variables. The actual moment matching involves numerical integrations and convolutions (using FFT) [6] or approximation using a Fourier series along with a table lookup [7] or fitting of a quadratic model along with moment matching [8]. There is a significant increase in computational complexity and in most cases, the cross terms are ignored to speedup computations. However, the quadratic delay models in [7], [8] are very general models that can handle non-Gaussian process parameter variations.

Of these methods, computationally the most efficient is the method proposed in [8]. It is a two step process that involves fitting of the MAX of two random variables to a quadratic function of their difference, followed by reconstruction of the quadratic canonical form through moment matching. The computational complexity of reconstruction is $O(n^3)$, where n is the number of principal components. If the cross terms in the quadratic canonical form are ignored, the semi-quadratic canonical form is obtained. This model has $O(n)$ computational complexity for the reconstruction.

III. Skew normal random variables

A standard skew normal random variable has a PDF given by [10]

$$f(z;\lambda) = 2\phi(z)\Phi(\lambda z) \tag{2}$$

where $\phi(z)$ and $\Phi(z)$ are the PDF and the CDF of the standard normal random variable. The parameter λ determines the skewness of the distribution. The moments can be adjusted using a location and a scale parameter. The CDF is given by

$$F(z;\lambda) = \Phi(z) - 2T(z;\lambda), \qquad \lambda > 0 \tag{3}$$

where $T(z;\lambda)$ is the Owen's T-function. Standard computer routines are available to compute this function. The properties of the Owen's T-function can be used to get the CDF for $\lambda < 0$ (negative skewness).

There are several representations of the skew normal random variable that have this PDF [10], [15]. The representation that we are interested in is

$$Z = \alpha + \beta X \tag{4}$$

where

$$X = \frac{\lambda}{\sqrt{1+\lambda^2}}|U| + \frac{1}{\sqrt{1+\lambda^2}}V \tag{5}$$

where U and V are independent standard normal random variables. The moments as well as several interesting properties of the skew normal random variable can be found in [10], [15]. The properties that are of interest to us are

1. The sum of a skew-normal and a normal random variable is also a skew normal random variable.
2. If $X_1 = a_1V_1 + b_1|U|$ and $X_2 = a_2V_2 + b_2|U|$, then $X_3 = X_1 + X_2 = a_3V_3 + (b_1+b_2)|U|$, where $a_3 = \sqrt{a_1^2 + a_2^2}$. V_1, V_2, V_3 and U are independent standard normal random variables. X_3 is also a skew-normal random variable.
3. The conditional distribution $f(x|u)$ is a normal distribution.
4. $X_4 = X_1 - X_2$ is therefore also a skew-normal random variable and $P(X_1 > X_2)$ can be found from the CDF of the skew-normal distribution [15], [16].
5. The maximum skewness of the skew-normal random variable is 0.995272.

These properties also hold true when the skew-normal random variables have an arbitrary location and scale factor.

IV. An equivalent skew-canonical model

The quadratic model described in section II can be expanded as:

$$x_q = x_o + \sum_{i=1}^{n} a_{x_i}\xi_i + \sum_{i=n+1}^{n+m} a_{x_i}\xi_i + s_{x\eta}\eta_x + \sum_{i=1}^{n+m} b_{x_i}\xi_i^2 + \sum_{i=1}^{n+m}\sum_{j=i+1}^{n+m} c_{x_{ij}}\xi_i\xi_j \tag{6}$$

where

$$c_{x_{ij}} = \begin{cases} 2b_{x_i}b_{x_j}, & \text{if } \xi_i,\ \xi_j \text{ belong to same process parameter} \\ 0, & \text{otherwise} \end{cases}$$

The variations in delay are caused by both inter- and intra-die process variations. Typically, both these variations are small and can be decoupled [17]. In the above equation, the principal components $\xi_1 \cdots \xi_n$ model the (correlated) intra-die variations. The m inter-die process variations are modelled by the components ξ_{n+1} to ξ_{n+m}, which are common to all gates within a die.

The mean and variance of the PDF of the gate delay can be written as follows.

$$\mu_q = x_o + \sum_{i=1}^{n+m} b_{x_i} \tag{7}$$

$$\begin{aligned} \sigma_q^2 &= \sum_{i=1}^{n} a_{x_i}^2 + s_{x\eta}^2 + \sum_{i=n+1}^{n+m} a_{x_i}^2 \\ &\quad + 2\sum_{i=1}^{n+m} b_{x_i}^2 + \sum_{i=1}^{n+m}\sum_{j=i+1}^{n+m} c_{x_{ij}}^2 \\ &= \sigma_{intra}^2 + \sigma_{inter}^2 + \sigma_{quad}^2 \end{aligned} \tag{8}$$

Here σ_{intra}^2, σ_{inter}^2 and σ_{quad}^2 denote the contributions of the intra-die, inter-die and quadratic variations to the variance of the gate delay. The third central moment is given by

$$\begin{aligned} \kappa_q &= \sum_{i=1}^{n+m} 8b_{x_i}^3 + \sum_{i=1}^{n+m}\sum_{j=i+1}^{n+m} 6c_{x_{ij}}^2(b_{x_i} + b_{x_j}) \\ &\quad + \sum_{i=1}^{n+m} 6a_{x_i}^2 b_{x_i} + \sum_{i=1}^{n+m}\sum_{j=i+1}^{n+m} 6c_{x_{ij}}a_{x_i}a_{x_j} \\ &\quad + 6\sum_{k=1}^{n}\sum_{j=k+1}^{n}\sum_{i=j+1}^{n} c_{x_{ij}}c_{x_{jk}}c_{x_{ik}} \end{aligned} \tag{9}$$

