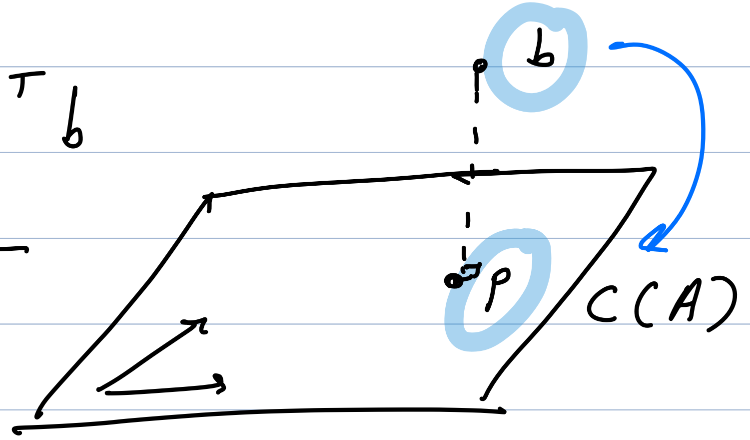
$$\hat{x} = (A^T A)^{-1} A^T b$$



What is p ?

$$p = A\hat{x}$$

$\cancel{(x-\hat{x})}$

$$\left\{ \quad p = \underbrace{A(A^T A)^{-1} A^T}\, b \quad \right\}$$

$(A^T A)^{-1} = A^{-1}(A^T)^{-1}$ $\times$ Not defined.

Projection operation; i/p : b
o/p : p

operator : P
matrix

↳ Take a numerical e.g. $\quad \underline{v = u + at}$.

$t = [-1, 1, 2]$

$v = [1, 1, 3]$

(1) Model may be wrong

(2) Measurements may have error.

$$\begin{array}{cc} 1 & t \end{array}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$A \rightarrow \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 3 \end{bmatrix} \leftarrow b$

$\rightarrow$ is $b \in C(A)$?

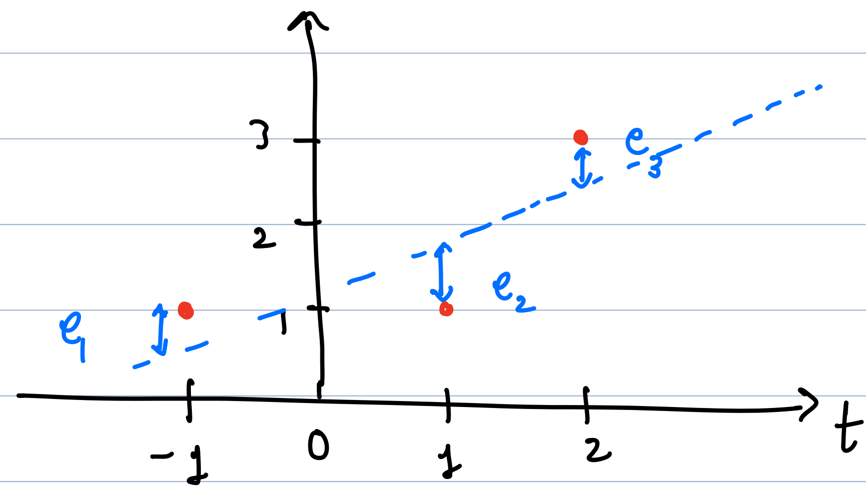
$\Rightarrow$ go for a least squared error soln.

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

solve: $\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 9/7 \\ 4/7 \end{bmatrix}$

soln pic



p ? $\quad p = A\hat{x} = \begin{bmatrix} 5/7 \\ 13/7 \\ 17/7 \end{bmatrix}$ $\leftarrow$ ideal soln.

fix

1) $v = u + a(t-1) = (u-a) + at$

2) Increase the order:

$$v = u + at + bt^2$$

Linear ✓

Variables $\underline{u, a, b}$

Not $1, t, t^2$

$$\begin{array}{ccc} 1 & t & t^2 \end{array}$$
$$\begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} u \\ a \\ b \end{bmatrix}$$

$$v = u + at + b \ln t$$

3) More # measurements. ✓

$$\boxed{P = A(A^TA)^{-1}A^T}$$

$$P \rightarrow P^2 = P \checkmark$$

$$P^T = \left[A(A^TA)^{-1}A^T\right)^T$$

$$= A\left((A^TA)^{-1}\right)^T A^T$$

$$= A\left((A^TA)^T\right)^{-1} A^T$$

$$= A(A^TA)^{-1}A^T = P$$

Projection onto a line.

Case when A is a col.

a

$$P = \frac{1}{(a^T a)} a a^T$$ ← Matrix of rank 1.

