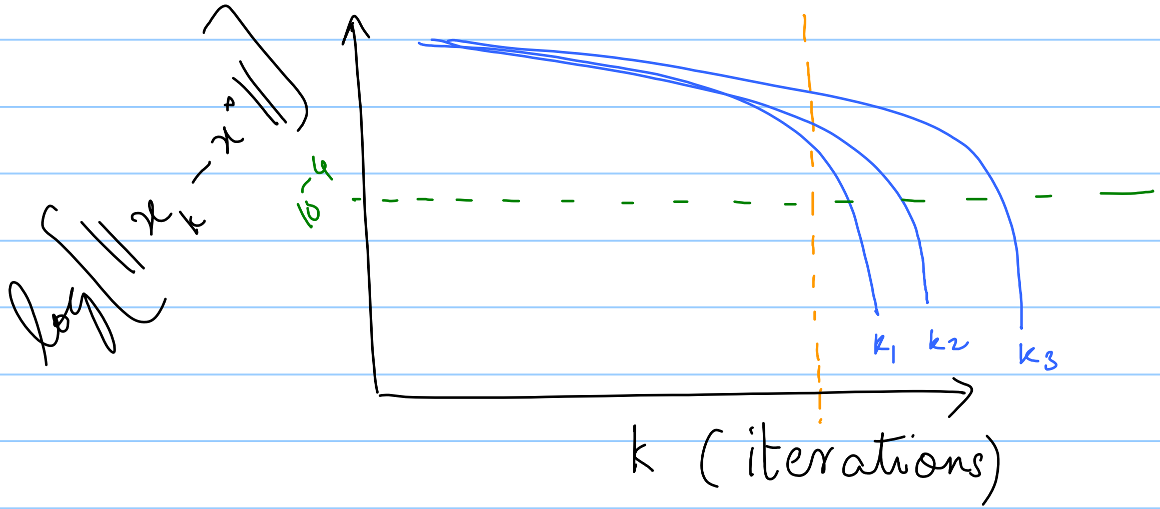
Preconditioned CG



$\kappa_1 < \kappa_2 < \kappa_3$

(1) For the same error higher κ (cond no)
$\Rightarrow$ more iterations

(2) For a fixed no of iters higher κ (cond no)
$\Rightarrow$ more error.

Keep as fixed $\rightarrow$ A must be sym P.D.

$$Ax = b$$

idea

$$TAx = Tb \qquad \boxed{\kappa(TA) < \kappa(A)}$$

TA may not be sym P.D.

ideal world $\rightarrow$ $T = A^{-1}$

$$Ax = b$$

R.M by some L^T

$$L^T A x = L^T b$$

$$\underbrace{L^T A L}\,\underbrace{L^{-1} x} = \underbrace{L^T b}$$

(L be invertible)

$\hat{A}$ $\hat{x}$ $\hat{b}$

$\rightarrow \hat{A}\hat{x} = \hat{b}$

Is $\hat{A}$ Sym P.D

$z^T \hat{A} z > 0 \quad \forall z \neq 0$

$L^{-1} x = \hat{x}$

$z^T L^T A L z > 0$ ✓

How do we work this with min effort.

① Conjugate dirs: $\boxed{\hat{p}_i^T \hat{A} \hat{p}_j} = \delta_{ij} \tau_i \Rightarrow = p_i^T A p_j$

$\underbrace{\hat{p}_i^T L^T}_{p_i^T} A \underbrace{L \hat{p}_j}_{p_j} = \text{"}$

$\therefore L \hat{p}_j = p_j \Rightarrow \hat{p}_j = L^{-1} p_j$

(2) Residual: $\hat{r}_k = \hat{A}\hat{x}_k - \hat{b} = L^T A L L^{-1} x_k - L^T b$

$$= L^T (A x_k - b) = L^T r_k$$

(3) $$\hat{\alpha}_k = \frac{\hat{r}_k^T \hat{r}_k}{\hat{p}_k^T \hat{A} \hat{p}_k} = \frac{r_k^T L L^T r_k}{p_k^T A p_k} = \frac{r_k^T y_k}{p_k^T A p_k}$$

