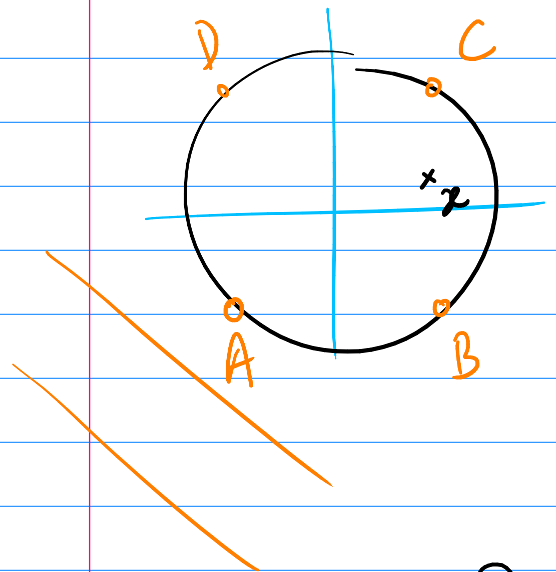
Single inequality constraint

$$f(x) = x_1 + x_2 \quad , \quad c_2(x) = 2 - x_1^2 - x_2^2 \geqslant 0$$



Soln = A

Q1. How do I improve? I'm at x

want to go to $x+s$ $\Rightarrow$ $c_2(x) \geqslant 0$

$c_2(x+s) \geqslant 0$

$$c_2(x+s) = c_2(x) + \nabla c_2^T(x) s$$

2 cases $\to$ Case I: x lies inside. $\to$ Inactive.

$c_2(x) > 0$

$$\left[\underbrace{c_2(x)}_{>0} + \underbrace{\nabla c_2^T(x) s} \geqslant 0 \right] \leftarrow$$

decrease $\nabla f^T(x) s < 0$

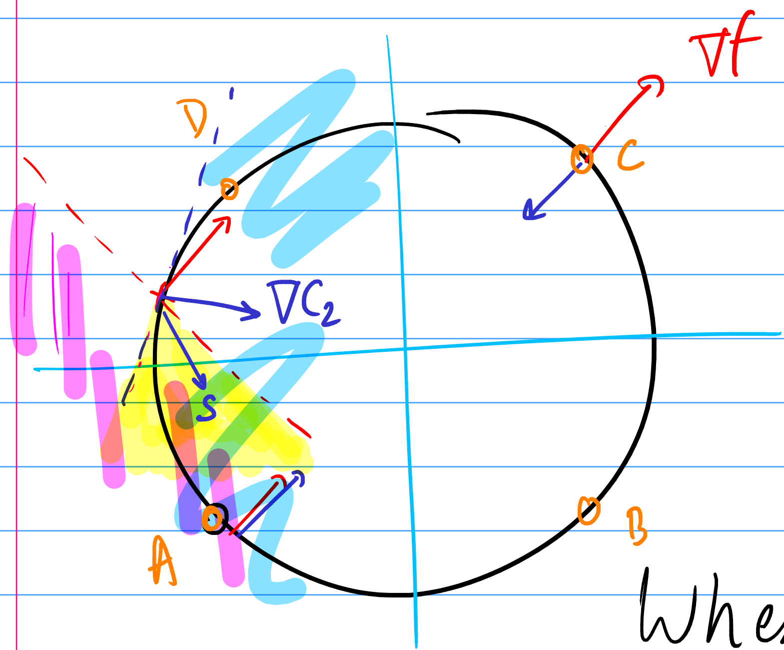
Any small s will work!

$$s = -\alpha \nabla f$$

Candidate
Not unique.

Case 2 Constraint active, $c_2(x) = 0$

$$c_2(x) + \nabla c_2^T(x)\, s \geq 0$$



$$\Rightarrow \nabla c_2^T(x)\, s \geq 0 \qquad \text{feasibility}$$

$$\nabla f^T(x)\, s < 0 \qquad \text{decrease}$$

$$\nabla c_2 = -2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

When do we stop improving?

