In this case:

optimal primal x^* = optimal dual z^*

equal

$\hookrightarrow$ No gap, strong duality.

— x —

Recal defn of $q(\lambda) = \min_x L(x, \lambda)$

Update: It can happen that for some λ
$q(\lambda) \to -\infty$.

$\therefore$ Domain of q: $\mathcal{D} \triangleq \{ \lambda \mid q(\lambda) > -\infty \}$

$\hookrightarrow$ Two imp results:

$\left.\begin{array}{l} \text{① 'q' is concave} \\ \text{② } \mathcal{D} \text{ is convex} \end{array}\right\}$ for any f, c

Implication: Solving the dual problem is a convex optimization problem!

proof of ① $\rightarrow$ Consider λ_1, λ_2 and $\alpha \in [0,1]$

Recal: $\mathcal{L} = f(x) - \lambda c(x)$ $\rightarrow$ $q = \min_x \mathcal{L}$

$\downarrow$

$$\mathcal{L}(x, (1-\alpha)\lambda_1 + \alpha\lambda_2) = (1-\alpha)\mathcal{L}(x,\lambda_1) + \alpha\mathcal{L}(x,\lambda_2)$$

get q.

$$\min_x (\ ) = \min_x (\ )$$

$$\left[\min(a+b) \boxed{\geq} \min a + \min b \right.$$
$$\left. \text{e.g } a = x, b = -x \right]$$

$$q((1-\alpha)\lambda_1 + \alpha\lambda_2) \geq (1-\alpha) q(\lambda_1) + \alpha q(\lambda_2)$$

$\Rightarrow$ concave fn.

— x —

Proof of ②: If $\lambda_1, \lambda_2 \in \mathcal{D} \Rightarrow q(\lambda_1) > -\infty$,
$q(\lambda_2) > -\infty$.

$$\underbrace{q((1-\alpha)\lambda_1 + \alpha\lambda_2)} \geq (1-\alpha) q(\lambda_1) + \alpha q(\lambda_2)$$

(by concavity of q)

$$> -\infty$$

$$\Rightarrow (1-\alpha)\lambda_1 + \alpha\lambda_2 \in \mathcal{D}$$

$\therefore$ A convex comb of λ_1 & λ_2 is also in $\mathcal{D}$

$\Rightarrow \mathcal{D}$ is a convex set.

— x —

Implication - primal problem → anything!
dual problem → always concave.

— x —

<u>Weak duality</u>: $\min_x f(x)$, s.t. $c(x) \geq 0$

For any feasible $\bar{x}$, and any $\bar{\lambda} \geq 0$, then the following holds true:

$$q(\bar{\lambda}) \leq f(\bar{x})$$

So what?

Imp corollary: $f^* = \inf\{ f(x) : x \in \Omega \}$

$$q^* = \sup\{ q(\lambda) : \lambda \geq 0 \}$$

<u>Then</u>, $\underbrace{q^*}_{\text{optimum value of dual problem}} \leq \underbrace{f^*}_{\text{optimal value of the primal problem.}}$

<u>proof</u>:

$$\Big[\, q(\bar{\lambda}) = \min_{x} \mathcal{L}(x, \bar{\lambda})$$

$$= \min_{x} \underbrace{f(x) - \bar{\lambda}\, c(x)}$$

$$\leq \underbrace{f(\bar{x}) - \bar{\lambda}\, c(\bar{x})} \quad \geq 0$$

But $\bar{\lambda} \geq 0$ and $c(\bar{x}) \geq 0$

$$\leq f(\bar{x}) \,\Big]$$

QED.

↳ We saw geometrical problem there was no gap between opt primal & opt dual.

1) If $q^* < f^*$, then $f^* - g^* > 0$ called a duality gap.

2) If $q^* = f^*$, called strong duality.

[3) For convex optim problems, the duality gap is always 0! under constraint qualification (eg. LICQ)]

— $*$ —