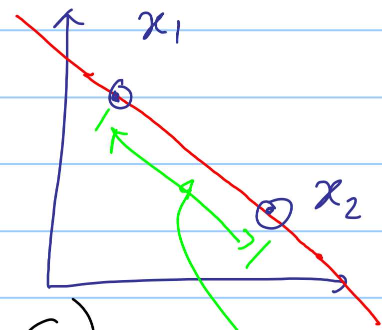
Optimization - Summary of background material.

① **Convexity.** $x_1, x_2 \in \mathbb{R}^n$, scalars $\alpha_1, \alpha_2 \in \mathbb{R}$



Combination of points:

a) $\alpha_1 x_1 + \alpha_2 x_2 \longrightarrow$ Linear Combination (LC)

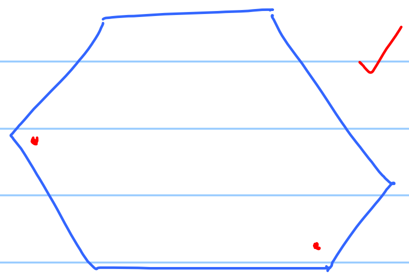
b) $\alpha_1 x_1 + \alpha_2 x_2$, $\alpha_1 + \alpha_2 = 1 \longrightarrow$ Affine combination of x_1, x_2

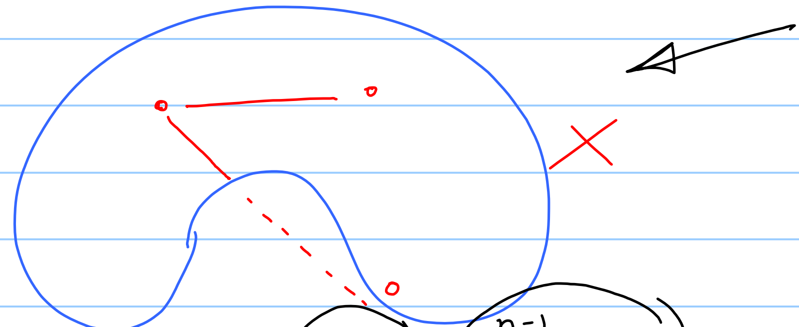
generalization $\sum \alpha_i x_i$, s.t. $\sum \alpha_i = 1$

c) $\alpha_1 x_1 + \alpha_2 x_2$, $\alpha_1 + \alpha_2 = 1$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$

$\rightarrow$ Convex Combination x_1, x_2

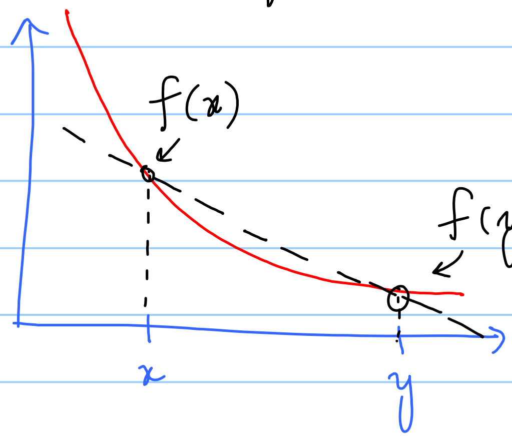
Sets: Convex set? For any $x_1, x_2 \in S$, their convex combination also $\in S$.





$$\sum \alpha_i x_i = \alpha_1 x_1 + \sum_{i=2}^{n-1} \alpha_i x_i$$

↳ Convex functions →



a) Domain must be a convex set

b) The foll must hold true: $x, y \in \mathbb{R}^n$

$$f(\underbrace{\alpha x + (1-\alpha) y}_{\text{Convex comb of } x,\, y}) \leq \underbrace{\alpha f(x) + (1-\alpha) f(y)}_{\text{c.c of } f(x),\, f(y).}, \quad \underbrace{\alpha \geq 0}_{0 \leq \alpha \leq 1}$$

e.g. $f(x) = \underbrace{x^T A x}_{A \in \mathbb{R}^{n\times n}} + \underbrace{b^T x}_{b \in \mathbb{R}^n} + c$, A is sym P.D. Is it convex?

