# Visualizing Quad forms (contd)

$A = Q \Lambda Q^T$ (assume $A$ is real sym)

$x^T A x = \underbrace{x^T Q}_{y^T} \Lambda \underbrace{Q^T x}_{y} = \sum y_i^2 \lambda_i$

$= [\quad] \Lambda \begin{bmatrix} \\ \\ \end{bmatrix}_y$

like an ellipse in $y$-frame.

Did $\boxed{y = Q^T x}$

Take

$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \rightarrow x^T A x = 5x_1^2 + 5x_2^2 + 8x_1 x_2$

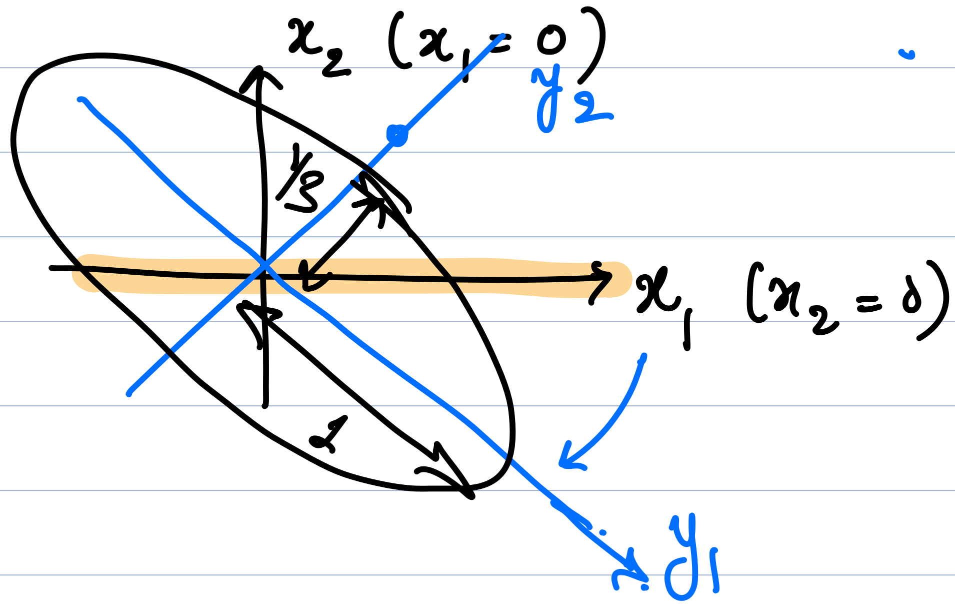
$= 1$

Is $A$ P.D ?

$a > 0$, $ac > b^2$ $\quad \lambda_1 = 1, \quad \lambda_2 = 9$

$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ eig : $\frac{1}{\ }\left(\ 1\ \right) \quad \frac{1}{\ }\left(\ 1\ \right)$

vectors : $\frac{1}{\sqrt{2}}\begin{pmatrix} \\ -1 \end{pmatrix}$ $\frac{1}{\sqrt{2}}\begin{pmatrix} \\ 1 \end{pmatrix}$




$$\longrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Q^T \begin{bmatrix} | & | \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\sum \lambda_i y_i^2$$

$$1 \times y_1^2 + \frac{y_2^2}{1/9} = 1$$

# Singular Value Decomposition

Any matrix, A can be expressed as

$$A = U \Sigma V^T$$

$V^T \longrightarrow$ n×n orthogonal matrix

$A$: tall/fat/sq real/

$U$: m×m orthogonal

$\Sigma$: m×n diagonal matrix with non negative real/ complex $m \times n$ orthogonal matrix values. "singular values"

Proof: Assume $m > n$ (tall).

$AA^T : m \times m$
$A^TA : n \times n$

1) Let $B = A^T A \to n \times n$.

$v$ is an eigvec of $B$. $\Rightarrow B v_i = \lambda_i v_i$, $i = 1, \ldots, n$

$v_i$'s span $\mathbb{R}^n$ (we have a basis for $\mathbb{R}^n$)

2) $A^T A v = \lambda v$

L.M with A

$A A^T A v_i = \lambda_i A v_i$

$\Rightarrow A v_i$ is an eigenvec of $\underline{AA^T}$ with eigval $\lambda_i$

3) $u_i' = A v_i$ (intermediate vector.)

e.g. $u_j'^T u_i' = v_j^T A^T A v_i = \lambda_i v_j^T v_i$

$= \lambda_i \delta_{ij}$

$\Rightarrow$ $u'$ are orthogonal.

4) Construct $u_i = \dfrac{u_i'}{\sqrt{\lambda_i}}$ $\Rightarrow$ Makes $u$'s to be orthonormal.

5) How many $u$'s do I have? $n$

How many entries of $u$ vector? $\rightarrow$ $m$

They span '$n$' dim subspace of $\mathbb{R}^m$.

6) $A$ rank: $n$

$\searrow m \times n$

| | R.S | C.S |
|---|---|---|
| | $n$ | $n$ |

$m \begin{bmatrix} & \end{bmatrix}_n$ N.S $(A^T)$ $m-n$ | N.S $0$

$A^T \rightarrow$ fat $\begin{bmatrix} & & \end{bmatrix} \Rightarrow \underline{A^T x} = 0$ & $x \neq 0$

There are $m-n$ vectors

7) $A^T x_i = 0 \qquad i = 1, \dots, m-n$

defn of null sp.

L.M by $A$

$$\underline{AA^T} x_i = 0$$

$\Rightarrow$ we have $m-n$ eigvecs of $AA^T$ with eigvalue $0$.

8) $AA^T \rightarrow$ size $m \times m$

$\Rightarrow$ EVD says $\rightarrow$ $m$ eigvals
$\rightarrow$ $m$ orthonormal eigvecs

We have found all $m$ of them
$\rightarrow$ $n$ from eigvectors of $A^TA$
$\rightarrow$ $m-n$ from n.s. of $A^T$

$U = [u_1 \cdots u_n \; u_{n+1} \cdots u_m]$ basis for $\mathbb{R}^m$

$\longleftrightarrow$ $m$

9) $A[v_1 \cdots v_n] = \boxed{[\underbrace{u_1' \cdots u_n'}]}$

Left Multiply by $U^T$

$$\begin{bmatrix} u_1^T & — \\ u_2^T & — \\ u_n^T & \end{bmatrix} A \begin{bmatrix} v_1 & \cdots & v_n \\ | & | & | \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & \cdots \\ & \ddots & 0 & \\ 0 & & & \end{bmatrix}$$

$A = U\Sigma V^T$
$U^TA = \Sigma V^T$

$u_i' = \sqrt{\lambda_i}\, u_i$

$$\begin{bmatrix} u_m^T & \text{—} \end{bmatrix} \qquad \underbrace{\begin{bmatrix} 0 & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \\ & 0 & \end{bmatrix}}_{m \times n} \quad \sqrt{\lambda_n} \rightarrow \sigma_n$$

$\Sigma$

$$U^T A V = \Sigma$$

$$\Rightarrow \quad A = U \Sigma V^T \qquad \underline{\underline{\text{SVD}}}$$

$U$: left singular vecs

$V^T$: Right singular vecs.

**Comments.**

- Singular values $\sigma_i \geq 0$
- Cols of $V$ are eigvecs of $A^T A$

  Cols of $U$ are eigvecs of $A A^T$

P.D matrix $A = Q \Lambda Q^T$

+ve nos

EVD & SVD is the same.

$Ax = \lambda x$

$Av_i = \sigma_i u_i$

$A = U \Sigma V^T$

$A^T = V \Sigma U^T$