Constrained Optimization

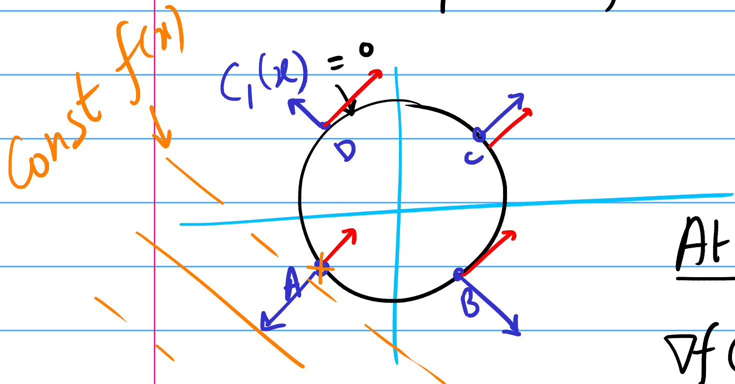
UnConstrained $\rightarrow \quad \min_{x} f(x)$

Constrained $\rightarrow \quad \min_{x \in \Omega} f(x), \quad \Omega = \left\{ x \;\middle|\; \begin{array}{l} c_i(x) = 0, \; i \in \mathcal{E} \\ c_i(x) \geq 0, \; i \in \mathcal{I} \end{array} \right.$

$\Omega \downarrow$ feasible set

Equality constrained problem

$$f(x) = x_1 + x_2 \,, \qquad c_1(x) = x_1^2 + x_2^2 - 2 = 0$$



$$\nabla c_1(x) = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

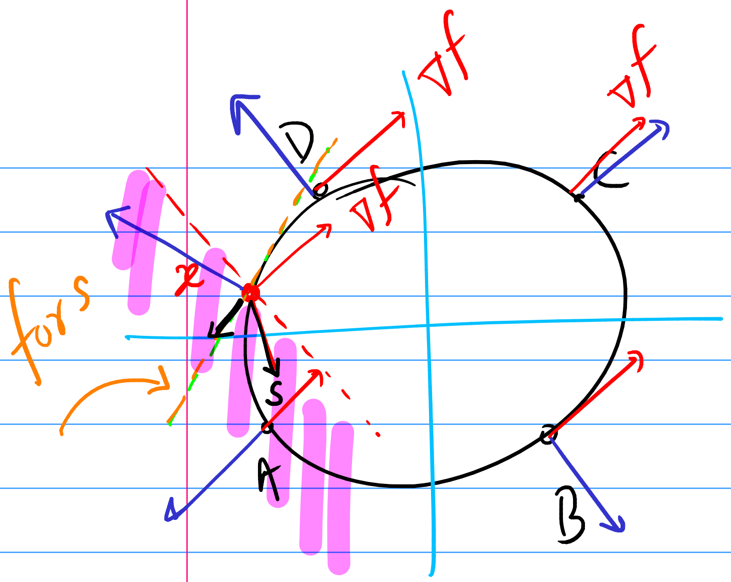
$$\nabla f(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Soln is A

At A

$$\nabla f(x^*) = \lambda \nabla c_1(x^*)$$

true also at pt C.



↳ I am at a <u>feasible</u> point, x

↳ I want to move to a better point. $\rightarrow$ $x + s$.

$$c_1(x) = 0 \longrightarrow c_1(x+s) = 0$$

Taylor's thm

$$c_1(x+s) = c_1(x) + \nabla c_1^T(x)\, s$$

$$\Rightarrow \nabla c_1^T(x)\, s = 0$$

Better:

$$f(x+s) < f(x) \longrightarrow \text{Required}$$

Taylor's thm

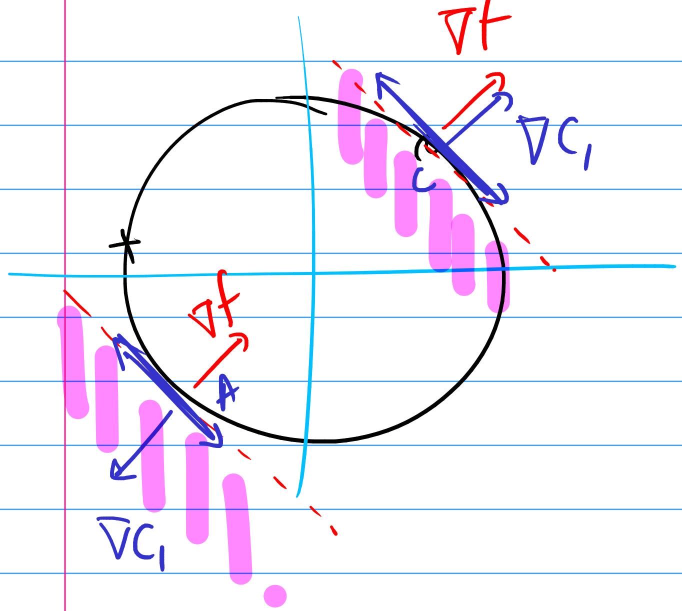
$$f(x+s) = f(x) + \nabla f(x)^T s$$

$$\Rightarrow \nabla f^T(x)\, s < 0$$

strict

Thus: $\left[\ \nabla c_1^T(x)\, s = 0 \text{ AND } \nabla f^T(x)\, s < 0\ \right]$

↳ When do we stop / when does improvement stop.



