4 fundamental subspaces in Linear Algebra.

③ Row space $C(A^T)$

→ Dim: r (rank)

→ Subspace of $\mathbb{R}^n$

→ Non zero rows of U or R form basis

$\perp^r$

① Col space $C(A)$

→ Dim: r (rank)

→ Subspace of: $\mathbb{R}^m$

→ Also called range (A)

→ Pivot cols form basis

$\perp^r$

$A x$ ↳ $m \times n$, $n \times 1$

$A^T y$ ↳ $n \times m$, $m \times 1$

② Null space $N(A)$.

↳ Dim: $n-r$ (no of free variables)

(nullity)

↳ Subspace of $\mathbb{R}^n$

↳ Kernel of A.

④ Left Null space $N(A^T)$

↳ Dim: $m-r$

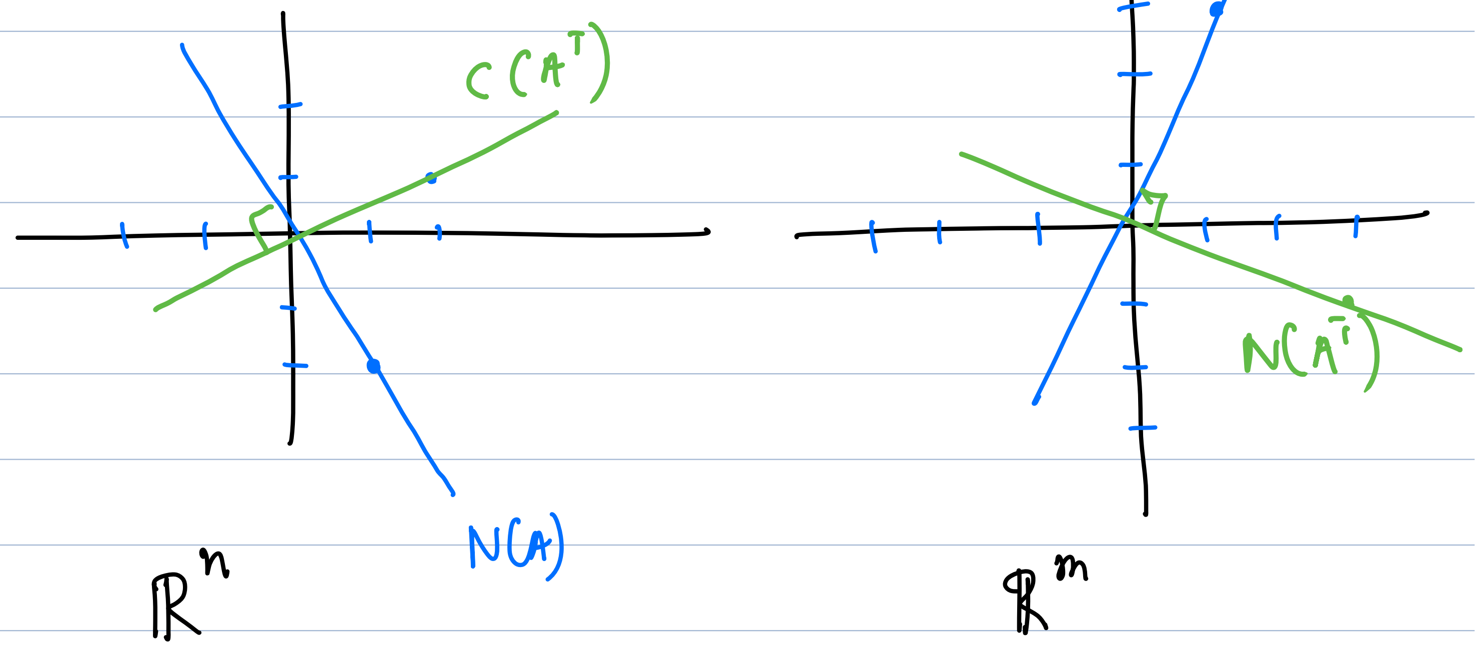
↳ Subspace: $\mathbb{R}^m$

e.g. $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$, $A^T = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$

rank = 1

$C(A^T) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$C(A) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
$N(A) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$	$N(A^T) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

, $C(A)$



Q1: Are $C(A^T)$ & $N(A)$ $\perp^r$? ✓

Q2: Are $C(A)$ & $N(A^T)$ $\perp^r$? ✓

$$C(A) \to c \qquad N(A^T) \to d$$

$c^T d = 0$, done!

$$(Ax)^T d$$

$$= x^T \underbrace{A^T d}$$

$$= x^T (\underbrace{A^T d}_{=0}) = 0$$

N.S.

