Scalar electromagnetic integral equations

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Abstract

This brief note derives the two dimensional scalar electromagnetic integral equation starting from Maxwell's equations, and shows how this equation can be used to formulate the forward and inverse scattering equations for microwave imaging.

I. Maxwell's equations

We start with our beloved Maxwell's equations [1]. These are:

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial B(\vec{r},t)}{\partial t}, \quad \nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial D(\vec{r},t)}{\partial t}, \quad \nabla \cdot \vec{B} = 0, \quad \nabla \cdot \vec{D} = \rho_e \tag{1}$$

along with the constitutive relations

$$\vec{J} = \vec{J}_i + \sigma \vec{E}, \quad \vec{D} = \epsilon \vec{E}, \quad \vec{B} = \mu \vec{H}, \quad \text{with} \quad \epsilon = \epsilon_0 \epsilon_r(r), \quad \mu = \mu_0 \mu_r(r) \tag{2}$$

where all symbols have their standard meaning, and J_i is an impressed current.

In order to eliminate one variable from the above equations, we take a curl of the first equation and substitute the second equation, while noting the following vector calculus identity:

$$\nabla \times (\nabla \times \vec{P}) = \nabla(\nabla \cdot \vec{P}) - \nabla^2 \vec{P}.$$

In a divergence free field[1] ($\nabla \cdot \vec{E} = 0$) this becomes:

$$\nabla^2 \vec{E} = \mu \left(\frac{\partial \vec{J}}{\partial t} + \frac{\partial^2 \vec{D}}{\partial t^2} \right). \tag{3}$$

We get the following equation (assuming a time invariant medium (i.e. ϵ, μ, σ are not functions of time)):

$$\nabla^2 \vec{E} = \mu \frac{\partial \vec{J}_i}{\partial t} + \sigma \mu \frac{\partial \vec{E}}{\partial t} + \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{4}$$

After this general derivation, we specialize to the time-harmonic case, i.e. when the time dependence of E, H, J is of the type $e^{j\omega t}$. This gives us :

$$\nabla^2 \vec{E}(r) + k^2(r) \vec{E}(r) = j\omega\mu \vec{J}_i(r) \tag{5}$$

where the squared complex wave vector[2] is given by $k^2(r) = \omega^2 \epsilon(r) \mu(r) [1 - j\frac{\sigma(r)}{\mu(r)\epsilon(r)}]$. It is convenient to express this for a non-magnetic medium ($\mu_r = 1$) as $k^2 = k_0^2 \epsilon_r(r)$, where $k_0^2 = \omega^2 \epsilon_0 \mu_0$ is the squared wave vector in vacuum, and the (complex) ϵ_r has been modified to incorporate conductivity σ.

Any arbitrary solution to these equations can be expressed in terms of the solution to two orthogonal polarizations. To make matters concrete, consider a TM (transverse magnetic) polarization to be characterized[3] by the following non-zero quantities: (E_z, H_x, H_y), and a TE (transverse electric) polarization by: (H_z, E_x, E_y). Effectively, we can solve a scalar equation by using only E_z or H_z, instead of the vector form in (5).

There is an important clarification in the derivation of (5): in deriving the $\vec{E}$ equation, we made the assumption that $\nabla \times \vec{B}(r) = \mu \nabla \times \vec{H}(r)$, which is true only if μ is not a function of space; in the general case, we would get the following relation:

$$\nabla \times \left(\frac{1}{\mu_r(r)} \nabla \times \vec{E}(r) \right) - k_0^2 \epsilon_r(r) \vec{E}(r) = -j\omega \vec{J}_i(r) \tag{6}$$

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[1]For e.g., consider TM polarization, $\vec{E} = \hat{z} E_z(x,y)$.

[2]The opposite time convention of $e^{-j\omega t}$ would have given k a positive imaginary component.

[3]Note that this characterization is not consistent in the literature, and interchanged definitions of TE/TM are common.

