$$\Sigma_{m\times n} = \begin{bmatrix} \ddots & \end{bmatrix} \qquad \Sigma^T_{n\times m} = \begin{bmatrix} \ddots & \end{bmatrix}$$

$$A = U\Sigma V^T$$

$$Av_i = \sigma_i u_i$$

$$AA^T = (U\Sigma V^T)(V\Sigma^T U^T) = U\Sigma\Sigma^T U^T$$

$$(AA^T)U = U\underbrace{\Sigma\Sigma^T}_{m\times m} \rightarrow \begin{bmatrix} \sigma_i^2 & 0 \\ 0 & \ddots \end{bmatrix}$$

$$(AA^T)\begin{bmatrix} u_1 & \cdots \\ | & \end{bmatrix} = U\begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_m^2 \end{bmatrix}$$

Eigen.value prob

Similarly $(A^TA) = (V\Sigma^T U^T)(U\Sigma V^T)$

$$(A^TA)V = V\underbrace{\Sigma^T\Sigma}_{\text{diag}(\sigma_i^2)}$$

$$\begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix}_{n \times n}$$

Tall: $m \times n$ $\quad A : U \Sigma V^T$ $\quad$ l. singular, r. singular

$m > n$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \\ & & \\ & & \end{bmatrix} \quad n$$

$$\Sigma \Sigma^T = \begin{bmatrix} \diagdown \\ \\ \end{bmatrix} \begin{bmatrix} \diagdown & \end{bmatrix} = \begin{bmatrix} \sigma_i^2 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \end{bmatrix} \quad n$$

$$\Sigma^T \Sigma = \begin{bmatrix} \diagdown & \end{bmatrix} \begin{bmatrix} \diagdown \\ \\ \end{bmatrix} = \begin{bmatrix} \diagdown \end{bmatrix} \quad n$$

Cols

$$AB = \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_n \end{bmatrix} \rightarrow \text{L.C. of cols of } A$$

$$= \begin{bmatrix} - a_1^T B - \\ - a_2^T B - \\ - a_m^T B - \end{bmatrix}$$

Rows $\rightarrow$ L.C. of rows of B

$AA^T \longrightarrow$ Cols are L.C. of cols of A

$A^TA \longrightarrow$ rows are L.C. of rows of A.

If rank(A) = r $\Rightarrow$ r lin indepn rows/cols of A.

$A, AA^T, A^TA \Rightarrow$ Have same rank r.

$\sqrt{\lambda_i} = \sigma_i$

$\Rightarrow$ No of singular values = $\underline{\underline{r}}$

By defn. : $\sigma_1^2 \geq \sigma_2^2 \geq \sigma_3^2 \ldots \geq \sigma_r^2$

$$A \begin{bmatrix} \overset{\longleftrightarrow\, r}{v_1} & v_2 & \cdots & v_r \\ \updownarrow n & | & & \end{bmatrix} = \underbrace{\begin{bmatrix} \overset{\longleftrightarrow\, r}{u_1} & u_2 & u_r \\ \updownarrow m & | & \end{bmatrix}}\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}_{r \times r} \quad \text{①}$$

$$A^T = V \Sigma^T U^T$$

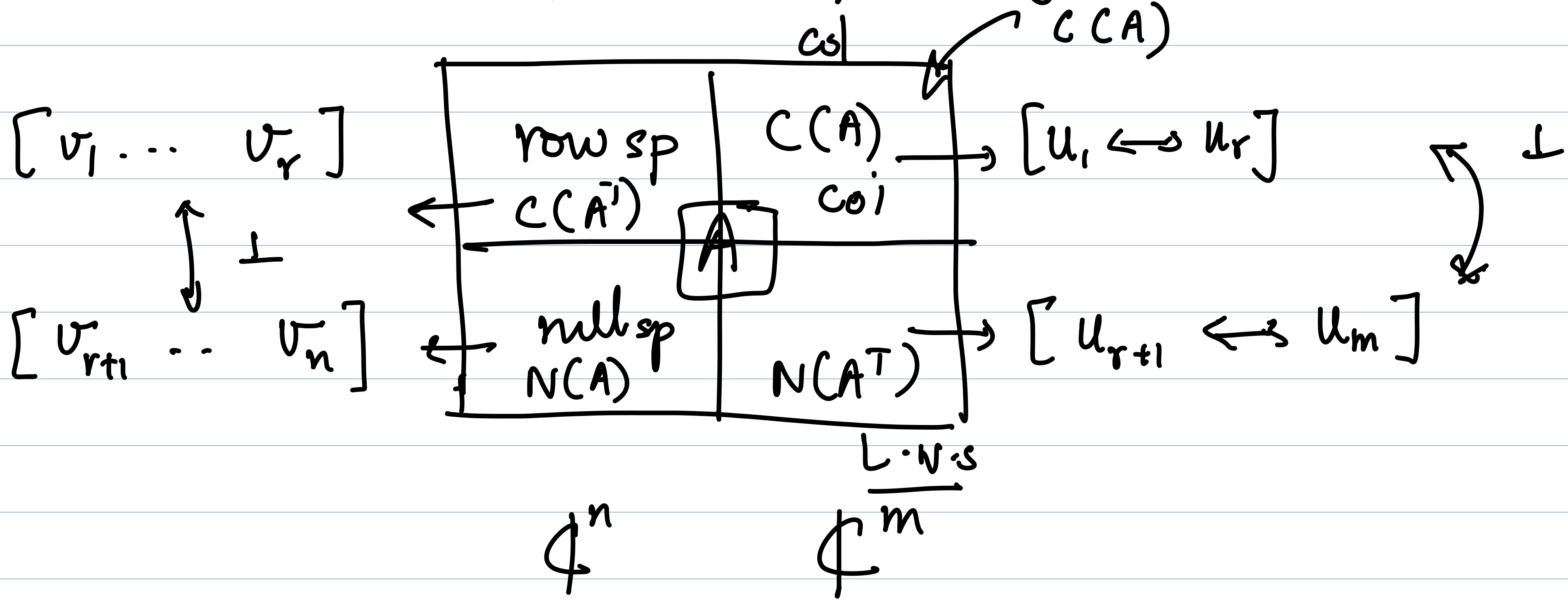
$$A^T \begin{bmatrix} u_1 & \cdots & u_r \\ \updownarrow m & & | \end{bmatrix} = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_r \\ | & \updownarrow & & | \end{bmatrix}}_{n}\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}_{r \times r} \quad \text{②}$$