What is the projection of b onto a?

$$p = Pb = \frac{1}{a^T a} \underbrace{a a^T}\underbrace{b}$$

↳ What is $\|p\|$?

$a^T a = \|a\|^2$

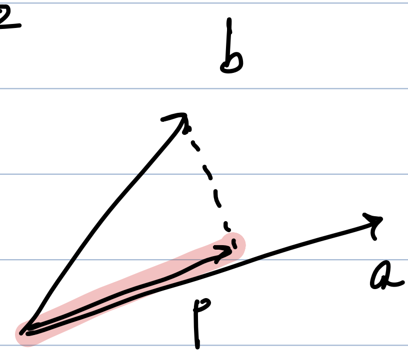
$$p^T p = \frac{1}{\|a\|^4} b^T (a a^T)^T \times a a^T b$$

$$= \frac{1}{\|a\|^4} b^T a a^T \cdot a a^T b$$

$$= \frac{1}{\|a\|^4} \|a\|^2 \underbrace{b^T a}\,\underbrace{a^T b}$$

$$= \frac{1}{\|a\|^2} [a^T b]^2$$

$$\therefore \|p\| = \frac{1}{\|a\|} |a^T b|$$



Can proj length be > length of b?

$$\|p\| \leq \|b\|$$

$$\Rightarrow \quad |a^T b| \leq \|a\| \|b\|$$

Cauchy Schwarz ineq.

↳ <u>Vector analysis</u> $\vec{a}, \vec{b}$

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos\theta$$

$$|\cos\theta| \leq 1 \quad \Rightarrow \quad |\vec{a}| |\vec{b}| \geq |\vec{a} \cdot \vec{b}|$$

$$\|a\| \|b\| \geq |a^T b|$$

$$a^T b = \|a\| \|b\| \cos\theta$$

$$\therefore \quad \cos\theta = \frac{a^T b}{\|a\| \|b\|} \qquad \sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

defn for $\cos\theta$ in higher dims.

↳ optional in sec 3.3 of G.S.: weighted least squares.

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$e = A\hat{x} - b$$

$$\|e\|^2 = w_1\left(\hat{x}_1 - \hat{x}_2 - 1\right)^2 + w_2\left(\hat{x}_1 + \hat{x}_2 - 1\right)^2 + w_3\left(\hat{x}_1 + 2\hat{x}_2 - 3\right)^2$$

Some measurement have trust issues. Then?

— x —.

Under Determined System of Eqns.

$\boxed{Ax = b}$ $\underline{m < n}$ "fat" matrix.

↳ Lesser measurements than variables.

↳ Assumption: rows are linearly indepn.

↳ fundamental spaces of a matrix

$\mathbb{R}^n$ $\xrightarrow{A}$ $\mathbb{R}^m$ • y

$\xleftarrow{A^T}$

	$\mathbb{R}^n$	$\mathbb{R}^m$	
$C(A^T)$	dim = m	dim = m	$C(A)$
$N(A)$	dim = $n-m$	dim = 0	$N(A^T)$

↳ $C(A) = \mathbb{R}^m$, what about b & $C(A)$

$\Rightarrow \quad b = A\hat{x}$ for any b !

$e = b - A\hat{x}$?

0 ← even in noise !

Least squared error soln → Make sense ?

↳ What else can we do?

$x \in \mathbb{R}^n$, $[\, x = x_r + x_n . \,]$

because $N(A) \neq \{0\}$

$x_r^T x_n = 0$

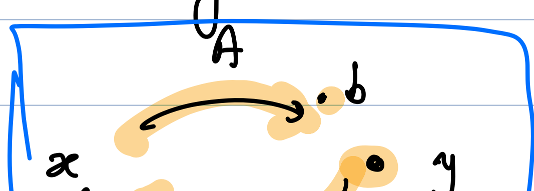
1) Let y be any vector in $\mathbb{R}^m$

$$x_r = A^T y \quad — ①$$

2) Multiply by A: $\quad A x_r = A A^T y. \quad — ②$

$$Ax = b = A(x_r + x_n) = A x_r$$

$$\Rightarrow \boxed{A x_r = b = A A^T y.}$$



3) $(AA^T) \rightarrow$ sq, $m \times m$, sym.

Invertible if A has indepn rows.
(proof exactly similar to that for $(A^TA)^{-1}$)

4) $y = (AA^T)^{-1} b$? ✓

5) $\Rightarrow x_r = A^T y = \underbrace{A^T (AA^T)^{-1}}_{\text{right pseudo inverse of } A} b$

$A \times C = I$

$A \times [A^T (AA^T)^{-1}]$

$= (AA^T) \times (AA^T)^{-1} = I$

[unique.
Of all solns to $A\hat{x} = b$, which soln has least energy?]