$\left[\begin{array}{l} \to \nabla f^T s < 0 \\ \nabla c_1^T s = 0 \end{array}\right] \begin{array}{l} \to \text{Improvement} \\ \to \text{feasibility} \end{array}$

They are not satisfied.

$\underbrace{\nabla f(x^*) = \lambda_1 \nabla c_1(x^*)}$ further improvement ✗

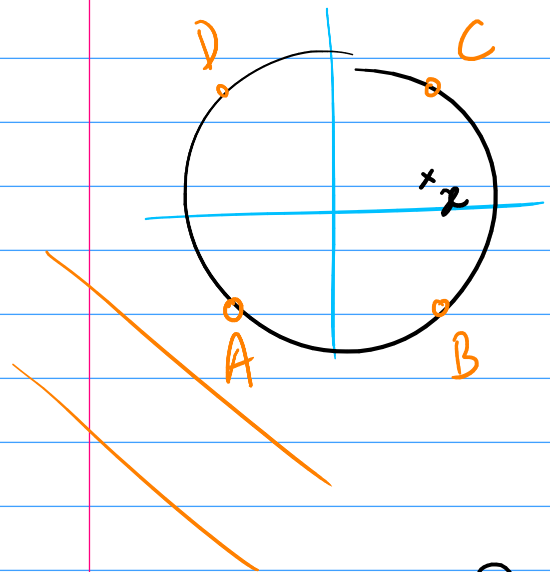
<u>Define</u>: $\mathcal{L}(x, \lambda_1) = f(x) - \lambda_1 c_1(x)$ [Lagrangian]

↳ λ_1: Lagrange multiplier.

∴ Stationary pt: $\underbrace{\nabla_x \mathcal{L}(x^*, \lambda_1^*) = 0}$

Single inequality constraint

$f(x) = x_1 + x_2$, $c_2(x) = 2 - x_1^2 - x_2^2 \geq 0$



Soln = A

Q1. How do I improve? I'm at x

want to go to $\underline{x+s}$. $\Rightarrow$ $c_2(x) \geq 0$

$c_2(x+s) \geq 0$

$$c_2(x+s) = c_2(x) + \nabla c_2^T(x) s$$

2 cases $\rightarrow$ Case 1: x lies inside. $\rightarrow$ Inactive.

$c_2(x) > 0$

decrease $\nabla f^T(x) s < 0$

$$\left[\underbrace{c_2(x)}_{>0} + \underbrace{\nabla c_2^T(x) s} \geq 0 \right] \leftarrow$$

Any small s will work!

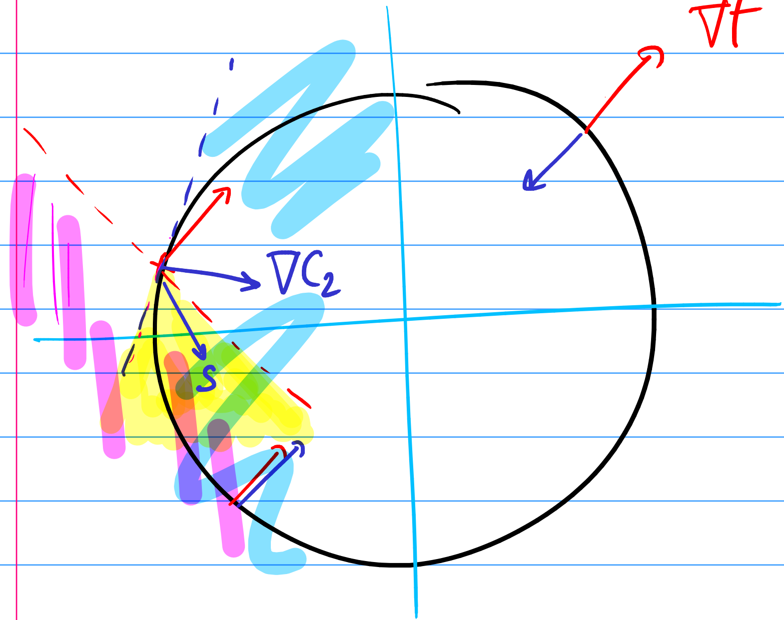
$$s = -\alpha \nabla f$$

Candidate

Not unique.

Case 2 → Constraint active, $c_2(x) = 0$

$$c_2(x) + \nabla c_2^T(x)\, s \geq 0$$



$$\Rightarrow \quad \nabla c_2^T(x)\, s \geq 0 \qquad \text{feasibility}$$

$$\nabla f^T(x)\, s < 0 \qquad \text{decrease}$$

$$\nabla c_2 = -2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$