Continuation, Determinant properties.

(4) If 2 rows of A are identical, $|A| = 0$

rule # 2, exchange identical rows
$\rightarrow -|A|$. $\therefore |A| = -|A|$
$\Rightarrow |A| = 0$.

(5) A row transformation leaves $|A|$ unchanged.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \therefore R_1 \rightarrow R_1 - mR_2$$

$$A' = \begin{bmatrix} a - mc & b - md \\ c & d \end{bmatrix}$$

$$|A'| = \left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| + \left|\begin{bmatrix} -mc & -md \\ c & d \end{bmatrix}\right|$$

rule #3

$$= \quad " \quad " \quad + \quad 0$$

$\therefore |A| = |A'|$

$A = LU \rightarrow |A| = |U|$.

⑥ If one of the rows is 0, then $|A| = 0$

$$A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$$

⑦ Triangular matrix: $|A|$ = prod of diagonal terms.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} A'$$

$$|A'| = a_{11} \left| \begin{pmatrix} 1 & & \\ & a_{22} & \\ & & a_{33} \end{pmatrix} \right| = \prod_i a_{ii}$$

⑧ If A is singular, $|A| = 0$.

By RREF → reduce it to triangular
Singular → atleast 2 rows are linearly dep
∴ atleast 1 row : all 0s
∴ $|A| = 0$ by rule #6.

⑨ Product rule: $|AB| = |A|\,|B|$.

step ⓐ $|DB|$ — diagonal.

$$\begin{bmatrix} d_{11} & & \\ & d_{22} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} d_{11} \times \leftarrow B_{1row} \rightarrow \\ \vdots \\ d_{nn} \times \leftarrow B_{nrow} \rightarrow \end{bmatrix}$$

keep using rule #3,

$$= \prod d_{ii}\, |B|$$
$$= |D|\,|B|.$$

step ⓑ $AB = LUB$

$$|AB| = |LUB| = |UB|$$
$$\downarrow$$
$$|D|\,|B| \leftarrow |D \times B|$$

$|A| = |LU| = |U| = |D|$

$\therefore |AB| = |A|\,|B|$.

⑩ $|A| = |A^T|$

$A = LDU$ where L, U have 1 on the diagonals.

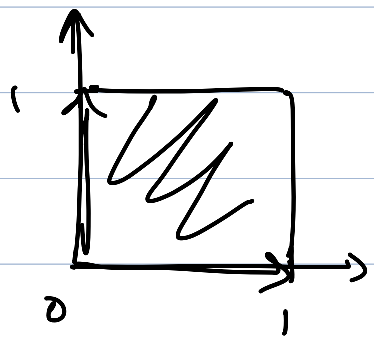
$|A| = |D|$

$A^T = U^T D^T L^T$

$|A^T| = |D|$ 2

Applications of Determinants

1) Volume of a parallelopiped (n-dims).

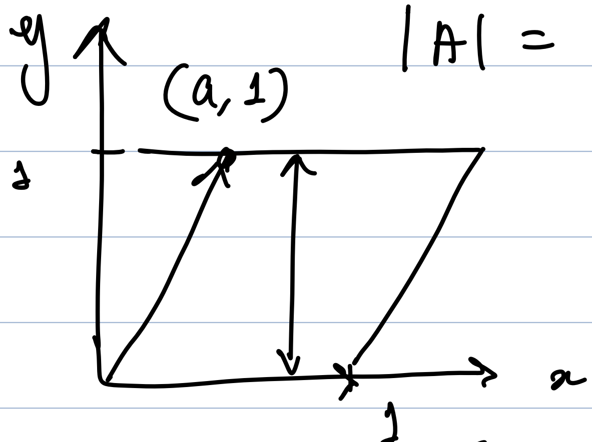


Put edge on a row

$$\boxed{A = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ x \quad\quad y \end{array}}$$

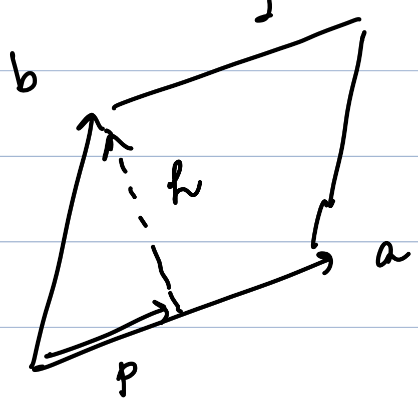
$\updownarrow$ edges.

$|A| = 1$



$A' = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

$|A'| = 1 = |A|$



$A = \begin{bmatrix} \leftarrow a \rightarrow \\ \leftarrow b \rightarrow \end{bmatrix}$

$$b' = b - \alpha a =$$

$$\alpha = \frac{\|p\|}{\|a\|}$$

$$\begin{bmatrix} \leftarrow a \rightarrow \\ \leftarrow h \rightarrow \end{bmatrix}$$

↳ n dimensions.

$$A = \begin{bmatrix} - e_1 - \\ - e_2 - \\ \vdots \\ - e_n - \end{bmatrix}$$

$$A A^T = \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \begin{bmatrix} | & | & | \end{bmatrix} = \begin{bmatrix} l_1^2 & & \\ & l_2^2 & \\ & & \ddots \end{bmatrix}$$

$$|A|^2 = \prod l_i^2$$

$$\Rightarrow |A| = \pm \prod l_i$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$