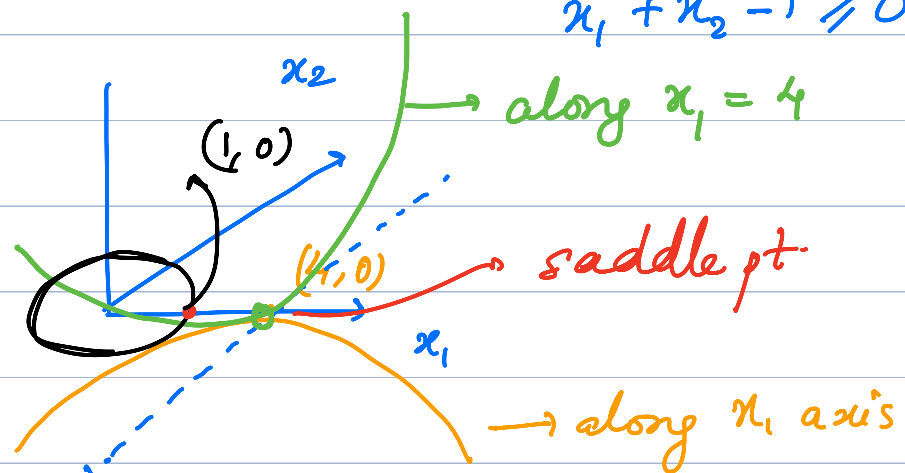
An e.g. $f(x) = -0.1(x_1-4)^2 + x_2^2$, s.t.

$x_1^2 + x_2^2 - 1 \geq 0$

f is non-convex.

fn is also not bounded from below



$$\nabla f = \begin{pmatrix} -0.2(x_1-4) \\ 2x_2 \end{pmatrix}, \quad \nabla c_1 = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathcal{L}(x,\lambda) = f - \lambda c_1$$

$$\nabla_x \mathcal{L}(x,\lambda) = \begin{pmatrix} -0.2(x_1-4) - \lambda \cdot 2x_1 \\ 2x_2 - 2x_2 \cdot \lambda \end{pmatrix}$$

$$\nabla^2_{xx}\mathcal{L}(x,\lambda) \rightarrow \begin{pmatrix} xx & xy \\ yx & yy \end{pmatrix} = \begin{pmatrix} -0.2 - 2\lambda & 0 \\ 0 & 2 - 2\lambda \end{pmatrix}$$

Check at $x_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\rightarrow \nabla_x \mathcal{L} = \begin{pmatrix} 0.6 - 2\lambda \\ 0 \end{pmatrix} = 0$

$\Rightarrow \lambda^* = 0.3$ works ($\lambda \geq 0$ ✓)

$c_1(x_A) = 0$. $\nabla c_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \neq 0$ LICQ ✓

$\Rightarrow$ comp ✓

What is the critical cone?

$$C(x^*, \lambda^*) = \{ w \mid \nabla c_1^T w = 0 \}$$

$$= \left\{ \begin{pmatrix} 0 \\ w_2 \end{pmatrix} \middle| \; w_2 \in \mathbb{R} \right\}$$

Thus to check the 2nd order sufficiency case:

$$w^T \nabla^2_{xx} \mathcal{L} \, w = \begin{pmatrix} 0 & w_2 \end{pmatrix} \begin{pmatrix} -0.8 & 0 \\ 0 & 1.4 \end{pmatrix} \begin{pmatrix} 0 \\ w_2 \end{pmatrix} = 1.4 w_2^2 > 0$$

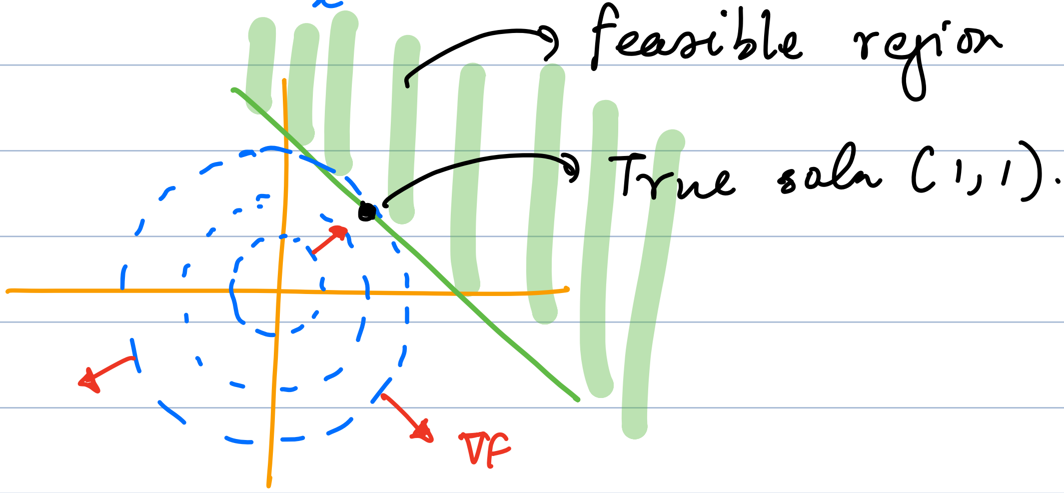
for any non zero w.