With this model, the covariance between gates x and y is given by

$$\begin{aligned} C_q(x,y) &= \sum_{i=1}^{n} a_{x_i}a_{y_i} + \sum_{i=n+1}^{n+m} a_{x_i}a_{y_i} \\ &\quad + 2\sum_{i=1}^{n+m} b_{x_i}b_{y_i} + \sum_{i=1}^{n+m}\sum_{j=i+1}^{n+m} c_{x_{ij}}c_{y_{ij}} \end{aligned} \tag{10}$$

Based on the representation of the skew-normal random variable, we propose a modification of the linear canonical form to a skew-canonical form as follows:

$$\begin{aligned} x_{sn} &= d_x + \sum_{i=1}^{n} a_{x_i}\xi_i + s_{x_\eta}\eta_x \\ &\quad + \sum_{i=1}^{m} s_{xw_i}\xi_{n+i} + a_{x_{n+m+1}}\xi_{n+m+1} + q_x|z| \end{aligned} \tag{11}$$

In this model, the sensitivities to the components corresponding to the linear intra-die variations $(a_{x_1} \cdots a_{x_n})$ as well as the independent component s_{x_η} are kept the same as the quadratic model. The sensitivities to the components corresponding to inter-die variations as well as the coefficients of the two random variables ξ_{n+m+1} and $|z|$ are determined by matching moments of the quadratic and skew-canonical model. The motivation for this form of the model will become clear later on in this section. The mean, variance and the third central moment for this model can be written as

$$\mu_{sn} = d_x + q_x\sqrt{\frac{2}{\pi}} \tag{12}$$

$$\sigma_{sn}^2 = \sum_{j=1}^{n} a_{x_i}^2 + s_{x_\eta}^2 + \sum_{i=1}^{m} s_{xw_i}^2 + a_{x_{m+n+1}}^2 + q_x^2\left(1 - \frac{2}{\pi}\right) \tag{13}$$

$$\kappa_{sn} = q_x^3 F, \quad F = \sqrt{\frac{2}{\pi}}\left(\frac{4}{\pi} - 1\right) \tag{14}$$

Since the third central moment of the skew-canonical delay PDF depends only on q_x, we can equate (9) and (14) to obtain its value as

$$q_x = \kappa_q^{\frac{1}{3}} F^{-\frac{1}{3}} \tag{15}$$

Using equations (7) and (12), the mean in the two cases can be matched by setting $d_x = \mu_q - q_x\sqrt{\frac{2}{\pi}}$. As seen from equations (8) and (13), both q_x and the quadratic terms also contribute to the variance. If the two models are exactly equivalent, both these contributions will be exactly the same and the two variances will automatically match. However, this is not the case in practice and scaling/introduction of an additional random variable is required to match the variance. A similar situation arises when moment matching is used to obtain the result of the MAX operator in linear SSTA and scaling of linear sensitivities [9] or an additional independent random variable [2] is used to match the variance. Since the contribution of q_x to the variance turns out to be larger than the contribution of the quadratic terms, introduction of an additional random variable is not an option. The sensitivities a_{x_i} can be scaled to match the variance, but this needs to be done carefully as is seen in the following example. Consider N identical gates in cascade. For simplicity, we assume that the intra-die variations are entirely uncorrelated and ignore the contributions of the quadratic terms to the variance. The path delay variance is therefore $\sigma_{P_q}^2 = N\sigma_{intra}^2 + N^2\sigma_{inter}^2$. With the SN model, assume that a fraction f of this variance is due to q_x i.e., $q_x^2\left(1 - \frac{2}{\pi}\right) = f\sigma^2$. To get the same gate delay variance with the SN model, we must have $\sigma_{inter}^2 + \sigma_{intra}^2 = (1-f)\sigma^2$ i.e., the linear sensitivities need to be scaled by a factor $\sqrt{1-f}$. With this the variance of the gate delay matches exactly, but the path delay variance using SN models will now be given by

$$\begin{aligned}\sigma_{P_{SN}}^2 &= N(1-f)\sigma_{intra}^2 + N^2(1-f)\sigma_{inter}^2 + N^2 f\sigma^2 \\ &= (N(1-f) + N^2 f)\sigma_{intra}^2 + N^2 f\sigma_{inter}^2\end{aligned}$$

since the random variable z is also common to all gates in the circuit. Therefore, one could potentially get a large error in the variance of the circuit delay even though the variance of the gate delay matches exactly. A solution to this problem is to scale the sensitivities due to inter-die variations alone. If $q_x^2\left(1 - \frac{2}{\pi}\right) = f_1\sigma_{inter}^2$ and the linear inter-die sensitivities are scaled by $\sqrt{1-f_1}$, it can be easily verified that both the gate and path delay variances obtained using the two models match exactly. Therefore, in the SN canonical model the coefficients s_{xw_i} are written as

$$s_{xw_i} = a_{x_{n+i}}\sqrt{1 - \frac{q_x^2}{\sigma_{inter}^2}\left(1 - \frac{2}{\pi}\right)}$$

