$q$: Lagrangian dual fn

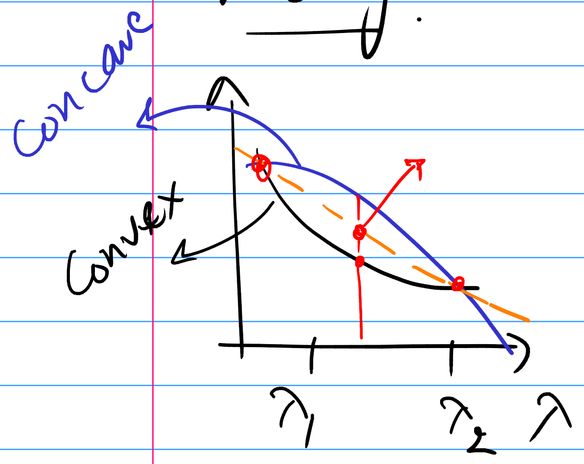
Update on defn of $q$. for some $\lambda$, $q(\lambda) \to -\infty$

Domain of $q$: $\mathcal{D} = \{\lambda \mid \underline{q(\lambda) > -\infty}\}$

2 results about the dual problem:

$$\left.\begin{array}{ll} \text{① } & q \text{ is concave} \\ \text{② } & \mathcal{D} \text{ is convex} \end{array}\right\} \text{ for } \underline{\underline{\text{any}}}\ f, c$$

Implication: Solving the dual problem is always a convex optimization problem.




**Proof:** ① $q$ is concave. $\qquad \alpha \in [0,1]$

define: $\rightarrow$ $q(\alpha\lambda_1 + (1-\alpha)\lambda_2) \geq \alpha\, q(\lambda) + (1-\alpha) q(\lambda_2)$

To prove

Defn of $\mathcal{L}$: $\mathcal{L}(x,\lambda) = f(x) - \lambda\, c(x)$

$$q = \min_x \mathcal{L}(x,\lambda)$$

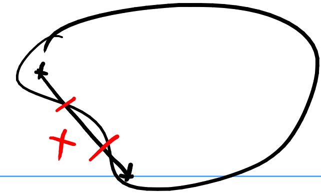
$$\mathcal{L}(x, (1-\alpha)\lambda_1 + \alpha\lambda_2) = \underbrace{(1-\alpha)\mathcal{L}(x,\lambda_1) + \alpha\,\mathcal{L}(x,\lambda_2)}_{\downarrow \min_x}$$

$$q((1-\alpha)\lambda_1 + \alpha\lambda_2) \geq (1-\alpha)\, q(\lambda_1) + \alpha\, q(\lambda_2)$$

$\longrightarrow$ $\left(\min(a+b) \geq \min(a) + \min(b)\right)$

$\Rightarrow$ $q$ is concave.

② Domain of $\mathcal{D}$ is <u>convex</u>.



<u>defn</u>: $\left[\begin{array}{ll} \text{if } \lambda_1 \in \mathcal{D} \text{ and } & \lambda_2 \in \mathcal{D} \\ \text{then } & \alpha\lambda_1 + (1-\alpha)\lambda_2 \in \mathcal{D} \end{array}\right]$

Using the fact that $q$ is concave:

$$q(\alpha\lambda_1 + (1-\alpha)\lambda_2) \geq \alpha q(\lambda_1) + (1-\alpha) q(\lambda_2)$$

Now, $\left.\begin{array}{l} \lambda_1 \in \mathcal{D} \Rightarrow q(\lambda_1) > -\infty \\ \lambda_2 \in \mathcal{D} \Rightarrow q(\lambda_2) > -\infty \end{array}\right]$ facts

$$\Rightarrow q(\alpha\lambda_1 + (1-\alpha)\lambda_2) > -\infty$$

$$\Rightarrow \alpha\lambda_1 + (1-\alpha)\lambda_2 \in \mathcal{D}.$$

$\Rightarrow \mathcal{D}$ is a convex set!

$\left(\begin{array}{l} q(\lambda) = \ln \lambda \\ \mathcal{D} = (0, \infty) \end{array}\right)$

⇒ Implication: primal problem → Anything (convex/nonconvex)

But dual problem → Always convex.