$\Rightarrow$ By second order sufficiency x_A is a strict local soln.

KKT Conditions & Duality

Start with an e.g. $f(x) = \frac{2}{5}(x_1^2 + x_2^2)$, and solve

P1 : $\min_x f(x)$ s.t. $c(x) = x_1 + x_2 - 2 \geq 0$



The Lagrangian : $\mathcal{L}(x, \lambda) = f(x) - \lambda c(x)$

As per KKT, at an optimal x^*, $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$

with $\lambda^* \geq 0$, $\lambda^* c(x) = 0$

$$\nabla f(x) = \lambda \nabla c(x) \iff 0.8 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What if $x = 5/4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \lambda = 1$, $c(x) \neq 0$

$\Rightarrow$ comp fails.

What if $x^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \lambda = 0.8$, $c(x) = 0$

$\underbrace{\lambda = 0.8}_{> 0}$

KKT condns are satisfied at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let's solve this differently.

→ Define a Lagrangian dual fn:

$$q(\lambda) = \min_x \mathcal{L}(x,\lambda) = \min_x f(x) - \lambda\, c(x)$$

• Soln to $\min_x \left(f(x) - \lambda\, c(x) \right)$ happens when?

$$\nabla_x \mathcal{L}(x,\lambda) = 0 \Rightarrow \frac{4}{5}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = 5/4 \begin{pmatrix} \lambda \\ \lambda \end{pmatrix}$$

• $$q(\lambda) = \frac{2}{5}\left(x_1^{*2} + x_2^{*2}\right) - \lambda\left(x_1^* + x_2^* - 2\right)$$

$$= \frac{4}{5}\left(\frac{5\lambda}{4}\right)^2 - \lambda\left(\frac{10}{4}\lambda - 2\right)$$

$$= -5/4\, \lambda^2 + 2\lambda$$

→ Next, Find $\lambda \geq 0$ which maximizes $q(\lambda)$

$$\max_\lambda q(\lambda) \Rightarrow -\frac{10}{4}\lambda + 2 = 0 \Rightarrow \lambda^* = 4/5$$

(got lucky).

$$\Rightarrow \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What did we do?

Primal problem: $\min_x f(x)$ s.t. $c(x) \geq 0$

Instead

dual problem: $\max_\lambda q(\lambda)$ s.t. $\lambda \geq 0$

where $q(\lambda) = \min_x \mathcal{L}(x, \lambda)$

↳ Lagrangian dual problem.

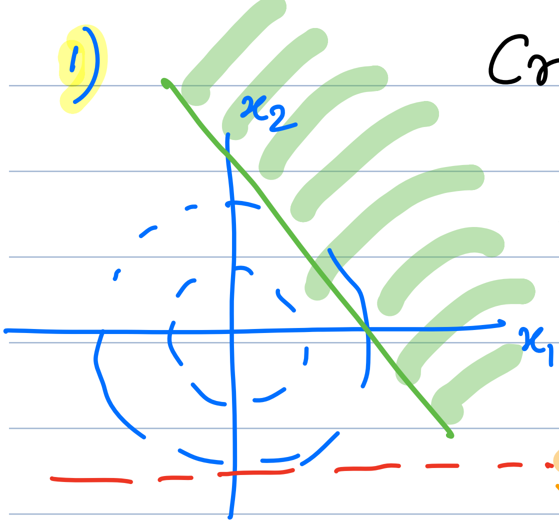
(In general when we have equalities then (↓) becomes: $\max_\lambda q(\lambda)$ s.t. $\lambda_i \geq 0 \ \forall i \in \mathcal{I}$.)