All the quadratic terms are absorbed into the additional component ξ_{n+m+1} and the corresponding sensitivity is given by

$$a_{x_{m+n+1}} = \sqrt{2\sum_{i=1}^{n+m} b_{x_i}^2 + \sum_{i=1}^{n+m}\sum_{j=i+1}^{n+m} c_{x_{ij}}^2}$$

In this model, the covariance between two gates x and y is given by

$$\begin{aligned}C_{sn}(x,y) = &\sum_{i=1}^{n} a_{x_i}a_{y_i} + q_x q_y\left(1 - \frac{2}{\pi}\right) \\ &+ \sum_{i=1}^{m} s_{xw_i}s_{yw_i} + a_{x_{m+n+1}}a_{y_{m+n+1}}\end{aligned} \tag{16}$$

Comparing equations (10) and (16), it is seen that the first term is identical, but there is an error that depends on the magnitude of the last three terms in equation (16) relative to the covariance due to the linear inter-die $\left(\sum_{i=n+1}^{n+m} a_{x_i}a_{y_i}\right)$ and the quadratic component of the variation. For typical process variations seen today, the contribution of the quadratic components to the covariance is quite small. Noting that $F^{-\frac{2}{3}}\left(1 - \frac{2}{\pi}\right) \approx 1$, the error can be approximated as

$$\sum_{i=n+1}^{n+m} a_{x_i}a_{y_i}\left[1 - \frac{1}{\rho_{inter-die}}\frac{(\gamma_x\gamma_y)^{\frac{1}{3}}\sigma_x\sigma_y}{x_w y_w} - \sqrt{\left(1 - \frac{\gamma_x^{\frac{2}{3}}\sigma_x^2}{x_w^2}\right)\left(1 - \frac{\gamma_y^{\frac{2}{3}}\sigma_y^2}{y_w^2}\right)}\right] \tag{17}$$

where $\rho_{inter-die} = \frac{\sum_{i=n+1}^{n+m} a_{x_i}a_{y_i}}{x_w y_w}$ is the correlation coefficient between gate delays due to the inter-die variation alone. We expect $\rho_{inter-die}$ to be close to one. Since typically the skewness of the gate delay PDFs is small [8] and this error also turns out to be quite small. Practically the error was found to be less than 2% in all the benchmarks. Note that this model automatically reduces to the linear model when the PDF of the gate delay has zero skewness.

A. Skew-canonical equivalent of semi-quadratic models

The semi-quadratic delay model is a simplification of the quadratic delay model achieved by ignoring the cross-components. The delay model is thus

$$x_{sq} = x_o + \sum_{i=1}^{n} a_{x_i}\xi_i + \sum_{i=n+1}^{n+m} a_{x_i}\xi_i + \sum_{i=1}^{n+m} b_{x_i}\xi_i^2 + x_\eta \eta_x \tag{18}$$

The equivalent model can be written as

$$x_{sn} = d_x + \sum_{i=1}^{n} a_{x_i}\xi_i + s_{xw}w + \sqrt{2}\sum_{i=n+1}^{2n+m} b_{x_{i-n}}\xi_i + q_x|z| + x_\eta \eta_x \tag{19}$$

where

$$s_{xw} = \sqrt{\sum_{i=n+1}^{n+m} a_{x_i}^2 - q_x^2\left(1 - \frac{2}{\pi}\right)}$$

Therefore, here we introduce additional "principal components" with sensitivities corresponding to the diagonal elements of $\mathbf{B}$. The reason is as follows. The covariance in the two cases are given by

$$C_{sq}(x,y) = \sum_{i=1}^{n} a_{x_i}a_{y_i} + \sum_{i=n+1}^{n+m} a_{x_i}a_{y_i} + 2\sum_{i=1}^{n+m} b_{x_i}b_{y_i} \tag{20}$$

$$C_{sn}(x,y) = \sum_{i=1}^{n} a_{x_i}a_{y_i} + s_{xw}s_{wy} + 2\sum_{i=1}^{n+m} b_{x_i}b_{y_i} + q_x q_y\left(1 - \frac{2}{\pi}\right) \tag{21}$$

The errors in this case are similar to that of the quadratic case. However, it can be shown that the two covariances match exactly if the sensitivities (both linear and quadratic) are a linear function of the gate delay. The details are included in Appendix A.

V. SUM and MAX operations

The two operations that are required for SSTA are the SUM and MAX operation. If x_{sn} and y_{sn} are two delay variables represented in the skew-canonical form and $S_{sn} = x_{sn} + y_{sn}$, the coefficients of S_{sn} can be computed as,

$$\begin{aligned} S_{sn} &= d_s + \sum_{i=1}^{n} a_{si}\xi_i + s_{s\eta}\eta_s + s_{sw}w + q_s|z| \\ d_s &= d_x + d_y \\ a_{si} &= a_{xi} + a_{yi} \quad \text{for} \quad i = 1, 2\cdots n \\ s_{s\eta} &= \sqrt{s_{x\eta}^2 + s_{y\eta}^2} \\ s_{sw} &= s_{xw} + s_{yw} \\ q_s &= q_x + q_y \end{aligned}$$

We wish to represent the result of the MAX operation in the skew-canonical form. As in linear SSTA, we use a moment matching technique. This requires the moments of the joint PDF of two skew normal random variables. The method in [4] uses a technique similar to the one proposed by Clark for normal random variables. This turns out to be complicated.