All are L.C.s of cols of A.

These r- linearly indepn vectors form the basis of col space of A.

All are L.C.s of rows of A (cols of A^T)

These r - linearly indepn vectors form the basis of row space of A.



$A = U \Sigma V^T$ → no of non zero = rank (r)

U → $\mathbb{C}^m$ → first r, $m-r$

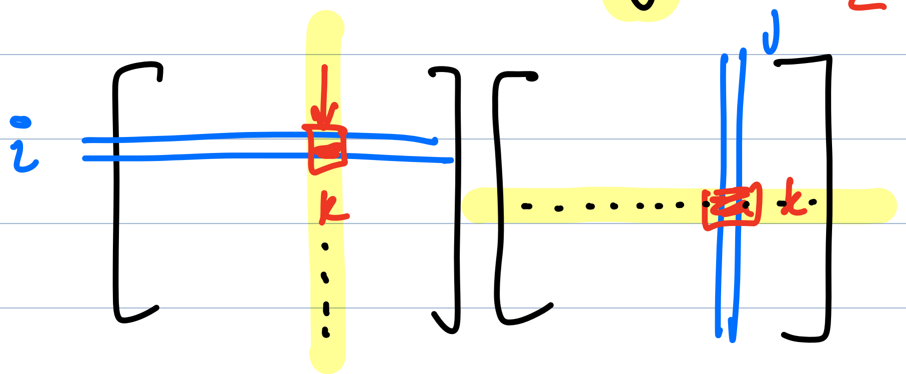
V^T → $\mathbb{C}^n$ → first r, $n-r$

$O(n^3)$

$C(A) \quad N(A^T) \qquad C(A^T) \quad N(A)$

$\hookrightarrow$ Outer product notation of SVD

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$



$$= \sum_k \left(a_k b_k^T\right)_{ij}$$

k^{th} col of A $\qquad$ k^{th} row of B.

$$AB = \sum a_k b_k^T$$

outer product

$$pq^T$$

$$\begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix} \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} = \begin{bmatrix} p_1 q_1 & \cdots & p_1 q_n \\ \vdots & & \\ p_m q_1 & \cdots & p_m q_n \end{bmatrix}$$

$$(pq^T)_{ij} = p_i q_j$$

k k k

$$\underset{A}{\begin{bmatrix} | & | & | & | \end{bmatrix}} \underset{B}{\begin{bmatrix} \text{—} \\ \text{—} \\ \text{—} \end{bmatrix}}$$

k^{th} col A $\quad$ k^{th} row of B $\quad$ rank = 1.

$m \times m$ $\quad$ $m \times n$

$\hookrightarrow$ SVD $\quad A = U \Sigma V^T$

$$= \begin{bmatrix} u_1\sigma_1 & u_2\sigma_2 & \cdots & u_n\sigma_n \\ | & | & & \end{bmatrix} \begin{bmatrix} — v_1 — \\ — v_2 — \\ \vdots \\ — v_m — \end{bmatrix}$$

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$ outer product

$\hookrightarrow$ Sum of r 1-rank matrices.

e.g. $\quad 1000 \times 1000$ pixel image

10^6 numbers

$u \to 10^3$, $v \to 10^3$ $\to$ send only 10

$$\left(10^3 + 10^3\right) \times 10 + 10$$

$$20 \times 10^3 + 10 \approx \cancel{20}\ 2 \times 10^4$$

$$10^6 / 2 \times 10^4 = 50$$