Eigenvectors & values.

Eigenvalue problem: Given a square matrix A, find vector(s)/scalar(s) x, λ, that satisfy

$$Ax = \lambda x$$

$$\nabla^2 E = c\frac{\partial^2 E}{\partial t^2} = -c\omega^2 E$$

$$E(\_, t) \propto e^{j\omega t}$$

$$\frac{\partial f}{\partial x} = \frac{f(x+k) - f(x)}{k}$$

By finite differencing,

$$\nabla^2 E \qquad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\downarrow$$

$$AE$$ where A: Matrix for finite differencing

$$AE = \lambda E$$

$$\lambda \rightarrow -c\omega^2$$

— x —

$$Ax = \lambda x$$

$$\underbrace{(A - \lambda I)}\,\underbrace{x} = 0$$

If there is a non trivial soln, it lives in the null space of $(A - \lambda I)$.

$$A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$$

$$(A - \lambda I)^{-1}(A - \lambda I)x = 0$$

$$\Rightarrow \quad x = 0$$

$\Rightarrow$ $A - \lambda I$ should be singular

$\therefore$ $\det(A - \lambda I) = 0.$

$$\det\begin{pmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{pmatrix} = 0$$

$$\lambda = -1, 2.$$

$$(4-\lambda)(-3-\lambda) + 10 = 0$$

$$\underbrace{\lambda^2 - \lambda - 2 = 0}$$

$\rightarrow$ Characteristic polynomial

Roots are the eigenvalues of the problem.

A: $n \times n$

→ c.poly will be n^{th} degree
⇒ Always n solns.
⇒ More computational resources.

$A - \lambda I$

$$\begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\lambda = -1$ $\quad \begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 2}$ $\quad v = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

↳ Triangular matrix. $A = \begin{pmatrix} a_{11} & — & \\ 0 & a_{22} & — \\ & & a_{33} \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & & \\ & a_{22} - \lambda & \\ & & a_{3} - \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \prod^{n} (a_{ii} - \lambda) = 0$$

$i = 1$

Char poly of A

Solns are $\underline{\lambda_i = a_{ii}}$.

↳ Projection operator onto a line (a)

$$P = \frac{a a^T}{a^T a} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

<u>Donkey.</u>

$$\left| \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 1 = 0$$

$$(1-\lambda) = \pm 1$$

$$\Rightarrow \lambda = 0, 2.$$

with $1/2 \rightarrow \underline{\lambda = 0, 1.}$

↳ $\begin{pmatrix} 1/2 - \lambda & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \end{pmatrix} = 0$

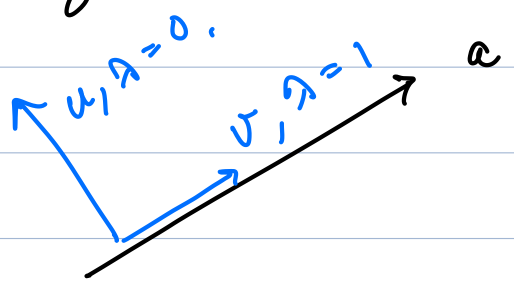
$$\begin{pmatrix} \cdots & \cdots \\ 1/2 & 1/2-\lambda \end{pmatrix}\begin{pmatrix} \cdots \\ x_2 \end{pmatrix}$$

$\lambda = 0 \longrightarrow$ $\quad u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 1 \longrightarrow$ $\quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Owl's way.

Eigvalues are $0, 1$
Eig vectors are a, $a^{\perp}$



For **Projection operators.**

→ Eig values are always **0 or 1.**

$$\lambda^{a}(1-\lambda)^{b} = 0$$

$(1-\lambda)^b$: repeated b times

λ^a: repeated a times

$a+b = n$.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cdots \end{pmatrix}$$ → Projection op.
$\lambda = 0, 1$

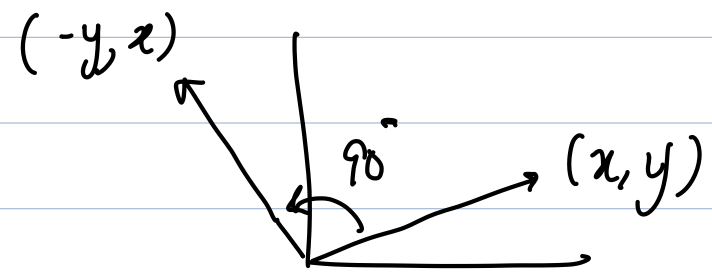
$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$\lambda = 0$ $\rightarrow$ $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 1$, eig vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

- Rotation matrix

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$



$$Rx = \lambda x$$

$\mathbb{R}^2$ $\quad$ $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$

$$|R - \lambda I|$$

$$= \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm j$$

$\therefore$ $\lambda = +j$, $\begin{pmatrix} -j & -1 \\ 1 & -j \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} +j \\ \end{pmatrix}$ ✓

$$\lambda = -j \quad \begin{pmatrix} j & -1 \\ 1 & j \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} j \\ -1 \end{pmatrix} \checkmark$$

Real matrices can
$\Rightarrow$ give us complex eigen vectors & values.

— x —

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \text{char poly.}$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

Say, λ_1, λ_2 : $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

① $Tr(A) = \Sigma$ diag values.

$\therefore \Sigma$ eig values $= Tr(A)$

② $\det(A) =$ Prod. eigenvalues.

$A : 0, 1, 2, \ldots, n-1$.

Algebraic & Geometric Multiplicities of eigen values.

Given a matrix A:

$\rightarrow$

$\mu_A(\lambda_i)$ = No of times an eigvalue λ_i repeats itself.
"Algebraic multiplicity of λ_i"

$\updownarrow$

appears as $(\lambda - \lambda_i)^{\mu_A(\lambda_i)}$.

We can say $\displaystyle\sum_{i=1}^{d} \mu_A(\lambda_i) = n$

d distinct eigvalues.

$\gamma_A(\lambda_i)$ = No of linearly indepn eigvectors for a given eig value.
"Geometric multiplicity of λ_i".

$$\boxed{1 \leq \sum_i \gamma_A(\lambda_i) \leq n}$$

$A = I \rightarrow (1-\lambda)^n = 0 \Rightarrow \lambda = 1$

$$\mu_I(1) = n$$
$$\gamma_I(1) = n$$

$$\begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix}$$

$1 \to 1 \longrightarrow 1$

$0 \to (n-1) \to n-1.$