Instead, we can use the fact that the conditional distributionof the skew normal model, given z, is a normal distribution. This makes it possible to use Clark's formulas to get the conditional moments. Since $z \sim N(0,1)$, the actual moments can be found by integrating the conditional moments along with the PDF of the standard normal. Analytical forms of all moments can be obtained in terms of the Owen's T function and the CDF of the standard normal. We illustrate this for the first moment. If $p_{sn} = MAX(x_{sn}, y_{sn})$, the conditional expectation given the value of z is obtained from Clark's formula as

$$E\{p_{sn}|Z\} = \mu_{p_{sn}|Z} = \mu_{x_{sn}|Z}T_{x_{cond}} + \mu_{y_{sn}|Z}(1 - T_{x_{cond}}) + \sigma_{(x_{sn}-y_{sn})|Z}\phi(\theta) \tag{22}$$

where $\theta = \frac{\mu_{(x_{sn}-y_{sn})|Z}}{\sigma_{(x_{sn}-y_{sn})|Z}}$ and

$$\mu_{(x_{sn}-y_{sn})|Z} = (d_x - d_y) + (q_x - q_y)|z| \tag{23}$$

$$\sigma_{(x_{sn}-y_{sn})|Z} = \sqrt{\sum_{i=1}^{n}(a_{xi} - a_{yi})^2 + s_{x\eta}^2 + s_{y\eta}^2 + (s_{xw} - s_{yw})^2} \tag{24}$$

$$T_{x_{cond}} = \Phi\left(\frac{\mu_{(x_{sn}-y_{sn})|Z}}{\sigma_{(x_{sn}-y_{sn})|Z}}\right) \tag{25}$$

To evaluate the first moment, we need to evaluate integrals of the form

$$\int_0^\infty \phi(z)\ \Phi(a+bz)\ \mathrm{d}z, \quad \int_0^\infty z\ \phi(z)\ \Phi(a+bz)\ \mathrm{d}z$$
$$\int_0^\infty \phi(z)\ \phi(a+bz)\ \mathrm{d}z \tag{26}$$

The first integral can be evaluated using the analytical expression for the unconditional tightness probability given by property (4) of the skew-normal random variables. Therefore,

$$T_x = \int_{-\infty}^{\infty} T_{x_{cond}}\ \phi(z)\ \mathrm{d}z = 2\int_0^\infty \Phi(a+bz)\ \phi(z)\ \mathrm{d}z$$
$$= \Phi(\tau) + 2\ T(\tau, b) \tag{27}$$

where,

$$a = \frac{d_x - d_y}{\sigma_{(x_{sn}-y_{sn})|Z}}, \quad b = \frac{q_x - q_y}{\sigma_{(x_{sn}-y_{sn})|Z}}, \quad \tau = \frac{a}{\sqrt{1+b^2}}$$

and $T(h,g)$ is the Owen's T function [11], given by

$$T(h,g) = \frac{1}{2\pi}\int_0^g \frac{e^{\frac{-h^2}{2}(1+x^2)}}{1+x^2}\ \mathrm{d}x$$

The third integral can be evaluated in terms of $\Phi(\cdot)$. Since $\frac{d\phi}{dz} = -z\phi(z)$, the second integral can be obtained by integrating by parts. Using this, the first moment can be written as

$$\begin{aligned} M_1 &= (d_x - d_y)\Phi(\tau) + 2(d_x - d_y)\ T(\tau, b) \\ &+ \sqrt{\frac{2}{\pi}}(q_x - q_y)\Phi(a) + \frac{2b}{t}(q_x - q_y)\phi(\tau)\Phi(-b\tau) \\ &+ \frac{2\sigma_{(x_{sn}-y_{sn})|Z}}{t}\phi(\tau)\Phi(-b\tau) + d_y + 2q_x \end{aligned}$$

where $t = \sqrt{1+b^2}$. It can be easily verified that it reduces to Clark's formula when $q_x = q_y = 0$.

All the higher order moments can be obtained similarly and using the fact that $z^2\phi(z)$ and $z^3\phi(z)$ can be written in terms of $\phi(z)$ and its derivatives.

If the mean, standard deviation and the skewness of p_{sn} are denoted by μ_p, σ_p and γ_p, the coefficients of the canonical form can be found as follows

$$q_p = \gamma_p^{\frac{1}{3}}\sigma_p F^{-\frac{1}{3}} \tag{28}$$
$$d_p = \mu_p - q_p\sqrt{\frac{2}{\pi}} \tag{29}$$
$$a_{pi} = a_{xi}\ T_x + a_{yi}\ (1 - T_x) \quad \text{for} \quad i = 1, 2\cdots n \tag{30}$$
$$s_{pw} = s_{xw}\ T_x + s_{yw}\ (1 - T_x) \tag{31}$$
$$s_{p\eta} = \sqrt{(s_{x\eta}T_x)^2 + (s_{y\eta}(1-T_x))^2} \tag{32}$$

where F is defined in equation (14). The coefficients a_{pi}, s_{pw} and $s_{p\eta}$ are scaled to match the standard deviation. The scaling factor s is found as follows

$$\sigma_p' = \sqrt{\sigma_p^2 - q_p^2\left(1 - \frac{2}{\pi}\right)} \tag{33}$$
$$S_0 = \sqrt{\sum_{i=1}^n a_{pi}^2 + s_{pw}^2 + s_{p\eta}^2} \tag{34}$$
$$s = \frac{\sigma_p'}{S_0} \tag{35}$$

VI. Criticality computation

Node/edge criticality is defined as the probability that the node/edge lies on a dominant critical path. Computation of the criticality requires both the path delay associated with the node (PD) and its complementary path delay (CPD). The PD for a node is the MAX of the delays of all paths passing through the node, while the CPD is the maximum delay distribution of all the paths that do not pass through the node. Criticality of a node is evaluated as the tightness probability of PD over CPD of the node.

$$\text{Criticality}(i) = P(PD_i \geq CPD_i) \tag{36}$$

For the skew canonical model, computation of node criticality is straight forward as the tightness probability in (36) can easily be evaluated using the Owen's T-function, if both PD and CPD are in the skew canonical form. To improve the accuracy of evaluating the criticality, the pruning algorithm using the K-center clustering described in [13] is also used.

