Matrix Computations

1) $$\underbrace{A}_{m\times n} = \underbrace{U}_{m\times m}\underbrace{\Sigma}_{m\times n}\underbrace{V^T}_{n\times n}$$

Say $m = n$

$$A^{-1} = \left(U\Sigma V^T\right)^{-1} = V\Sigma^{-1}U^T$$

$$\Sigma^{-1} = \text{diag}(1/\sigma_i)$$

when $\sigma_i \neq 0$.

Define:

$$A^{+} = \underbrace{V}_{n\times n}\underbrace{\Sigma^{+}}_{n\times m}\underbrace{U^T}_{m\times m}$$

$$(\Sigma^{+})_{ii} = \begin{cases} \frac{1}{\sigma_i}, & \sigma_i \neq 0 \\ 0 & \text{else} \end{cases}$$

<u>When $m > n$</u>.

$$\Sigma = \begin{bmatrix} \ddots \; \sigma_i & \\ & \end{bmatrix}, \quad \Sigma^{+} = \begin{bmatrix} \ddots \; 1/\sigma_i & \end{bmatrix}$$

$$\Sigma\Sigma^{+} = \begin{bmatrix} \ddots \; 1 & 0 \end{bmatrix} \qquad \Sigma^{+}\Sigma = \begin{bmatrix} \ddots & \end{bmatrix}$$

$$= I.$$

$$AA^+ = (U\Sigma V^T)(V\Sigma^+U^T) = U\Sigma\Sigma^+U^T$$

$$A^+A = (V\Sigma^+U^T)(U\Sigma V^T) = I.$$

$\Rightarrow$ A^+ is the left inverse of A. when $m>n$

$\Rightarrow$ A^+ is the right inverse of A when $m<n$.

$$\hat{x} = \underbrace{(A^TA)^{-1}A^T}\, b$$

$$A^TA = (V\Sigma^TU^T)(U\Sigma V^T) = \overbrace{V(\underbrace{\Sigma^T\Sigma}_{\text{diag}(\sigma_i^2)})V^T}^{n\times n}$$

$$\Rightarrow (A^T A)^{-1} = V \underbrace{(\Sigma^T \Sigma)^{-1}}_{\mathrm{diag}(1/\sigma_i^2)} V^T$$

$$(A^T A)^{-1} A^T = \left(V \underbrace{(\Sigma^T \Sigma)}^{-1} V^T \right) \left(V \underbrace{\Sigma}^T U^T \right)$$

$$= V \underbrace{\Sigma^+}_{n \times m} U^T$$

$$\boxed{Ax = b \longrightarrow x = A^+ b}$$

2) "Condition number"

$$Ax = b \quad \text{--- (1)}$$

due to errors

$$A(x + \Delta x) = b + \Delta b \quad \text{--- (2)}$$

$$\boxed{A\,\Delta x = \Delta b \Rightarrow \Delta x = A^{-1}\Delta b}$$

Quantify error by $\| \ \|_2$

Define: $\kappa = \dfrac{\|\Delta x\| / \|x\|}{\|\Delta b\| / \|b\|} = \left(\dfrac{\|A^{-1}\Delta b\|}{\|\Delta b\|}\right) \times \left(\dfrac{\|b\|}{\|A^{-1}b\|}\right)$

$$\frac{\|A^{-1}\Delta b\|_2^2}{\|\Delta b\|_2^2} = \frac{\|V\Sigma^{-1}U^T\Delta b\|^2}{\|\Delta b\|_2^2} = \frac{\|\Sigma^{-1}\overbrace{U^T\Delta b}^{y}\|_2^2}{\|\Delta b\|_2^2}$$

U^T

$$= \left(\frac{\sum_i (1/\sigma_i^2)\, y_i^2}{\sum y_i^2}\right)$$

$z = U^T b$;

$$\frac{\| A b \|}{\| b \|^2} = \frac{\sum_i \left(1/\sigma_i^2\right) z_i^2}{\sum z_i^2}$$

$$\sum_i \left(\frac{1}{\sigma_i^2}\right)\left(\frac{y_i^2}{\sum y_i^2}\right) = \boxed{\sum \left(\frac{1}{\sigma_i^2}\right) \eta_i}$$

$$\sum \eta_i = 1$$

$$\eta_i \geq 0$$

same form

$$K^2 = \frac{\overbrace{\sum \left(1/\sigma_i^2\right)\left[y_i^2 / \sum y_i^2\right]}}{\underbrace{\sum \left(1/\sigma_i^2\right)\left[z_i^2 / \sum z_i^2\right]}} \begin{array}{l} \longrightarrow \text{MAX} \\ \longrightarrow \text{MIN} \end{array}$$

$$\left(\frac{1}{\sigma_{max}^2}\right) \underbrace{\sum \eta_i} \leq \sum \left(\frac{1}{\sigma_i^2}\right) \eta_i \leq \left(\frac{1}{\sigma_{min}^2}\right) \underbrace{\sum \eta_i}$$

$$\kappa = \left(\frac{\sigma_{max}}{\sigma_{min}}\right) = \frac{\sigma_1}{\sigma_r}$$ Condition no of A.

e.g. $A = \begin{pmatrix} 1 & 1000 \\ 0 & 1 \end{pmatrix}$, $\lambda = 1, 1$

$\left(A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ Orig.} \right)$ $\left(A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0.01 \end{pmatrix} \text{ New} \right)$

Soln: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Soln: $\begin{pmatrix} -9 \\ 0.01 \end{pmatrix}$

$\sigma = 10^3, 10^{-3} \Rightarrow \kappa = 10^6$

Convention: $\kappa > 10^4 \longrightarrow$ A is ill conditioned.

$Ax = b \longrightarrow \kappa(A)$

$$A^T A x = A^T b$$

$$\text{SVD}\,(A^T A) : (V \Sigma^T U^T)(U \Sigma V^T)$$

$$= \left(V \underbrace{\Sigma^T \Sigma}_{\sigma_i^2} V^T \right)$$

bad condn no became worse.

— x —

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T \leftarrow \underline{A = U \Sigma V^T}$$

$$A x = b \rightarrow \hat{x} = A^+ b$$

$$V \Sigma U^T$$

$$= \sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^T$$

$$\Rightarrow \hat{x} = \sum_{i=1}^{r} \underbrace{\left(\frac{u_i^T b}{\sigma_i} \right)}_{\text{scalar}} \underbrace{v_i}_{\text{vector}}$$

$\Rightarrow$ Smallest σ_i has max influence on soln.

$\rightarrow$ Truncated SVD $\rightarrow$ d is $\leq r$.

$$\Rightarrow \hat{x} = \sum_{i=1}^{d} \left(\frac{u_i^T b}{\sigma_i} \right) v_i$$

Choosing d like cooking!

↳ (2) Tikhonov regularization.

$$\hat{x} = \sum_{i=1}^{r} \left(u_i^T b\right) \left(\frac{\sigma_i}{\sigma_i^2 + \lambda^2}\right) v_i$$