(4) $$\hat{\beta}_{k+1} = \frac{\hat{r}_{k+1}^T \hat{r}_{k+1}}{\hat{r}_k^T \hat{r}_k} = \frac{r_{k+1}^T L L^T r_{k+1}}{r_k^T L L^T r_k} = \frac{r_{k+1}^T y_{k+1}}{r_k^T y_k}$$

(5) Introduce $L L^T r_k = y_k$

(6)

$$\hat{p}_k = -\hat{r}_k + \hat{\beta}_k \hat{p}_{k-1}$$

$$L^{-1} p_k = -L^T r_k + \hat{\beta}_k L^{-1} p_{k-1}$$

$$p_k = -L L^T r_k + \hat{\beta}_k \underbrace{L L^{-1}}_{I} p_{k-1}$$

$$p_k = -y_k + \hat{\beta}_k p_{k-1}.$$

(7)

$$\hat{p}_0 = -\hat{r}_0$$

$$L^{-1} p_0 = -L^T r_0$$

$$\Rightarrow \quad \underline{p_0} = -L L^T r_0 = -\underline{y_0}$$

(8)

$$\hat{x}_{k+1} = \hat{x}_k + \hat{\alpha}_k \hat{p}_k$$

$$L^{-1} x_{k+1} = L^{-1} x_k + \hat{\alpha}_k L^{-1} p_k$$

$$\Rightarrow \quad x_{k+1} = x_k + \hat{\alpha}_k p_k$$

Price to be paid $\Rightarrow$ $LL^T r_k = y_k$

↪ Choice of L? $\quad \hat{A} = L^T A L$

Choleski decomp of $A = CC^T$

C → Lower triangular.

$\hat{A} = L^T C C^T L$ ideal L?

$L^T = C^{-1}$

Incomplete Cholesky decomp

$A \approx K K^T$ sparse lower tri

$\| A - KK^T \|_F \neq 0$

$\Rightarrow$ Choose $L^T = K^{-1}$

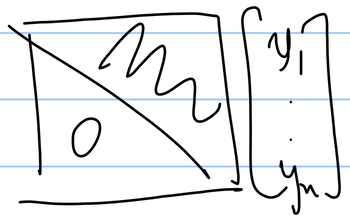
$L = (K^{-1})^T$

$$LL^T r_k = y_k \rightarrow (K^{-1})^T K^{-1} r_k = y_k$$

$$\boxed{r_k = \underbrace{KK^T}_{\text{Given by ichol}} y_k.}$$

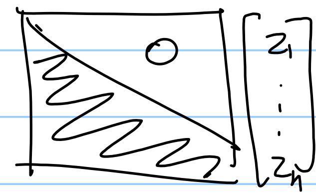
① No need for K^{-1}!

② $r_k = K\underbrace{K^T y_k}_{z_k} = K z_k.$



$K^T y_k = z_k$

back subst



fwd subst

get z_k cheaply.

Summary. Given A, b

① $A \approx KK^T$ (ichol) <cheap>

② Pick random/your choice of x_0

③ $p_0 = -y_0 \rightarrow$ Solving $r_0 = KK^T y_0$

$r_0 = Ax_0 - b$

$\boxed{\text{FB}}$

④ $$\hat{\alpha}_k = \frac{r_k^T y_k}{p_k^T A p_k}$$ ✓

⑤ $$x_{k+1} = x_k + \hat{\alpha}_k p_k$$ ✓

⑥ $$\hat{\beta}_{k+1} = \frac{r_{k+1}^T y_{k+1}}{r_k^T y_k}$$ ✓

⑦ $$p_k = -y_k + \hat{\beta}_k p_{k-1}$$