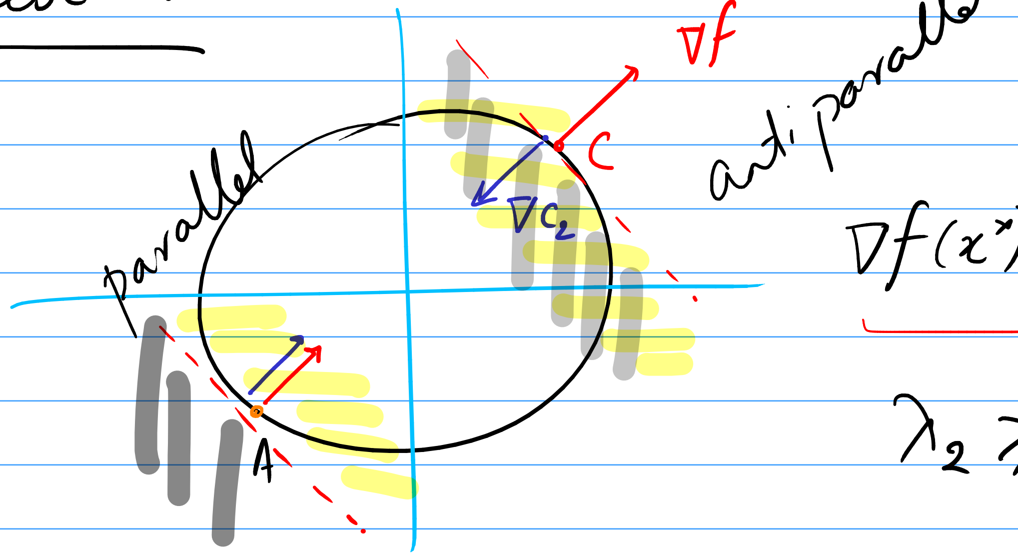
Case 1: $c_2(x) > 0$, no feasible dir

if $\nabla f(x) = 0$

$$\left.\begin{array}{l} \nabla_x \mathcal{L} = \underline{\nabla f(x)} - \lambda_2 \nabla c_2(x) \\ \text{Set } \lambda_2 = 0 \Rightarrow \underline{\nabla_x \mathcal{L} = 0} \end{array}\right. ✓$$

$$\mathcal{L}(x, \lambda) = f(x) - \lambda c_2(x).$$

Case 2



At pt A

$$\nabla f(x^*) = \lambda_2 \nabla c_2(x^*)$$

$\lambda_2 \geq 0$ captures A

$$\mathcal{L}(x,\lambda) = f(x) - \lambda_2 c_2(x), \quad \nabla_x \mathcal{L}(x,\lambda) = 0 \ \& \ \lambda_2 \geq 0$$

Let's combine these two:

① $\nabla_x \mathcal{L}(x^*,\lambda^*) = 0$ with $\lambda_2 \geq 0$, AND

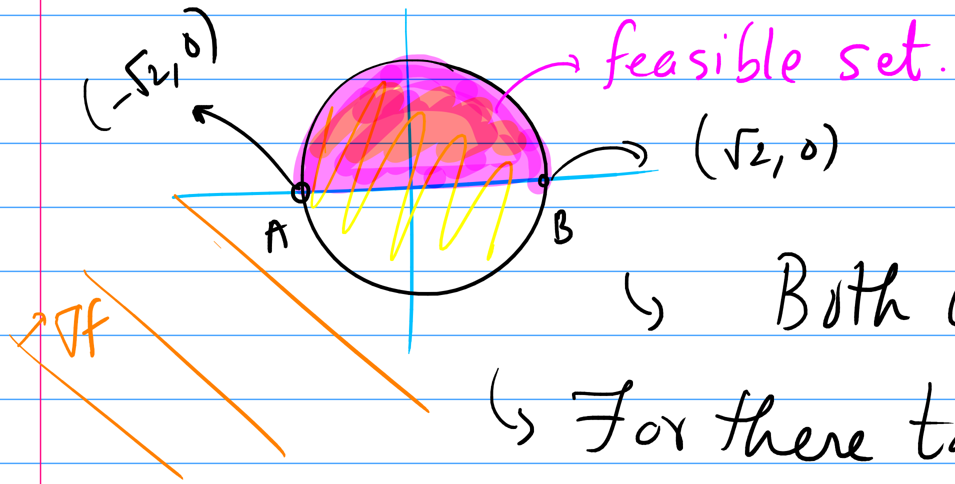
② $\underbrace{\lambda_2}_{=0 \text{ (case 1)}} \underbrace{c_2(x^*)}_{=0 \text{ (case 2)}} = 0$ → Complementarity condition

Central to C.O.

Take an e.g. $f(x) = x_1 + x_2$, s.t. $\begin{cases} x_1^2 + x_2^2 \leq 2 \\ x_2 \geq 0 \end{cases}$

Standard form:

$$\min_x f(x). \quad \text{s.t.} \begin{cases} c_1(x) = 2 - x_1^2 - x_2^2 \geq 0 \\ c_2(x) = x_2 \geq 0 \end{cases}$$



Consider pts A & B.

↳ Both constraints are active.