For the quadratic canonical form used in [8], a tightness probability has not really been defined. However, we can use the quadratic approximation used in performing MAX operation in [8] to compute an equivalent for the tightness probability. Consider two canonicals x and y in quadratic canonical form. To get the tightness probability, we need to evaluate $P(x \geq y)$. Let $v = (x - y)$ be represented as a quadratic function of a standard normal random variable w as follows.

$$v = (x - y) \approx c_2 w^2 + c_1 w + c_0 \tag{37}$$

The coefficients c_2, c_1 and c_0 can be computed using the first three moments of v [8]. Define t_1 and t_2 as:

$$t_1 = \min\left(\frac{-c_1 - \sqrt{c_1^2 - 4c_2c_0}}{2c_2}, \frac{-c_1 + \sqrt{c_1^2 - 4c_2c_0}}{2c_2}\right) \tag{38}$$

$$t_2 = \max\left(\frac{-c_1 - \sqrt{c_1^2 - 4c_2c_0}}{2c_2}, \frac{-c_1 + \sqrt{c_1^2 - 4c_2c_0}}{2c_2}\right) \tag{39}$$

Based on this, the tightness probability can derived as

$$P(x \geq y) \approx \begin{cases} \Phi(t_1) + \Phi(-t_2) & \text{if } c_2 > 0 \\ \Phi(t_1) - \Phi(t_2) & \text{if } c_2 < 0 \\ \Phi\left(\frac{c_0}{c_1}\right) & \text{if } c_2 = 0, c_1 > 0 \\ \Phi\left(\frac{-c_0}{c_1}\right) & \text{if } c_2 = 0, c_1 < 0 \end{cases}$$

VII. Results

In this section we compare the errors and run-time of the quadratic and the equivalent skew-canonical model against Monte Carlo simulations. The standard deviation of the intra and inter-die components of the process parameters is assumed to be 10% of their respective nominal values and the standard deviation of the independent random component is assumed to be 5% of the nominal delay for all gates. The quad-tree structure was used to represent spatial correlation. The equivalent skew canonical model for the quadratic canonical model is constructed as described in section IV.

To make sure the quantities are dimensionally the same, we report errors in the mean, standard deviation and the cube root of the third central moment. Table I shows the error μ, σ, $\kappa^{\frac{1}{3}}$ and the 95% yield point of the circuit delay distribution for the quadratic, semi-quadratic and skew canonical models. The errors are with respect to Monte-Carlo simulations of 10^6 iterations using the quadratic model for the edge delays. The error in μ in both the quadratic, semi-quadratic and skew-normal cases are almost the same, but the error in σ is lower in the case of skew-normal model than in the full quadratic or semi-quadratic model. The error in $\kappa^{\frac{1}{3}}$ for the skew-normal model is significantly higher than for the full-quadratic model and is comparable to the error obtained when the semi-quadratic model is used. So there is trade-off in terms of more accurate σ and a less accurate $\kappa^{\frac{1}{3}}$ on using the skew-normal model as opposed to the full quadratic model. Overall, the SN model performs better than the semi-quadratic model. Table II compares the error in μ, σ and 95% yield point between the skew canonical model and the linear model. As expected, the the mean and standard deviation is larger when the linear model is used. Interestingly, the error in the 95% yield point is comparable in all cases even though the use of semi-quadratic and skew-normal results in a larger skewness error and the use of the equivalent linear models results in a larger mean and σ error.In some sense, there seems to be a trade-off in the errors of various moments, resulting in a good estimate of the yield point. This is also illustrated in Figure 1, which contains the PDF and CDF of skew-normal random variables that have the same mean and variance, but different values of the third moment. If is seen that the differences in the PDF and CDF are marginal for considerable variations in the third moment. Also shown is the PDF of a Gaussian that has 2% shift in the mean and a 3% shift in the standard deviation. Although the PDF looks different, the 95% yield point is practically the same as that of the skew-normal random variables.

