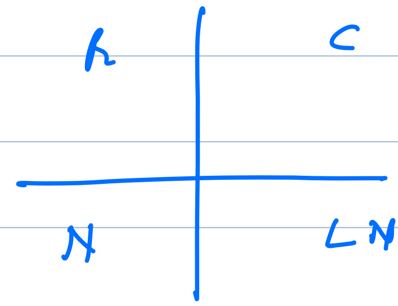
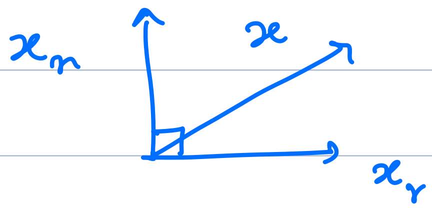
$Ax = y$

Min energy: $\|x\|_2^2$



$$x = x_r + x_n$$



$$\|x\|_2^2 = \|x_r\|^2 + \|x_n\|^2 .$$

Why does $Ax = b$ have ∞ solns?
Where is the source of ∞?

Min energy for x? Still satisfy $Ax = b$.

$\Rightarrow$ x_n set to 0.
only $x = x_r$.
when rows were indepn.

right inv. $= C$

$$\hat{x} = x_r = \underbrace{\boxed{A^T (AA^T)^{-1}} b}_{\text{min energy soln.}}$$

$$A \times C = A \left[A^T (AA^T)^{-1} \right] = I.$$

$A_{m \times n}$

Tall matrix	Fat matrix
1) $m > n$	$m < n$
2) cols are indepn $\Leftrightarrow (A^T A)^{-1}$ exists	rows are indepn $\Leftrightarrow (AA^T)^{-1}$ exists.

$$\hat{x} = A^{+} b$$

A^{+} ↓ pseudo inverse

3) left inv	right inv
4) Soln $\hat{x}$ is that soln which gives least squared error.	All solns have 0 error
5) N/A.	$\hat{x}$ soln is that soln which has least energy.

$$\min_x \|x\|_2^2 \quad s.t. \quad Ax = b$$

why 2 ?

$$\min_x \|x\|_1 \quad s.t. \quad Ax = b$$

The soln we get is sparse-st under some condns on A.

$\downarrow$

Study is compressive sensing.

— x —

Orthogonal basis fns, Gram-Schmidt.

Vectors space V, has a basis $\{v_i\}, i = 1, \dots, n$

Basis is:

* orthogonal if $v_i^T v_j = \|v_i\|^2 \delta_{ij}$
* orthonormal if $v_i^T v_j = \delta_{ij}$

↳ Orthogonal matrix Q is a sq matrix with orthonormal cols.

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \\ | & | & & \end{bmatrix}$$

Properties:

(1)

$$Q^T Q = \underbrace{\begin{bmatrix} \text{———} \\ q_i^T \ \text{———} \\ \text{———} \end{bmatrix}}_{\text{Row}} \underbrace{\begin{bmatrix} \cdots & q_j & \cdots \\ | & | & | \end{bmatrix}}_{\text{Cols}}$$

$$= I_{n\times n}$$

$$\Rightarrow \quad Q^{-1} = Q^T$$

② Start with x, $y = Qx$.

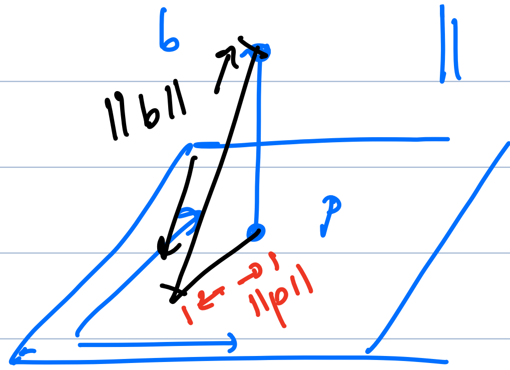
$\|x\|_2$ & $\|y\|_2$?

$$\|y\|_2^2 = y^T y = x^T \underbrace{Q^T Q}_{= I} x = \|x\|^2$$

$\Rightarrow$ Q preserves the length!

test: P is a projection matrix

$$\|Px\| \leq \|x\|$$



an eg. when length is reduced!

$\Rightarrow$ eg. rotation, reflection, permutation

③ Say x & y have θ angle between them.

Now apply Q. Angle between Qx, Qy?

$$x^T y = \|x\| \|y\| \cos\theta.$$

$$(Qx)^T(Qy) = \|Qx\| \, \|Qy\| \cos\theta'$$

$$x^T \underbrace{Q^T Q}_{I} y = \|x\| \, \|y\| \cos\theta'$$

$$= x^T y \implies \underline{\underline{\theta = \theta'}}$$

angles are <u>unchanged</u>.

(4) How to represent a vector in a Q basis.

