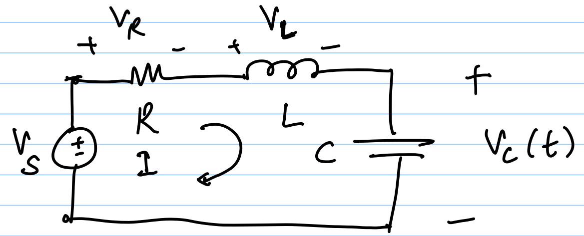
19/3/15

Lec 27

Second Order Systems



$I = C\frac{dV_c}{dt}$

$V_L = L\frac{dI}{dt}$

$= LC\frac{d^2V_c}{dt^2}$

$V_R + V_L + V_c = V_s$

$$\boxed{LC\frac{d^2V_c}{dt^2} + RC\frac{dV_c}{dt} + V_c = V_s}$$

Natural response : $V_s = 0$

$$LC\frac{d^2V_c}{dt^2} + RC\frac{dV_c}{dt} + V_c = 0$$

$$V_c = V_p \exp(pt)$$

$$\underbrace{(LC\cdot p^2 + RCp + 1)}_{=0} V_p \exp(pt) = 0$$

$$LC\cdot p^2 + RC\cdot p + 1 = 0$$

2nd order ckt

$$p = \frac{-RC \pm \sqrt{(RC)^2 - 4LC}}{2LC}$$

1st order:

$$RC\frac{dV_c}{dt} + V_c = 0$$

$$V_c = V_p \exp(pt) : (RCp+1)\cdot V_p \exp(pt) = 0$$

$$p = \frac{-1}{RC}$$

$$p = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$= p_1$ & p_2

p_1, p_2
- $\rightarrow$ real & distinct
- $\rightarrow$ real & equal
- $\rightarrow$ complex conjugate

Real & distinct $\Rightarrow (R/2L)^2 > 1/LC \Rightarrow R^2 > \frac{4L}{C}$

$$V_c(t) = A_1 \exp(p_1 t) + A_2 \exp(p_2 t)$$

$A_1, A_2 \rightarrow$ from initial conditions

$V_c(0)$ $\quad$ $I_L(0)$

Real & equal $\Rightarrow$ $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC} \Rightarrow R^2 = \frac{4L}{C}$

$p_1 = p_2 = p$

$$V_c(t) = (A_1 + A_2 t)\exp(pt)$$

complex conjugate $\Rightarrow$ $R^2 < 4L/C$

$$p_{1,2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \quad p_r \pm j p_i$$

$$V_c(t) = A_1 \exp(p_1 t) + A_2 \exp(p_2 t)$$

$$A_2 = A_1^*$$

$$V_c(t) = A_1 \exp(p_1 t) + A_1^* \exp(p_2 t)$$

real + imag } 2 initial conditions

$$p_{1,2} = p_r \pm j p_i$$

$A_0 \cos(p_i t + \varphi)$

$$V_c(t) = A_1 \exp(p_r t)\exp(p_j t) + A_1^* \exp(p_r t)\cdot\exp(-p_j t)$$

Sinusoidal solution, with amplitude exponentially decreasing over time.

$$LC\frac{d^2 V_c}{dt^2} + RC\frac{dV_c}{dt} + V_c = V_s$$

$$\frac{d^2 V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c}{LC} = \frac{V_s}{LC}$$

std. form $\rightarrow$ $\dfrac{d^2 V_c}{dt^2} + 2\zeta\omega_n\dfrac{dV_c}{dt} + \omega_n^2 V_c = \omega_n^2 V_s$

ζ $\downarrow$ 'zeta'

ζ = damping factor

ω_n = natural frequency

$$\omega_n = \frac{1}{\sqrt{LC}}$$

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\frac{d^2 V_c}{dt^2} + \frac{\omega_n}{Q}\cdot\frac{dV_c}{dt} + \omega_n^2 V_c = \omega_n^2 V_s$$

Q = quality factor $= \dfrac{1}{2\zeta} = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$

Characteristic polynomial

$LC\cdot p^2 + RC\cdot p + 1 = 0$

$p^2 + 2\zeta\omega_n p + \omega_n^2 = 0$

&

$p^2 + \frac{\omega_n}{Q}\cdot p + \omega_n^2 = 0$

$p_{1,2} = \omega_n\left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]$

$= \omega_n\left[\frac{-1}{2Q} \pm \sqrt{\left(\frac{1}{2Q}\right)^2 - 1}\right]$

$p_{1,2}$ → real & distinct

$\zeta > 1$; $Q < 1/2$

overdamped system

→ real & equal

$\zeta = 1$, $Q = 1/2$

critically damped system

→ complex conjugate

$\zeta < 1$; $Q > 1/2$

underdamped system