# Diagonalization of a matrix.

$A$: $n \times n$    assume : '$n$' linearly indepn eigvectors.

$$S = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$$

$$AS = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$\Rightarrow \boxed{A = S \Lambda S^{-1}}$$

R. mult by $S^{-1}$

$$\boxed{S^{-1} A S = \Lambda}$$

Called the diagonalization of $A$.

→ Distinct eigvalues → ✓

Repeated eigvalues

↓

$\gamma_A = \mu_A \longrightarrow$ ✓

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad |A - \lambda I| = \lambda^2 = 0$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ only eigvec

<u>NO DIAG</u>

## Powers of a Matrix.

$$A^k = (S\Lambda S^{-1}) \cdots\cdots (S\Lambda S^{-1})$$

$$\leftarrow k \rightarrow$$

$$= S\Lambda^k S^{-1}$$

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$$

## Eigen decomposition of A

$A^{-1} \longrightarrow \quad A = S \Lambda S^{-1}$

$$A^{-1} = S \Lambda^{-1} S^{-1}$$

$A^{-1}$ has eigvalues = reciprocal of eigs $(A)$.

only when $\lambda_i \neq 0 \ \forall i$.

$$A x = \lambda x$$

$$x = \lambda A^{-1} x$$

$$\frac{1}{\lambda} x = A^{-1} x$$

— x —

$AB$

$x, \lambda \qquad y, \mu \qquad By = \mu y.$ (2)

(1) $\quad Ax = \lambda x$

$$ABy = \mu A y$$

$BAx = \lambda Bx$

→ Diagonalizable matrices share the same eigenvectors $S$ iff $\underbrace{AB = BA}$.

↪ Proof.

① Assume same $S$ for $A, B$.

$$A = S\Lambda_1 S^{-1}, \quad B = S\Lambda_2 S^{-1}$$

$$AB = S\Lambda_1\Lambda_2 S^{-1} = S\Lambda_{1,2}S^{-1}$$

$= BA$

$(\lambda_1 \times \lambda_2)$ on diags

② Assume $AB = BA$

Say $\boxed{Ax = \lambda x}$

$$BAx = \lambda \underbrace{Bx} = A\underbrace{Bx}$$

$\Rightarrow$ $Bx$ is an eigvector of $A$ with eigenvalue $\lambda$.

[$*$ Assume: all eigvalues are distinct]

$\therefore$ eigvalue $\lambda \to$ must have a fixed eig vect

$\Rightarrow$ $Bx$ & $x$ are multiples of each other, i.e

$$Bx = \mu x$$

$\Rightarrow$ $x$ is also an eigvector of $B$

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$\to$ Uncertainty Principle.

$x \to$ wave fn.

$P \to$ Pos

$Q \to$ Momentum

$P^H = P$

$Q^H = -Q$

$$PQ - QP = I$$

$\dfrac{\|Qx\|}{\|x\|}$ : momentum uncertainty

$\dfrac{\|P_x\|}{\|x\|}$ : position uncertainty

$$\boxed{x^H y \leq \|x\| \, \|y\| \quad \text{C.S.}}$$

$$\|x\|^2 = x^H x = x^H (PQ - QP) x$$

$$= x^H PQx - x^H QPx$$

$$\leq \underbrace{|x^H PQx| + |x^H QPx|}$$

$$\leq 2 \|Qx\| \, \|Px\|$$

$$\frac{\|Qx\|}{\|x\|} \frac{\|Px\|}{\|x\|} \geq \frac{1}{2}$$

— x —

Cayley Hamilton Thm.

Every sq matrix satisfies its own characteristic eqn.

$$p(\lambda) = |A - \lambda I|$$

$n^{th}$ order polynomial.

$$\Rightarrow p(\lambda_i) = 0$$

eg. $5\lambda^2 + 3\lambda - 1 = 0$

Thm says $5A^2 + 3A - \underline{I} = \vec{0}$

0 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$p(\lambda) = \lambda^2 - \underbrace{Tr(A)}_{(a+d)} \lambda + \underbrace{|A|}_{(ad-bc)} = 0$$

$$\Rightarrow A^2 - (a+d)A + (ad-bc)I = \vec{0}$$

Proof: Distinct eigvalues

$A^k = S \Lambda^k S^{-1}$. Substitute A into

$$S\Lambda^2 S^{-1} - (a+d) S \Lambda^1 S^{-1} + (ad-bc) I \qquad \boxed{P(\lambda)}$$

$$S\Big[\underbrace{\Lambda^2 - (a+d)\Lambda + (ad-bc) I}\Big] S^{-1}$$

$$S \begin{bmatrix} \lambda_1^2 - (a+d)\lambda_1 + (ad-bc) & 0 \\ 0 & \lambda_2^2 - (a+d)\lambda_2 + (ad-bc) \end{bmatrix} S^{-1}$$

$\lambda_1, \lambda_2$ satisfy $P(\lambda) = 0$

$\therefore \quad = 0$

Fibonacchi Numbers

$$f_{n+2} = f_n + f_{n+1}, \quad f_0 = 0, \; f_1 = 1$$

$f_{100}$ ?

$$\begin{bmatrix} f_{n+2} \\ f_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A^1} \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$$

$$u_{n+1} = A^1 \, \underline{u_n}$$

$$f_{100} = u_{99}$$

$$u_{n+1} = A^2 u_{n-1} \cdots = A^{n+1} u_0$$

Need $A$ to be diagonalizable.

$$-\lambda(1-\lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0 \qquad \boxed{\lambda = \frac{1 \pm \sqrt{5}}{2}.}$$