Benchmark	Skew normal model			Linear canonical model		
	$\delta\mu$	$\delta\sigma$	$\delta Y_{95\%}$	$\delta\mu$	$\delta\sigma$	$\delta Y_{95\%}$
s953	0.28	-1.13	0.81	0.32	-2.20	-0.96
s1196	0.26	0.21	1.12	0.27	-0.92	-0.74
s1238	-0.36	-1.46	0.41	-0.27	-2.84	-1.54
s1423	0.25	0.15	1.30	0.24	-0.85	-0.55
s1488	-0.27	-1.38	0.52	-0.19	-2.80	-1.46
s1494	-0.18	-2.00	0.54	-0.05	-3.99	-1.67
s5378	1.36	-2.18	1.39	1.29	-4.25	-0.88
s9234	1.36	0.01	1.84	1.37	-0.97	0.18
s13207	0.76	-0.91	1.27	0.70	-1.45	-0.47
s15850	0.52	-0.17	1.10	0.54	-1.32	-0.73
s35932	-3.08	-0.93	-2.02	-2.75	-2.55	-3.50
s38417	6.49	-2.74	4.66	6.85	-5.23	3.36
s38584	2.17	-5.02	1.78	1.55	-4.36	-0.64
Avg. error	1.33	1.41	1.44	2.18	2.59	1.63

TABLE II: Percentage error in μ, σ and 95% Yield point of the circuit delay distribution for equivalent skew-canonical model, equivalent linear model and linear model with respect to Monte-Carlo simulations of 10^6 iterations with the quadratic model for ISCAS89 benchmarks using quad-tree QT

Table III has the comparison of the overhead in CPU times as compared to the linear model for computing the circuit delay of the ISCAS85 and 89 benchmarks using the three models. Clearly, the skew-normal model is only marginally slower than the linear model and has a significant advantage over both the quadratic models. The average speed-up over the quadratic SSTA is about two to three orders of magnitude. The speed-up obtained depends on the number of principal components and will be larger as this number increases.

Benchmark	Quadratic canonical model				Semi-quadratic canonical model				Skew normal model			
	$\delta\mu(\%)$	$\delta\sigma(\%)$	$\delta(\kappa^{\frac{1}{3}})(\%)$	$\delta Y_{95\%}(\%)$	$\delta\mu(\%)$	$\delta\sigma(\%)$	$\delta(\kappa^{\frac{1}{3}})(\%)$	$\delta Y_{95\%}(\%)$	$\delta\mu(\%)$	$\delta\sigma(\%)$	$\delta(\kappa^{\frac{1}{3}})(\%)$	$\delta Y_{95\%}(\%)$
c17	0.22	-2.37	-3.08	-0.38	0.18	-3.72	-28.84	-1.22	0.19	-1.47	9.46	0.21
c432	0.61	-1.99	-1.04	0.16	0.57	-3.01	-19.58	-0.44	0.69	-0.74	26.95	1.35
c499	-0.17	-2.39	-1.20	-0.60	-1.06	-5.71	-24.83	-2.44	0.26	-3.10	20.63	0.35
c880	0.23	-1.39	-0.69	-0.06	0.23	-1.75	-20.29	-0.49	0.29	0.03	26.08	1.14
c1355	-0.56	-2.75	-2.27	-0.97	-1.29	-5.25	-23.91	-2.47	-0.08	-2.63	23.61	0.32
c1908	0.79	-1.46	-1.43	0.31	0.77	-2.06	-19.14	-0.15	0.82	-0.56	27.23	1.42
c2670	0.20	-1.47	-1.71	-0.19	0.17	-2.05	-21.31	-0.71	0.19	0.09	23.88	0.97
c3540	-0.24	-2.36	-1.59	-0.61	-0.32	-3.54	-20.55	-1.27	-0.14	-1.17	25.99	0.61
c5315	0.47	-3.88	-4.10	-0.46	0.44	-5.11	-24.74	-1.18	0.54	-0.39	23.12	1.31
c6288	0.62	-2.50	-1.72	0.07	0.60	-3.98	-20.67	-0.60	0.38	-0.83	28.64	1.24
c7552	0.03	-1.00	-0.62	-0.17	0.02	-1.33	-18.46	-0.57	0.04	0.37	29.22	1.08
Avg. error	0.38	2.14	1.77	0.36	0.51	3.41	22.03	1.05	0.33	1.03	24.07	0.91

(a)ISCAS85 benchmarks

Benchmark	Quadratic canonical model				Semi-quadratic canonical model				Skew normal model			
	$\delta\mu(\%)$	$\delta\sigma(\%)$	$\delta(\kappa^{\frac{1}{3}})(\%)$	$\delta Y_{95\%}(\%)$	$\delta\mu(\%)$	$\delta\sigma(\%)$	$\delta(\kappa^{\frac{1}{3}})(\%)$	$\delta Y_{95\%}(\%)$	$\delta\mu(\%)$	$\delta\sigma(\%)$	$\delta(\kappa^{\frac{1}{3}})(\%)$	$\delta Y_{95\%}(\%)$
s953	0.22	-2.54	-2.93	-0.35	0.17	-3.71	-23.72	-1.05	0.28	-1.13	21.19	0.81
s1196	0.24	-1.90	-1.58	-0.23	0.23	-2.50	-20.88	-0.73	0.26	0.21	25.04	1.12
s1238	-0.52	-3.55	-1.63	-1.14	-0.61	-5.29	-20.80	-1.95	-0.36	-1.46	28.28	0.41
s1423	0.23	-0.66	0.67	0.17	0.23	-0.86	-16.22	-0.16	0.25	0.15	31.34	1.30
s1488	-0.39	-3.94	-3.48	-1.10	-0.51	-5.87	-23.63	-1.99	-0.27	-1.38	25.35	0.52
s1494	-0.59	-4.67	-3.96	-1.47	-0.69	-7.16	-23.79	-2.47	-0.18	-2.00	27.80	0.54
s5378	0.83	-4.08	-5.18	-0.31	0.56	-6.12	-29.71	-1.49	1.36	-2.18	17.04	1.39
s9234	1.36	-0.85	-2.32	0.94	1.34	-1.53	-22.92	0.41	1.36	0.01	20.28	1.84
s13207	0.57	-1.37	-1.06	0.23	0.57	-1.76	-23.71	-0.28	0.76	-0.91	22.54	1.27
s15850	0.38	-1.86	-2.95	-0.19	0.36	-2.62	-25.48	-0.81	0.52	-0.17	20.67	1.10
s35932	-1.08	-3.21	-5.41	-1.56	-2.03	-5.75	-28.79	-3.26	-3.08	-0.93	13.78	-2.02
s38417	8.23	-2.49	-3.67	5.81	6.89	-7.35	-26.15	3.31	6.49	-2.74	3.84	4.66
s38584	1.82	-2.92	-3.22	0.76	1.50	-5.79	-29.17	-0.67	2.17	-5.02	26.23	1.78
Avg. error	1.27	2.62	2.93	1.10	1.21	4.33	24.23	1.43	1.33	1.41	21.80	1.44