$Ax = 0$

L N.S

$A^T y = 0$

$\xrightarrow{T}$ $y^T A = 0$

$C(A^T) \perp N(A)$? ✓

$e \qquad f$

$$e^T f =$$

$$(A^+ y)^T f = y^T \underbrace{A f}_{0} = 0.$$

Matrix Inverses.

① $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$

What matrix exists s.t. $\underset{2\times 3}{A} \cdot \underset{3\times 2}{C} = \underset{2\times 2}{I}$. ?

$$\overset{A}{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}} \overset{C}{\begin{bmatrix} 1/4 & 0 \\ 0 & 1/5 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$c_{31} \quad c_{32} \neq 0$

↳ Is it unique → No

↳ Family of inverse matrices.

Could we have found $\underset{3\times 2}{C'} \underset{2\times 3}{A} = \underset{3\times 3}{I}$?

$$\begin{bmatrix} 1/4 & 0 \\ 0 & 1/5 \\ ? & ? \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\times$

$\neq$

$$\begin{bmatrix} 1/4 & 0 \\ 0 & 1/5 \\ ? & ? \end{bmatrix} \begin{bmatrix} 4 & 0 & A_{13} \\ 0 & 5 & A_{23} \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1 & 0 & A_{13}/4 \\ 0 & 1 & A_{23}/5 \end{bmatrix}$$

$\times$

$\Rightarrow$ Right inverse was possible.
Left inverse not possible.

Full row rank:

- $r = m$
- All rows have pivots
- All rows are lin indepn.

$\Rightarrow$ Statement w/o proof (for now)
if A has full row rank then A has a right inverse

$$\Rightarrow \quad \underset{m\times n}{A} \cdot \underset{n\times m}{C} = I_{m\times m}$$

↳ When is full row rank possible: $m \leq n$.

↳ when full row rank, $Col(A) = \mathbb{R}^m$

↳ What does this mean for $Ax = b$?

$\Rightarrow$ At least one soln, since $b \in Col(A)$ (existence prop).

case: $m < n$

here: $n - m > 0$ free vars

$\Rightarrow N(A) \neq \{\vec{0}\}$

$\Rightarrow$ solns not unique.

Only stating.

$$C = A^T (AA^T)^{-1}.$$

Check if $AC = I$?

$$AC = AA^T (AA^T)^{-1} = I.$$

pseudo inverse.

↳ obtained by setting $c_{3_1} = c_{3_2} = 0$.

Right inverse in general, is not unique.

↳ Left inverses.

$$AC = I \checkmark$$

Say: $A = \begin{pmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{pmatrix}$ Let us try $\underset{2\times 3}{B}\,\underset{3\times 2}{A} = \underset{2\times 2}{I}$

B_{13}

$$\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

B_{23}

found B, a left inv of A!

Like before, we have a left inverse (exist) it is not unique.

In this case we need full col. rank.

when is full col rank, when $\underline{m \geq n}$

↳ What does this mean for $Ax = b$?

Cols are indep n ⇒ rank = no of cols. ✓ (because full col rank).

∴ No of free vars $= n - r = 0$

∴ Only 2 possibilites, 0 or 1 soln. (atmost one soln)

(uniqueness

property).

w/o proof:

$$B = (A^TA)^{-1}A^T$$ still reserved for sq matrices.

See $B \cdot A = I$

practically:

$Ax = b$ (A has full col rank).

$A^TAx = A^Tb$

$$x = \underbrace{(A^TA)^{-1}A^T}b$$ if $(A^TA)^{-1}$ exists.

ONLY A TRICK TO REMEMBER THE EXPR.

it suggest a soln to $Ax=b$ exists even when $b \notin C(A)$. $\therefore$ Not correct.

Left / Right inverses $\rightarrow$ pseudo inverses

A^+