Recap of fundamental spaces ...

$$A = \begin{array}{c} \overbrace{}^{r} \\ \begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$r \cdot x$

↳ $C(A)$: 2 dim subspace of $\mathbb{R}^3$

$R(A)$: entire $\mathbb{R}^2$

$\Rightarrow N(A)$: ϕ zero vector only.

$N(A^T)$: 1 dim subspace of $\mathbb{R}^3$ $\perp$ $C(A)$ $\neq \mathbb{R}^1$

$$\overbrace{B : \begin{pmatrix} 1 & 0 & 1 \\ 5 & 4 & 1 \\ 2 & 4 & -2 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 10 \\ 4 \end{pmatrix}}^{Bx = b}$$

$C(B) = C(A)$ because $\text{Col} 3 = \text{Col} 1 - \text{Col} 2$

$N(B) = ?$

$Bx = 0 \Rightarrow x_1 + x_3 = 0 \rightarrow x_3 = c$

$5x_1 + 4x_2 + x_3 = 0$

$\Rightarrow x_1 = -c$. $\Rightarrow 5(-c) + 4x_2 + c = 0 \Rightarrow x_2 = c$

R3 $\Rightarrow$ $2x_1 + 4x_2 - 2x_3 \stackrel{?}{=} 0$

$2(-c) + 4(c) - 2(c) = 0$ ✓

$\Rightarrow c\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \rightarrow$ vector generates $\underline{\underline{N(A)}}$.

→ To summarize, when solve $Ax = b$, 2 tasks:

① $b \in C(A)$, gives a soln.

② Check $N(A)$. if $\neq \phi \Rightarrow \infty$ solns

if $= \phi \Rightarrow 1$ soln.

$x_n \in N(A) \neq 0$

∞ solns. eg. $x = x_p + c\, x_n$.

$x_p = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ satisfies $Bx = b$ & $\neq 0$.

$\Rightarrow$ soln to $Bx = b$ → ∞ solns

$$\boxed{x = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$$

Echelon form of A matrix.

Rectangular matrix: Can we do LU decomp?

$\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ✓ $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ ✓

$A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{pmatrix}$

pivot A'?

1st col pivot

$$\left[\begin{aligned} R_2 &= R_2 - 2R_1 \\ R_3 &= R_3 + R_1 \end{aligned}\right. \rightarrow \begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{pmatrix}$$

No pivots

3rd col

A': Echelon matrix

$\Rightarrow$ 0's to the left of the first non zero entry of each row.

$$\begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 + b_1 \end{pmatrix} \rightarrow \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{pmatrix}$$

$$\left[\, R_3 = R_3 - 2R_2 \right.$$

$$\rightarrow \begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A''$$

U

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}_{3\times 3}$$

$\Rightarrow$ U is 3×4

$LU = A$

L: 3×3, U: 3×4, A: 3×4

echelon matrix

One more step $\rightarrow$

Reduced Row Echelon form (RREF)

1) all pivots are 1
2) all entries above pivot are zero

$$\begin{pmatrix} 1 & 3 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \boxed{\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

$R_1 = R_1 - 3R_2$ $\qquad R$

Why is this useful?

↳ Null space $\quad Ax = 0 \Rightarrow \boxed{Rx = 0} \Rightarrow Ux = 0$

$$LUx = 0$$
$$L^{-1}LUx = L^{-1}0 = 0 = Ux.$$

Row 1: $x_1 + 3x_2 + \boxed{0} - x_4 = 0$

Row 3: $0 + 0 + x_3 + x_4 = 0$

Allows us to split the variables into 2:

→ pivot variables → cols with pivots 1, 3

→ free variables. → other cols : 2, 4.

$$\begin{pmatrix} x_1 = -3x_2 + x_4 \\ x_3 = \quad - x_4 \end{pmatrix}$$

from R1

from R2.

$$\underbrace{\begin{pmatrix} x_1 \\ x_3 \end{pmatrix}}_{\text{Pivot vars}} = \underbrace{\begin{pmatrix} -3 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} x_4}_{\text{free vars}}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_2 + x_4 \\ x_2 \end{pmatrix}$$

substituted

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \qquad \begin{pmatrix} -x_4 \\ x_4 \end{pmatrix}$$

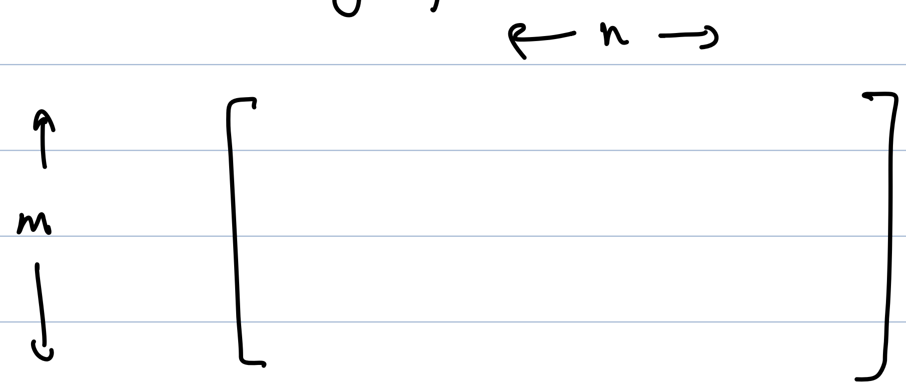
n.s vectors $= \left\{ \underbrace{\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{n_1} x_2 + \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{n_2} x_4 \right\}$

defines N.S. $\qquad$ 2 dim subspace of $\mathbb{R}^4$

$\neq \mathbb{R}^2$

We can generalize this to a $m \times n$ matrix.

Let's try fat matrix $m < n$.



no of pivots $\leq m$

no of free $\geq n - m$.

↳ for $Ax = b$ with $m < n$

always has a $N(A) \neq \phi$

Null space: has the same dim as the no of free vars.

Colspace : has the same dim as the no of pivots.

$$
\begin{array}{cccc}
v_1 & v_2 & v_3 & v_4
\end{array}
$$

$$
\begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{pmatrix}
$$

$$Ax = b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$Ux = b' = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for a soln to exist $b \in C(A)$

2 dim subsp of $\mathbb{R}^3$

There is a possibility of $b \notin C(A)$.

$$\Rightarrow b_3 - 2b_2 + 5b_1 = 0$$

$\vec{b} \cdot \hat{n} = 0$

plane thru $\vec{0}$

$\vec{n} = (5, -2, 1)$

Take $\vec{v_i} \cdot \vec{n}$ for $i = 1, \dots, 4. = 0$