(b)ISCAS89 benchmarks

TABLE I: Error in μ, σ, $\sqrt[3]{\kappa}$ and 95% Yield point of the circuit delay distribution for quadratic canonical model, semi-quadratic canonical model and equivalent skew-canonical model-I with respect to Monte-Carlo simulations of 10^6 iterations with the quadratic model for ISCAS85 and ISCAS89 benchmarks using quad-tree QT

Node criticalities are evaluated using the quadratic and skew-normal models as described in the section VI. Figure 2 shows the maximum error in node criticalities for the two models with respect to the Monte-Carlo simulations. It is clear from the figures that both the models tend to have similar errors. Figure 3 shows the ratio of run-times between the quadratic and skew-normal models. On an average, the run time using skew canonical model is about two orders of magnitude lower than using quadratic delay models.

VIII. Conclusion

In this paper, we have proposed a skew-canonical form for the gate delay and arrival times to take into account the skewness in the gate delay variation and the inherent non-linearity of the MAX operator. We also propose a method to obtain a skew canonical model for the gate delay from the quadratic model. While it matches the first three moments, there is an error in the covariance between gates. In practice, this error was found to be of the order of one percent.

The advantage of using the skew-canonical form is that analytical expressions for the moments of the MAX operation can be obtained in terms of the CDF of the standard normal and the Owen's T-function. Standard computer routines are available for both these functions. The computational complexity of evaluating these moments is marginally more than using Clark's formulas. As a result, SSTA using these models is two to three orders of magnitude faster than SSTA using the quadratic models. Another advantage is that the computational complexity of criticality computation is comparable to that of linear models. It gives the same accuracy as the quadratic models at a fraction of the run-time.

However, it has the same limitations and advantages as the canonical form. If the variations have a strong independent component, an extended (skew) canonical form may be essential to account for reconvergent paths in the circuit. Some more work is also required to see the effect of non-Gaussian process parameter variations.

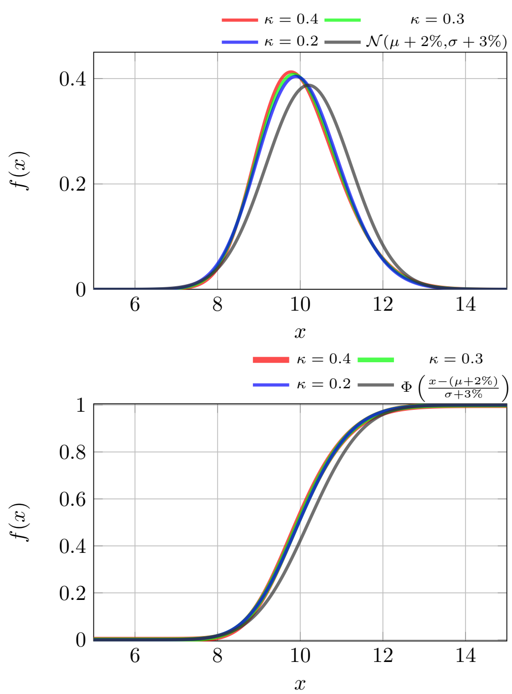

Fig. 1: The PDF and CDF of skew normal random variables that have the same mean and variance, but different values of the third moment. Also shown is the PDF and CDF of a Gaussian with 2% error in the mean and 3% error in the standard deviation.

Benchmark	Quadratic	Semi-quadratic	Skew-normal
c17	1849.43	558.27	16.17
c432	3979.62	296.32	4.76
c499	6957.15	381.69	4.84
c880	1897.07	323.81	0.99
c1355	8051.10	354.75	12.43
c1908	5479.02	306.34	9.64
c2670	2798.14	363.86	3.12
c3540	4054.15	305.73	5.61
c5315	5509.44	389.65	4.31
c6288	6316.96	305.93	3.60
c7552	1802.14	280.81	0.96
Average	4897.77	359.27	7.57

(a) ISCAS85 benchmarks

Benchmark	Quadratic	Semi-quadratic	Skew-normal
s953	2658.94	322.87	4.34
s1196	6019.55	351.50	2.50
s1238	6511.69	325.51	4.06
s1423	1938.97	289.17	8.02
s1488	4686.62	343.83	5.70
s1494	5109.86	340.57	4.67
s5378	4000.22	343.80	6.60
s9234	6334.19	380.50	5.90
s13207	1780.39	332.98	5.33
s15850	2306.96	298.20	4.11
s35932	13294.85	475.28	5.12
s38417	2870.79	327.76	4.05
s38584	3852.92	296.72	5.76
Average	5560.83	343.70	5.26