II. Solving an inhomogeneous differential equation

A slight diversion now: Assume we *want* to solve a differential equation of the form

$$\nabla^2\Phi(r) + k^2\Phi(r) = f(r) \tag{7}$$

where $f(r)$ is a finite function, and we *know* the solution to the differential equation

$$\nabla^2 G(r, r') + k^2 G(r, r') = -\delta(r, r') \tag{8}$$

How can (8) help us in solving (7)? If we multiply (8) by $f(r')$ and integrate over primed coordinates (noting that ∇ only acts on unprimed coordinates, thus allowing $f(r')$ and the integration operator to 'move' past it), we get the following relation:

$$\nabla^2\left(\int_{V'} G(r, r')f(r')\, dr'\right) + k^2\left(\int_{V'} G(r, r')f(r')\, dr'\right) = -f(r). \tag{9}$$

In other words, the solution to (7) can be expressed in terms of the Green's function [2, Ch. 14], G, as:

$$\Phi(r) = -\int_{V'} G(r, r')f(r')\, dr'. \tag{10}$$

In the cases that we consider, the Green's function turns out to be of the form $G(r, r') = G(r - r')$, and the above equation is in effect, a convolution operation. Thus, it is also possible to derive the above using Fourier theory. See App. A for a derivation of an explicit Green's function for a homogeneous medium.

III. Towards an integral equation for the electric field

We continue from Section (I) and stick to the TM polarization, which gives: $\nabla^2 E_z + k^2 E_z = j\omega\mu J_i$.

Imagine a current source $J_i(r)$ that is generating a field $E_i(r)$ in the absence of any objects. In such a case, the above equation is simply:

$$\nabla^2 E_i(r) + k_0^2 E_i(r) = j\omega\mu J_i(r) \tag{11}$$

Now, imagine that we have brought an object, covering a 2D region Ω and we want to find the fields at any point in space due to the presence of this object. Well, this is simply the field E satisfying:

$$\nabla^2 E(r) + k_0^2\epsilon_r(r)E(r) = j\omega\mu J_i(r) \tag{12}$$

Since the exact nature of J_i can be quite complicated in general, we make life easier by subtracting the above two equations to elimiate J_i, and after a bit of algebra, we get:

$$\nabla^2(E(r) - E_i(r)) + k_0^2(E(r) - E_i(r)) = -k_0^2(\epsilon_r(r) - 1)E(r) \tag{13}$$

The above equation just looks like the inhomogeneous equation of the earlier section, Eq. (7), and so its solution is given by Eq. (10). Incidentally, $E(r) - E_i(r)$ is referred to as the scattered field, $E_s(r)$, and $(k^2(r) - k_0^2)/k_0^2$ is called the dielectric contrast, $\chi(r)$. The solution to the above equation is:

$$E(r) - E_i(r) = E_s(r) = \int_\Omega k_0^2\chi(r')G(r, r')E(r')dr' \tag{14}$$

This concludes the derivation of the scalar electric field integral equation (EFIE); this kind of an equation is called a Freedholm integral equation of the second kind, since the unknown of interest (E in this case) appears both inside and outside the integral sign.

IV. Putting the EFIE to use

The EFIE forms the basis for much work on electromagnetic scattering problems.

A. Forward problems

When $\chi(r)$ is known, and $E_s(r)$ is to be determined, it is referred to as a 'forward' problem. For instance, calculating the radar cross-section of an aircraft, or the radiation pattern of an antenna are examples of forward problems. The usual strategy for solving the forward problem is to discretize Ω and then:

1) Choose $r \in \Omega$ in Eq. 14 to obtain a set of equations that can be easily solved. As a result, the value of the field $E(r)$ is known at all the chosen points in Ω. Mathematically, solve this equation:
$$E(r) - \int_{\Omega} k_0^2 \chi(r') G(r, r') E(r') dr' = E_i(r), \quad r, r' \in \Omega \tag{15}$$
2) Next, we choose r to be any location away from the object,i.e. $r \notin \Omega$, where we want to calculate the field. Since $E(r')$ for $r' \in \Omega$ is already known from the previous step, $E(r)$ can be easily computed from Eq. 14. Mathematically, solve this equation
$$E_s(r) = \int_{\Omega} k_0^2 \chi(r') G(r, r') E(r') dr', \quad r \notin \Omega, r' \in \Omega \tag{16}$$

An example of this method has been demonstrated in Richmond's classic paper on computing the scattering from a dielectric cylinder [3].