↓

CVX

↳ Weak duality: Primal $\min_{x} f(x)$ s.t. $c(x) \geq 0$

↳ For any feasible $\bar{x}$, and any $\bar{\lambda} \geq 0$, then the following holds true: $q(\bar{\lambda}) \leq f(\bar{x})$

~~Implication:~~

$$f^* = \text{Inf}\{ f(x) : x \in \Omega \}$$
$$q^* = \text{Sup}\{ q(\lambda) : \lambda \geq 0 \}$$
$$\Rightarrow q^* \leq f^*$$

optimum dual

optimum primal

**Proof:**

$$q(\bar{\lambda}) = \min_{x} \mathcal{L}(x, \bar{\lambda})$$

$$= \min_{x} \left[ f(x) - \bar{\lambda} c(x) \right]$$

$$q(\bar{\lambda}) \boxed{\leq} f(\bar{x}) - \underbrace{\bar{\lambda} c(\bar{x})}_{\geq 0}, \quad \underline{\bar{x} \in \Omega}$$

$$\bar{\lambda} \geq 0, \quad c(\bar{x}) \geq 0$$

$$\underline{q(\bar{\lambda}) \leq f(\bar{x})}$$

(a) If $q^* < f^* \Rightarrow f^* - q^* > 0$, duality gap (weak duality)

(b) If $q^* = f^* \Rightarrow$ No gap, strong duality.

(c) For convex problems, $\rightarrow$ always <u>strong duality</u>.

* (under constraint qualification, e.g. LICQ).

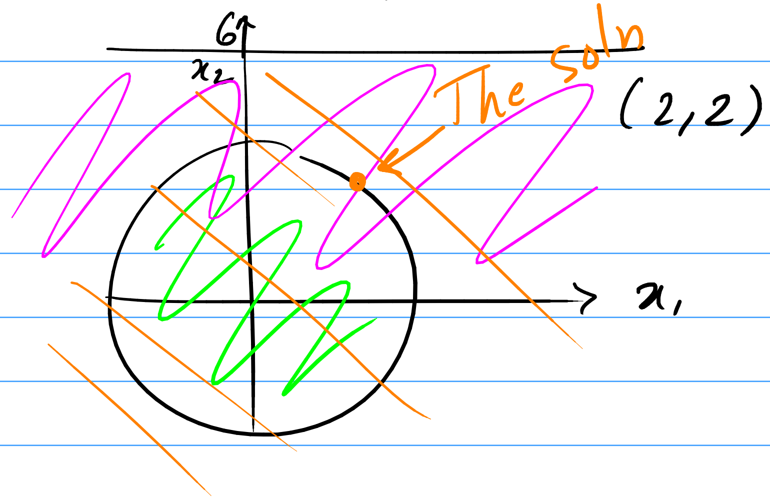
↳ Take an eg. $\min\left[-(x_1+x_2)\right]$ s.t. $8 - x_1^2 - x_2^2 \geq 0$

$6 - x_2 \geq 0$

graphically:

$f^* = -4$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$




## Dual problem:

① $\mathcal{L}(x,\lambda) = f(x) - \sum \lambda_i c_i(x)$

$= \left(-(x_1+x_2)\right) - \left(\lambda_1(8 - x_1^2 - x_2^2) - \lambda_2(6 - x_2)\right)$

$= x_1^2(\lambda_1) + x_2^2(\lambda_1) - x_1 - x_2(1-\lambda_2) - 8\lambda_1 - 6\lambda_2$

convex, ∴ makes sense to do $\min_x \mathcal{L}$

$q(\lambda) = \min_x \mathcal{L}(x,\lambda) \rightarrow \nabla_x \mathcal{L} = \begin{bmatrix} 2\lambda_1 x_1 - 1 \\ 2\lambda_1 x_2 - (1-\lambda_2) \end{bmatrix} = 0$

$$\left(x_1^* = \frac{1}{2\lambda_1},\quad x_2^* = \frac{1-\lambda_2}{2\lambda_1}\right),\quad q(\lambda) = \lambda_1(x_1^2+x_2^2) - x_1 + (\lambda_2 - 1)x_2 - 8\lambda - 6\lambda_2$$

$$\Rightarrow q(\lambda) = \left[\frac{1+(1-\lambda_2)^2 - 2 - 2(1-\lambda_2)^2}{4\lambda_1} - 8\lambda_1 - 6\lambda_2\right]$$

~~Step 2~~: Solve max problem. $\max_{\lambda \geq 0} q(\lambda)$, Checked concavity.