$\{q_i\}$ is a basis.

$b \in V$, $\boxed{b = \sum_{i=1}^{n} q_i x_i}$ (L.C.)

$$b = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \\ | & | & & \updownarrow \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} =$$

col. pic. of a matrix

$$= Qx.$$

↳ How to determine x_i's?

$$q_j^T b = q_j^T \left(\sum_{i=1}^{n} q_i x_i \right) = \|q_j\|^2 x_j = x_j$$

n n

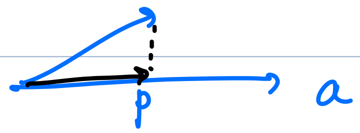
$$\therefore\ b = \sum_{i=1} q_i x_i = \sum_{i=1} q_i (q_i^T b)$$

$$b = \underbrace{\sum_{i=1}^{n}}_{\text{sum}} \underbrace{(q_i q_i^T)\, b}_{\text{projection}}$$

$A(BC)$
$= (AB)C$
$= ABC$

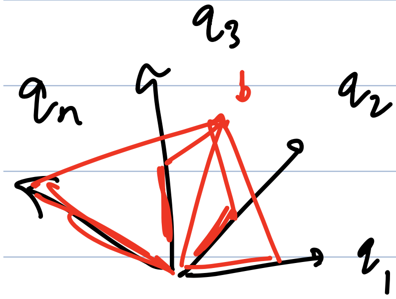
Hint: what is the projection of b onto a?

$$p = \frac{a a^T b}{a^T a} = \frac{a a^T b}{\|a\|^2}$$



Now if $\|a\| = 1$, then

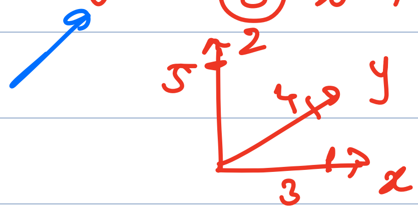
$$p = a(a^T b) = \underbrace{(a a^T)}\, b$$



b is the sum of 1D projections of itself onto the basis vectors.

eg. $\vec{v} = 3\hat{x} + 4\hat{y} + 5\hat{z}$

scalar × unit vect



$$b = \sum_{i=1}^{n} \underbrace{(q_i q_i^T)}_{\text{Matrix of rank 1.}} b$$

$=$

$$\|v\|^2 = 3^2 + 4^2 + 5^2$$

$$b = \sum_{i=1}^{n} q_i (q_i^T b)$$

unit vec scalar

$$\|b\|^2 = \sum_{i=1}^{n} [q_i^T b]^2$$ real nos

$$\|b\|^2 = \sum_{i=1}^{n} |q_i^H b|^2$$ complex nos

H: conj. transp.

Q? b?

$$= (q_1^T b)^2 + (q_2^T b)^2 + \cdots (q_n^T b)^2$$

$$\begin{bmatrix} q_1^T & \text{———} \\ q_2^T & \text{———} \\ & \\ q_n^T & \text{———} \end{bmatrix} \begin{bmatrix} | \\ b \\ | \end{bmatrix} = \begin{bmatrix} q_1^T b \\ q_2^T b \\ \vdots \\ q_n^T b \end{bmatrix}$$

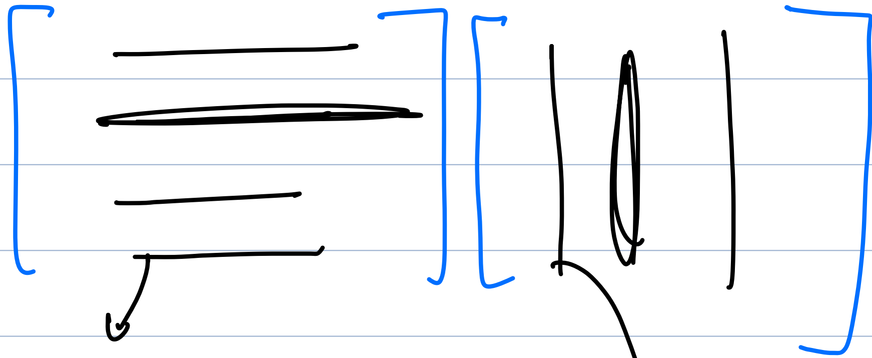
$$= \|Q^T b\|^2$$

If the cols of Q are orthonormal, are the rows also " " ? (square matrix)

Yes. $Q^T = B$

$$Q^T Q = I \Rightarrow Q^T = Q^{-1}$$

$$\boxed{\Rightarrow Q Q^T = I} \qquad \left(\begin{aligned} &Q \cdot \times Q^{-1} \\ = \; &Q^{-1} \times Q \\ = \; &I \end{aligned} \right)$$



$= I$

transp of cols of Q

Rows

Cols

rows of Q