B. Inverse problems

A harder problem is the 'inverse' problem, where $E_s(r)$ is measured at a few discrete points outside Ω, and the unknown permittivity $\chi(r)$ is to be determined. This has applications in problems such as breast cancer imaging, or buried object detection. To measure the scattered field, we place receivers on a contour Γ that encloses the object Ω. We thus have the following "data" equation for the measured field, $E_s(r)$ for $r \in \Gamma$:
$$E_s(r) = \int_{\Omega} k_0^2 \chi(r') G(r, r') E(r') dr' + n(r), \quad r \in \Gamma, r' \in \Omega \tag{17}$$
where $n(r)$ is the noise in the measurements. The optimization problem to solve is:
$$\min_{\chi} \sum_{r} \left| E_s(r) - \int_{\Omega} k_0^2 \chi(r') G(r, r') E(r') dr' \right|^2 \tag{18}$$

A solution to the above problem gives χ, which can be used to update the estimate of the internal fields by solving the forward problem as in Eq. (15). A common method of solving the inverse problem, called the *Born Iterative Method (BIM)* [4], is outlined below.

1) Make the initial estimate of the internal fields to be equal to the incident fields. This is also called the Born approximation.
2) Solve Eq. (18) to obtain $\chi(r)$.
3) Use $\chi(r)$ to update the internal fields using Eq. (15).
4) Check convergence. If not adequate, go to step 2.

The above algorithm works for a limited range of contrasts. An advanced version of this is called the *Distorted Born Iterative Method (DBIM)* [5], where, in addition to the internal field update as in Step (3) above, the Green's function is also updated at each iteration. In fact, all updates are performed with respect to the previous iteration.

To see how to accomplish this, assume that we have estimated the fields $E_1(r)$ corresponding to the estimate of the relative permittivity at that particular iteration, $\epsilon_{r1}(r)$. The Green's function, $G_1(r, r')$ corresponding to the above permittivity is also known (numerically, see App. B). We would like to update to fields and permittivity, $E_2(r)$ and $\epsilon_{r2}(r)$, respectively. Since each iteration satisfies the general equation: $\nabla^2 E_z + k^2 E_z = j\omega\mu J_i$, we can subtract the equations for each case to get:
$$\nabla^2 [E_2(r) - E_1(r)] + k_0^2 \epsilon_{r1}(r)[E_2(r) - E_1(r)] = -k_0^2 [\epsilon_{r2}(r) - \epsilon_{r1}(r)] E_2(r). \tag{19}$$
With the knowledge of G_1, we can write the solution to the above equation as:
$$E_2(r) - E_1(r) = \int_{\Omega} k_0^2 [\epsilon_{r2}(r) - \epsilon_{r1}(r)] E_2(r) G_1(r, r')\, dr' \tag{20}$$

Adding and subtracting the incident field on the LHS gives, after some rearrangement:

$$E_{2s}(r) = E_{1s}(r) + \int_{\Omega} k_0^2[\epsilon_{r2}(r') - \epsilon_{r1}(r')]E_2(r')G_1(r,r')\,dr' \tag{21}$$

This gives the permittivity update equation for $\delta(r) = \epsilon_{r2}(r) - \epsilon_{r1}(r)$ as follows;

$$\min_{\delta} \sum_r \left| E_s(r) - E_{1s}(r) - \int_{\Omega} k_0^2 \delta(r')G_1(r,r')E(r')dr' \right|^2 \tag{22}$$

We can now describe the DBIM as follows;

1) Apply the Born approximation; set internal fields to be equal to the incident fields.
2) For the first iteration, solve Eq. (18) to obtain $\chi(r)$.
3) Use $\chi(r)$ to update internal fields using Eq. (15), and update $G(r,r')$ as per Eq. (29).
4) Use Eq. (16) with current estimate of $\chi(r)$ to estimate E_{1s} in Eq. (22). Then solve Eq. (22) with the internal fields as estimated from the previous iteration to obtain δ and then update the contrast $\chi(r)$.
5) If convergence in the previous step is not satisfactory, go to step 3.