Take the mid pt: $\alpha = \frac{1}{2}$. $\rightarrow f\left(\frac{x+y}{2}\right) - \left[\frac{1}{2} f(x) + \frac{1}{2} f(y)\right]$

$$\left(\frac{x+y}{2}\right)^T A \left(\frac{x+y}{2}\right) + \cancel{b^T\left(\frac{x+y}{2}\right)} + \cancel{c} - \left[\frac{1}{2} x^T A x + \frac{1}{2} y^T A y + \cancel{\frac{1}{2} b^T x} + \cancel{\frac{1}{2} b^T y} + \cancel{c}\right]$$

$$= -\frac{1}{4} x^T A x - \frac{1}{4} y^T A y + \frac{1}{4} x^T A y + \frac{1}{4} y^T A x$$

$$= -\frac{1}{4}\left[\underbrace{(x-y)^T A (x-y)}\right] \leq 0 \quad \forall x, y \Rightarrow \text{Convex fn.}$$

③ Calculus – Continuity & Differentiability

$f: A \to B$

$A \to$ domain, $B \to$ Range

a) A fn f is continuous at $x \in dom(f)$ if for all $\varepsilon > 0$

there exists a δ s.t.

$y \in dom(f)$, $\|y - x\|_2 \leq \delta \Rightarrow \|f(y) - f(x)\| \leq \varepsilon$

↳ The constant δ, depends on ε, x, y

ⓑ But if δ depends only on ε

→ Uniformly continuous.

ⓒ The fn f is Lipschitz continuous if

→ $\|f(x) - f(y)\| \leq L \|x - y\| \quad \forall x, y \in dom(f)$.

L: finite positive scalar.

e.g. $f(x) = \sqrt{1 - x^2}$, $x \in [-1, 1]$.

$x, y \in [-1, 1]$. Say $|x - y| < \delta$.

Consider $|f(x) - f(y)|^2 = \left| \sqrt{1 - x^2} - \sqrt{1 - y^2} \right|^2$

$$= \left[\left| \sqrt{1 - x^2} - \sqrt{1 - y^2} \right| \left| \underbrace{\sqrt{1 - x^2}}_{a} - \underbrace{\sqrt{1 - y^2}}_{b} \right| \right]$$

$|a - b| \underline{\leq} |a + b|$ when $a, b \geq 0$

$$\leq \left[\left| \sqrt{1 - x^2} - \sqrt{1 - y^2} \right| \left| \sqrt{1 - x^2} + \sqrt{1 - y^2} \right| \right]$$

$$= |x^2 - y^2| = |x - y| |x + y| \leq 2 |x - y|$$

$$\leq 2\delta$$

$\Rightarrow \quad |f(x) - f(y)| \le \sqrt{28} \longrightarrow$ My ε

$\Rightarrow f$ is uniformly continuous.

<u>Lipschitz?</u> $\quad \to \quad |f(x) - f(y)| \le L\,|x-y| \quad \forall\, x, y \in [-1, 1]$

choose $y = 1 \quad \to \quad |f(x) - 0| = \left|\sqrt{1-x^2}\right| \overset{?}{\le} L\,|x-1|$

take $\displaystyle \lim_{x \to 1^-} \frac{\left|\sqrt{1-x^2}\right|}{|x-1|} = \frac{\left|\cancel{\sqrt{1-x}}\right|\left|\sqrt{1+x}\right|}{\left|\sqrt{1-x}\right|^{\cancel{2}}} = \frac{\sqrt{1+x}}{\sqrt{1-x}}$

limit doesn't exist

L can't be found.