↳ Not guaranteed to always work, works for a class of problems

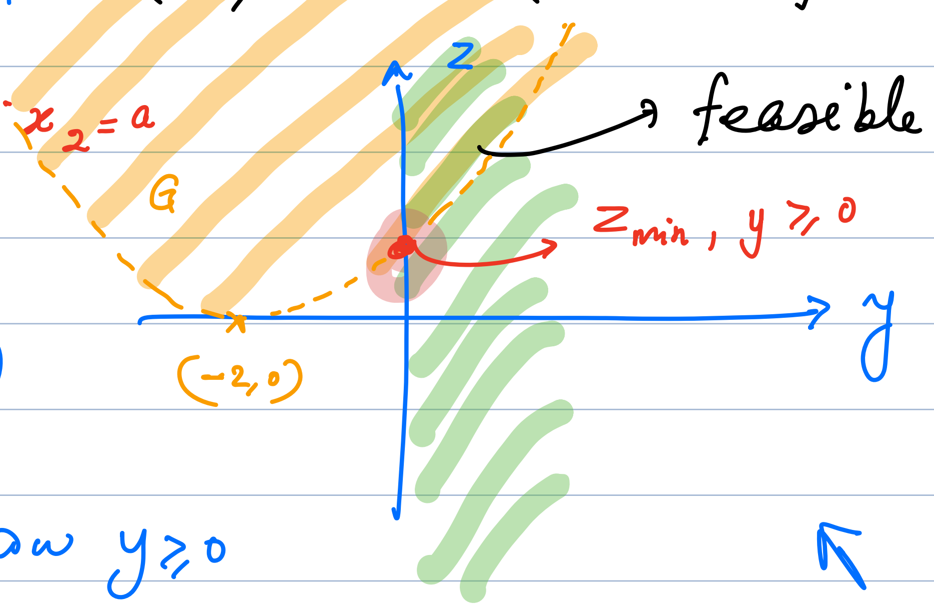
↳ When it works, leads to a simpler optim problem.

↳ proofs later, geometry now.

1) Create a map. $\mathcal{L} = f(x) - \lambda c(x)$



$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} y = c(x) \\ z = f(x) \end{pmatrix}$$



$y = c(x) = x_1 + x_2 - 2$

$z = f(x) = \frac{2}{5}(x_1^2 + x_2^2)$

↳ feasible is now $y \geq 0$

↳ Take $x_2 = a \Rightarrow y = x_1 + a - 2$

$$z = \frac{2}{5}(x_1^2 + a^2) = \frac{2}{5}\left((y + 2 - a)^2 + a^2\right)$$

Horizontal line $\longleftrightarrow$ parabola. When $a = 0$

$$z = \frac{2}{5}(y+2)^2$$

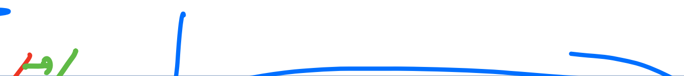
↳ Geometry is clear now

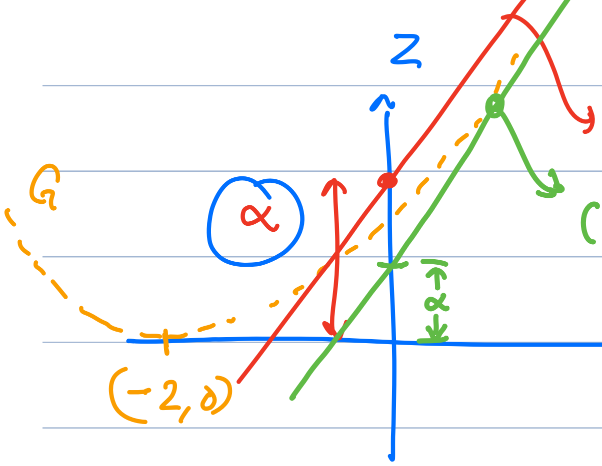
1) Ask: what is primal problem?

Min z, while $y \geq 0$

2) What is the dual problem?

<u>Visualize</u> $\mathcal{L}(x, \lambda) = f(x) - \lambda c(x)$





$\alpha = z - \lambda y$, $\lambda \geq 0$

$z = \lambda y + \alpha$

λy → slope → fixed

α → intercept

Lagrangian dual fn? $q(\lambda) = \min_{x} \mathcal{L}(x, \lambda)$

fixed λ

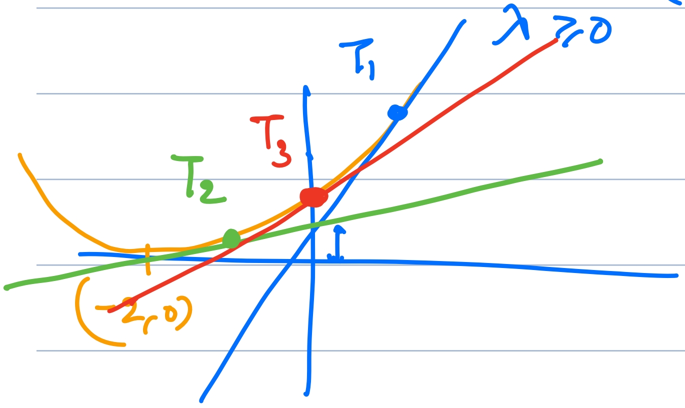
$= \min_{y, z \in G} (z - \lambda y)$ → α

In words: what is the min possible value of α while $y, z \in G$?

↳ Finally, to find dual optimum:

$\max_{\lambda} q(\lambda)$ s.t. $\lambda \geq 0$.

$= \max_{\lambda \geq 0} (z^+ - \lambda y^+)$



↳ maximize the intercept!

Happens when T_3

$z^* - \lambda^\circ y^* = q(\lambda^\circ)$

T_3 → soln to optimum dual problem.

In this case:

optimal primal z^* = optimal dual z^*

equal

No gap, strong duality.