Finally, in passing we mention that in addition to the BIM and DBIM, there is a family of methods for the inverse problem under the name of 'Contrast Source Inversion' [6] which does not solve the forward problem as an intermediate step (i.e. step 3 of (D)BIM).

APPENDIX A
EXPLICIT GREEN'S FUNCTIONS

In general, equations of the form (8) are very difficult to solve, particularly when the wavevector k is a function of space, i.e. $k(r)$. There are some special cases, however, when we can solve this equation and we will briefly consider a two-dimensional case (with k constant).

Recall the 2^{nd} order Bessel's differential equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0 \tag{23}$$

where α is some constant. We can recast the LHS of equation (8) into the form of the LHS of (23) by noting that $\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$. Since the RHS of (8) describes a point source at $r = r'$, the solution should not have any θ dependence. Multiplying (8) by r^2 and comparing with (23), we see that the mapping we seek is $x \to kr$, $y \to G$, $\alpha \to 0$.

Hankel functions of the first and second kind are the solutions to equation (23), i.e.

$$y = aH_{\alpha}^{(1)}(x) + bH_{\alpha}^{(2)}(x), \tag{24}$$

where a, b are constants that can be determined by asymptotic considerations. For $x \to \infty$, these functions are approximated as

$$H_{\alpha}^{(1)}(x) \approx \sqrt{2/(\pi x)}\exp\left[j(x - \frac{\alpha\pi}{2} - \frac{\pi}{4})\right],\ H_{\alpha}^{(2)}(x) \approx \sqrt{2/(\pi x)}\exp\left[-j(x - \frac{\alpha\pi}{2} - \frac{\pi}{4})\right] \tag{25}$$

Since we have a point source at $r = r'$, the only solution that makes physical sense is the one corresponding to an outgoing wave. Since we have assumed a time convention of the form $\exp(j\omega t)$, we must have $a = 0$, and b is some constant to be determined by normalization[4].

The astute reader will demand to know what happened of the delta function which appears on the RHS of (8) but not (23). The situation is salvaged by noting that one, the equations are identical when $r \neq r'$, and two, $H_0^{(2)}(x)$ contains a singularity at $x = 0$.

Thus, we have found the solution to (8) in the 2D case for a homogeneous medium as

$$G(r,r') = \frac{-j}{4}H_0^{(2)}(k|r - r'|). \tag{26}$$

[4]This can be found by integrating on a small ball centred on $r = r'$. Turns out that $b = -j/4$.

APPENDIX B
NUMERICALLY CALCULATING THE GREEN'S FUNCTION

Let us say that the Green's function, G, for a homogeneous medium, is known, i.e.

$$\nabla^2 G(r,r') + k_0^2 G(r,r') = -\delta(r-r'),$$

and we wish to find the Green's function, G', for a non-homogenous medium, i.e.

$$\nabla^2 G'(r,r') + k_0^2 \epsilon_r(r) G'(r,r') = -\delta(r-r'),$$

After subtracting the two equations, and with a bit of algebra, we get:

$$\nabla^2 [G'(r,r') - G(r,r')] + k_0^2 [G'(r,r') - G(r,r')] = -k_0^2 \chi(r) G'(r,r') \tag{27}$$

Once again, this reminds us of Eq. (7). To ensure that the RHS remains finite (as was the requirement), we must ensure that $r \neq r'$; let us choose $r \in \Omega$ (denoting the object extent) and $r' \in \Gamma$ (denoting a contour outside the object), and invoke the solution to the above equation:

$$G'(r,r') - G(r,r') = \int_\Omega k_0^2 \chi(r'') G'(r'',r') G(r,r'')\, dr'' \tag{28}$$

On rearranging, we get an equation very similar to the forward problem we encountered in Eq. (15);

$$G'(r,r') - \int_\Omega k_0^2 \chi(r'') G'(r'',r') G(r,r'')\, dr'' = G(r,r'), \quad r, r'' \in \Omega, r' \in \Gamma \tag{29}$$

Discretizing Eqns. (15), (29) give similar matrix equations; in fact, the only difference is in the RHS, where the former equation contains the incident field, while the latter equation contains the homogeneous Green's function.

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