↳ For there to be a feasible dir to move in

$\nabla f^T(x) d < 0$ (decrease)

$\nabla c_i^T(x)^T d \geq 0$ (feasibility) (like case 2)

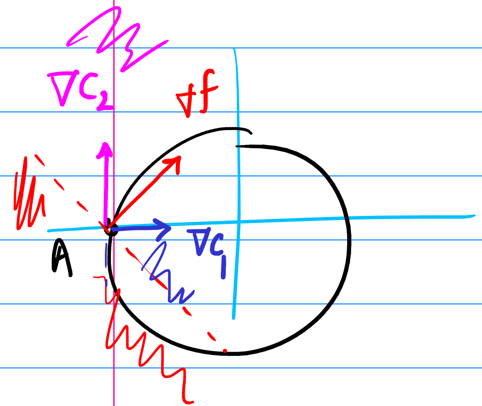
b) Define Lagrangian: $\mathcal{L}(x,\lambda) = f(x) - \lambda_1 c_1(x) - \lambda_2 c_2(x)$

$$= f(x) - \underbrace{\sum \lambda_i c_i(x)}_{\lambda^T c}$$

(1) $\nabla_x \mathcal{L}(x,\lambda) = 0$ & $\lambda_1 \geq 0, \lambda_2 \geq 0$

(2) $\lambda_1 c_1(x) = 0$, $\lambda_2 c_2(x) = 0$ complementarity

$$\nabla_x \mathcal{L}(x,\lambda) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} - \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda_1 x_1 \\ 1 + 2\lambda_1 x_2 - \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



<u>At A</u> $(-\sqrt{2}, 0)$

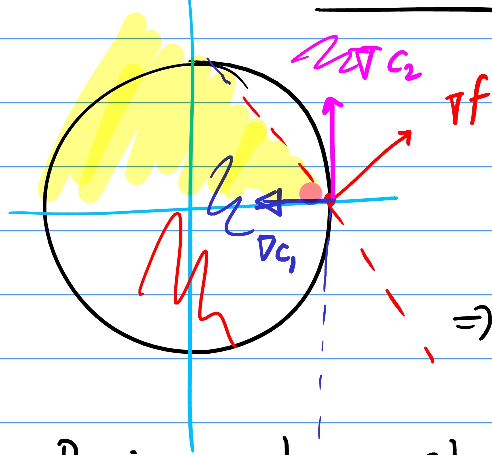
No feasible dir.

$1 - 2\sqrt{2}\lambda_1 = 0$

$1 - \lambda_2 = 0$

$\Rightarrow \lambda_1 = 1/2\sqrt{2}$, $\lambda_2 = 1$ ✓

A is a candidate for a minima pt.



<u>At B</u> $(\sqrt{2}, 0)$

$1 + 2\sqrt{2}\lambda_1 = 0$

$1 - \lambda_2 = 0$

$\Rightarrow \lambda_1 = -1/2\sqrt{2}$ ✗

$\lambda_2 = 1$

B is not a stationary pt.

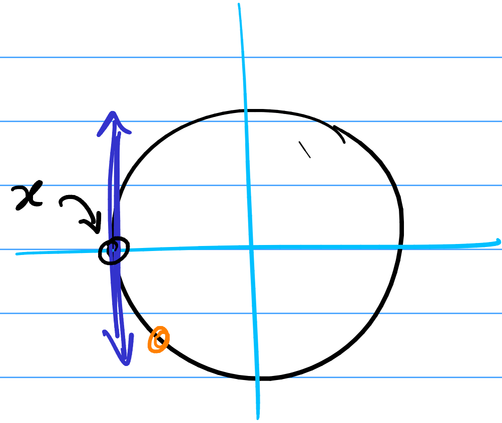
Linearized feasible directions: at (x)

$$\underline{\mathcal{F}(x)} = \left\{ d \;\middle|\; \begin{array}{ll} d^T \nabla c_i(x) = 0 & \forall i \in \overbrace{\mathcal{E}}^{\text{Equalities}} \\ d^T \nabla c_i(x) \geq 0 & \forall i \in \underbrace{\mathcal{A}(x) \cap \mathcal{I}}_{\text{Active inequalities}} \end{array} \right\}$$

Revisit older e.g.

$$f(x) = x_1 + x_2$$

s.t. $x_1^2 + x_2^2 - 2 = 0$



$$\mathcal{F}(x) = \{ d \mid d^T \nabla c_1 = 0 \} \longrightarrow [d_1 \;\; d_2] \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = 0 \Rightarrow -2\sqrt{2}\, d_1 = 0$$

Sub $(-\sqrt{2}, 0)$

$$\Rightarrow \boxed{\mathcal{F}(x) = \left\{ \begin{pmatrix} 0 \\ d_2 \end{pmatrix} \;\middle|\; d_2 \in \mathbb{R} \right\}}$$

c)

$$C_1(x) = \left(x_1^2 + x_2^2 - 2 \right)^2 = 0$$

Relook feasible directions. $\quad d^T \nabla C_1 = 0$

$$\begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 2(x_1^2 + x_2^2 - 2)(2x_1) \\ 2(x_1^2 + x_2^2 - 2)(2x_2) \end{bmatrix} = 0 \quad \text{at } (\sqrt{2}, 0)$$

$$\begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \Rightarrow \mathcal{F}(x) = \mathbb{R}^2$$

Geometry → Unchanged

(Algebra → Changed) → Constraint qualification