↳ A is sym pos. def. : Do conj vects exist?

Yes. $p_i^T A p_j = 0 \quad i \neq j$

$$A = U \Lambda U^T$$ u_i's are orthogonal.

$$AU = U\Lambda$$

$$A u_i = \lambda u_i \xrightarrow[u_j^T]{\text{LM by}} u_j^T A u_i = 0 \quad i \neq j$$

⇒ eigvectors are conjugate directions.

$$\boxed{f(x) = \frac{1}{2} x^T A x - b^T x} \leftarrow k \quad (1)$$

$$\nabla f = (Ax - b)$$

$$\phi(x) = \left[\frac{1}{2} \|Ax - b\|^2 \right.$$ ← k^2 (2)

$$\left. \cancel{+ \lambda \|x - x_0\|^2} \right]$$

$$\lambda \|x\|_1$$

Regularization.

C-DM

$$f(x) = \frac{1}{2} \underbrace{x^T A x}_{\text{quadratic.}} - b^T x$$

1) Assume A is diagonal.

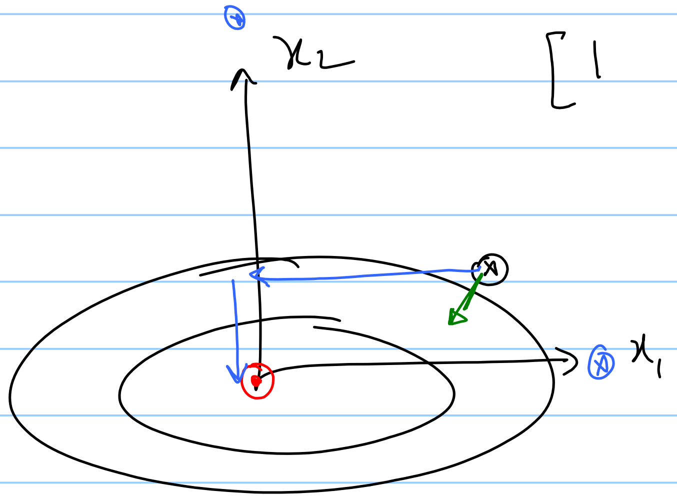
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$$

$$p_i^T A \, p_j = 0 \quad i \neq j$$

p_0

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} & \\ & \end{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

p_1



2) A is not diagonal.

$$x^T A x$$

$$I = P P^{-1} \qquad (P^T)(P^T)^{-1}$$

$$P = \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \\ | & | & & \end{bmatrix}$$

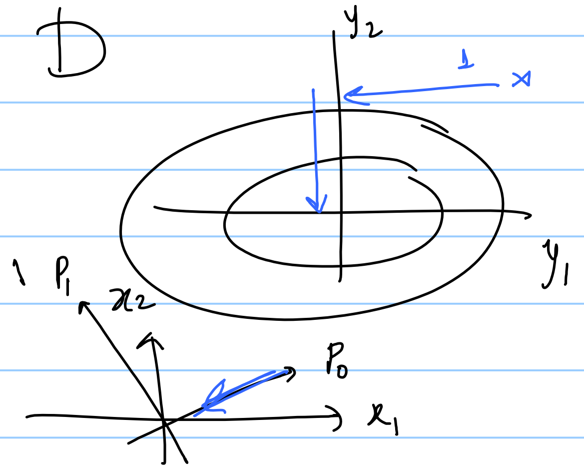
$$= \underbrace{x^T (P^T)^{-1}}_{y^T} \underbrace{(P^T) A P}_{} \underbrace{P^{-1} x}_{y} = \underbrace{y^T D y}_{}$$

$$P^T A P$$

$$\begin{bmatrix} p_0^T & \text{---} \\ p_1^T & \text{---} \\ & \end{bmatrix} A \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \\ | & | & & \end{bmatrix} = D$$

$$y = P^{-1} x \Rightarrow x = P y.$$

$$y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \Rightarrow x_1 = p_0$$



Ways of generating p's

① Eigen decomposition

② Modified Grahm Schmidt

$\Big\} O(n^3)$

eigvectors $\Rightarrow$ conjugacy ✓

Conjugacy $\Rightarrow$ eigvectors ✗

~~CDM~~ $\rightarrow$ $\boxed{\text{CGM}}$

Expanding Subspace Theorem.

Result: Using the CDM $\{x_k\}$, starting from x_0 minimizing ϕ

① $r_k^T p_i = 0$ for $i \in [0, k-1]$

IFF

② In an affine space $\{x \mid x_0 + \text{span}\{p_0, \ldots, p_{k-1}\}\}$, x_k is the minimizer of $\phi(x)$.

Proof: $r(x) = Ax - b = \nabla \phi(x)$

$$h(\sigma) = \phi\left(x_0 + \sum_{j=0}^{k-1} \sigma_j p_j\right) \rightarrow \text{Is it convex?}$$

vector

Yes & it has a unique minimizer

Chain rule.

$$\frac{\partial h}{\partial \sigma_i} = 0 \quad \forall\, i \in [0, k-1]$$

because these σ_i's $\equiv$ minimizer

$$\nabla \phi\left(x_0 + \sum \sigma_j p_j\right)^T p_i = 0$$

$$\Rightarrow \quad r(x_k)^T p_i = 0 \quad \forall\, i \in [0, k-1].$$