$$\boxed{x = A^{-1} b}$$

$$Ax = b \longrightarrow \text{full rank}$$

LM by $A^T \rightarrow \boxed{\underbrace{A^T A}\, x = A^T b}$

$$\downarrow$$

$$x^T (A^T A) x \qquad \text{P.D}$$

$$= \|Ax\|^2 = 0 \quad \text{only when } x = 0$$

Conjugate Gradient Method (CGM)

1) $$p_k = -r_k + \underbrace{\beta_k}_{\text{Scalar factor}} p_{k-1}$$

$$r_k \rightarrow (Ax_k - b)$$

2) $$p_0 = -r_0$$

Conjugacy $\rightarrow$ $$p_i^T A p_j = 0 \qquad i \neq j$$

LM $\rightarrow$ $$\cancelto{0}{p_{k-1}^T A p_k} = -p_{k-1}^T A r_k + \beta_k p_{k-1}^T A p_{k-1}$$

I want conjugacy.

$$\Rightarrow \quad \beta_k = \frac{p_{k-1}^T A r_k}{p_{k-1}^T A p_{k-1}}$$

$$x_{k+1} = x_k + \alpha_k p_k, \quad \alpha_k = \frac{-r_k^T p_k}{p_k^T A p_k}.$$

$$\begin{bmatrix} 2 \text{ vv}, & 1 \text{ mv} \\ 2 \text{ vv}, & 2 \text{ mv} \end{bmatrix}$$

Some more algebra.

$$r_{k+1} - r_k = A(x_{k+1} - x_k) = \alpha_k A p_k.$$

1 mv
2 vv

Final:

$$\alpha_k = \frac{r_k^T r_k}{p_k^T A p_k}, \quad \beta_k = \frac{r_k^T r_k}{r_{k-1}^T r_{k-1}}$$

To store $p_k = -r_k + \beta_k p_{k-1}$

to get: $\underbrace{r_k, p_{k-1}, r_{k-1}}$

Linear rate of convergence

A has eigvalues: $\lambda_1 \leq \lambda_2 \; \dots \leq \lambda_n$

Cond no $\kappa = \dfrac{\lambda_n}{\lambda_1}$

1) If I know the eigvalues of A

$$\|x_{k+1} - x^*\| \leq \left[\frac{\lambda_{n-k} - \lambda_1}{\lambda_{n-k} + \lambda_1}\right] \|x_0 - x^*\|$$

$n = 10$

$k = 1$

$\|x_2 - x^*\|$

$k = 5$

$(\lambda_9 - \lambda_1 \;/\; \|x_6 - x^*\| \to (\lambda_5 - \lambda_1) \xrightarrow{} 0$

2) If I don't know eigvalues of A

$\rightarrow K$.

$$\| x_{k+1} - x^* \| \leq 2 \left(\frac{\sqrt{k} - 1}{\sqrt{k} + 1} \right)^k \| x_0 - x^* \|$$

In SD $\sqrt{k} \longrightarrow K$ ✓

CGM

1) $r_k^T r_i = 0$ for $i = 0, 1, \ldots, k-1$

EST $\rightarrow$ $r_k^T p_i = 0$ for $i \in [0, k-1]$

$$\underbrace{p_i = -r_i + \beta_i p_{i-1}}$$

$$r_k^T(\;) = \underbrace{\cancel{r_k^T p_i}}_{0} = -r_k^T r_i + \beta_i \underbrace{\cancel{r_k^T p_{i-1}}}_{0}$$

$$\Rightarrow r_k^T r_i = 0 \quad , \; i \in [0, k-1]$$