(b) ISCAS89 benchmarks

TABLE III: Percentage increase in run-time for the quadratic, semi-quadratic and the skew-normal canonical models over the linear model

Appendix A

Consider two random variables x_{sq} and y_{sq} represented in the semi-quadratic canonical form shown in equation (18). Let x_{sn}, y_{sn} be the equivalent skew-normal canonicals of x_{sq} and y_{sq} as expressed in equation (19). The difference in covariance between the two models is given by:

$$C_{x_{sq}y_{sq}} - C_{x_{sn}y_{sn}} = \sum_{i=n+1}^{n+m} a_{x_i}a_{y_i} - s_{xw}s_{yw} - q_x q_y\left(1 - \frac{2}{\pi}\right) \quad (40)$$

With the assumption that the sensitivities are a linear function of the nominal delay, the sensitivities can be expressed as:

$$a_{x_i} = s_i\mu_x, \ a_{y_i} = s_i\mu_y \quad (41)$$

$$b_{x_i} = t_i\mu_x, \ b_{y_i} = t_i\mu_y \quad (42)$$

With this assumption, third cumulant of x_{sq} and the covariance between x_{sq} and y_{sq} becomes,

$$\kappa_x = \mu_x^3 \times \sum_{i=1}^{n}\left(6t_i s_i^2 + 8t_i^3\right) = k \times \mu_x^3 \quad (43)$$

Using (43), q_x and s_{xw_i} are given by:

$$q_x = \mu_x \times F \times \sqrt[3]{k} = \mu_x \times k_1 \quad (44)$$

$$s_{xw} = \mu_x \times \sqrt{\sum_{i=n+1}^{n+m} s_i^2 - k_1^2\left(1 - \frac{2}{\pi}\right)} \quad (45)$$

Using (41), (42), (44) and (45) in (40),

$$C_{x_{sq}y_{sq}} - C_{x_{sn}y_{sn}} = 0 \quad (46)$$

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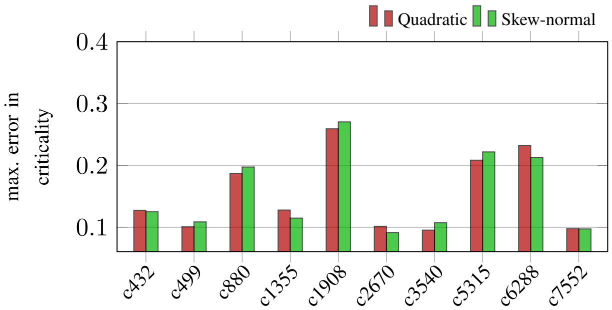
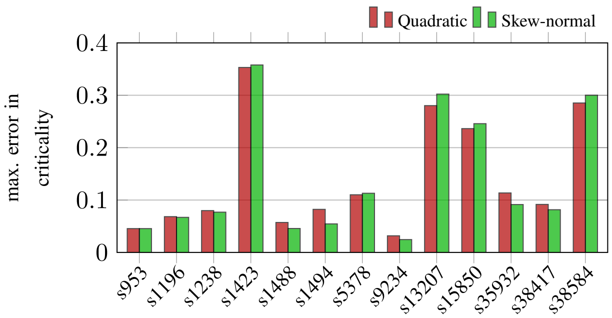
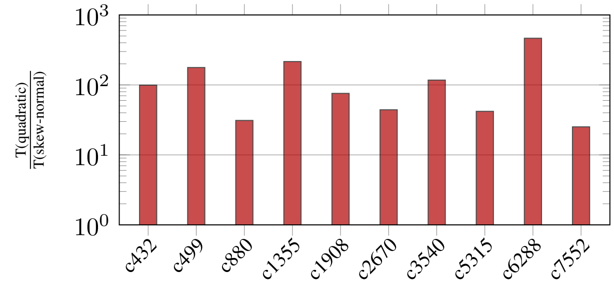


(a) ISCAS85 benchmarks

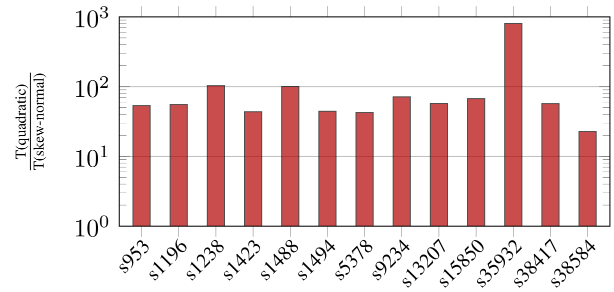


(b) ISCAS89 benchmarks

Fig. 2: Max. error in criticality computed using skew canonical model compared against the quadratic canonical models for ISCAS85 and ISCAS89 benchmarks. Average of the max. error is 0.14 for both the models and the maximum error of 0.35 and 0.36 for s1423 for the quadratic and skew-normal respectively



(a) ISCAS85 benchmarks



(b) ISCAS89 benchmarks

Fig. 3: Ratio of run-times for computing criticality using skew canonical model and quadratic canonical models for ISCAS85 and ISCAS89 benchmarks $\left(\frac{\text{T(quadratic)}}{\text{T(skew-normal)}}\right)$; Average improvement in run-time is close to $120\times$.

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