$$\max_{\lambda \geq 0} q(\lambda) = \min_{\lambda \geq 0}\left[\frac{1+(1-\lambda_2)^2}{4\lambda_1} + 8\lambda_1 + 6\lambda_2\right]$$

$\lambda_1 \geq 0,\ \lambda_2 \geq 0$ $\quad W = \overbrace{f(x)}\quad - \sum \lambda_i c_i(x)$

New lagrangian: $W(\lambda,\mu) = \left(\frac{1+(1-\lambda_2)^2}{4\lambda_1} + 8\lambda_1 + 6\lambda_2\right) - \mu_1\lambda_1 - \mu_2\lambda_2$

$\mu_1 \geq 0,\ \mu_2 \geq 0$

~~Solve~~ by KKT thm.

$\nabla_\lambda W = 0$, $\underbrace{\mu_1 \lambda_1 = 0}$, $\mu_2 \lambda_2 = 0$

$\lambda_1 = 0$ not allowed in domain of $q$

$\Rightarrow \mu_1 = 0$

$$\nabla_\lambda W = \begin{bmatrix} \dfrac{(1+(1-\lambda_2))^2}{-4\lambda_1^2} + 8 \\ \dfrac{-2(1-\lambda_2)}{4\lambda_1} + 6 - \mu_2 \end{bmatrix} = 0 \quad , \mu_2 \lambda_2 = 0$$

2 case $\hookrightarrow$ $\mu_2 > 0 \Rightarrow \underbrace{\lambda_2 = 0} \Rightarrow \dfrac{2}{-4\lambda_1^2} + 8 = 0 \Rightarrow \lambda_1 = 1/4$

$\dfrac{-1}{2\lambda_1} + 6 - \mu_2 = 0 \Rightarrow \mu_2 = 4$ ✓

$\hookrightarrow$ $\mu_2 = 0 \Rightarrow \dfrac{\lambda_2 - 1}{2\lambda_1} + 6 = 0 \Rightarrow \lambda_2 = 1 - 12\lambda_1$

$\Rightarrow 112\lambda_1^2 + 1 = 0$ ✗

—— x ——

The only soln is $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$

| | |
|---|---|
| $q(\lambda^*) = -4.$ | $f^* = -4$ |
| $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ | $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ |

dual = primal

<u>No duality gap</u>.

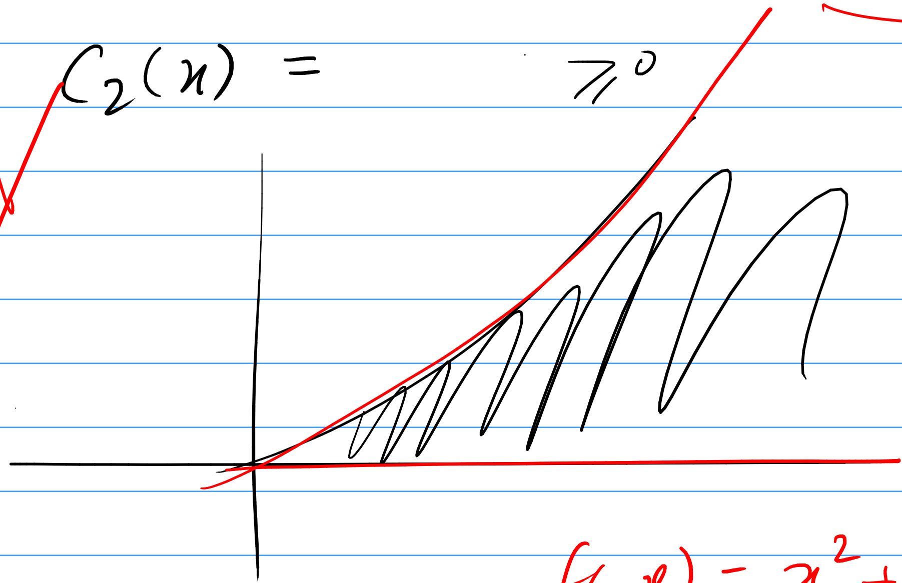
$C_i(x) \geq 0$

$C_1(x) = x_2 \geq 0$

$C_2(x) = \qquad \geq 0$

$C_2(x) = -x^2 \geq 0$

not convex.




$f(x) = x_1^2 + x_2^2$

$D =$