Combining case 1 & case 2.

$$\mathcal{L}(x,\lambda) = f(x) - \lambda_2 c_2(x)$$

→ When no 1^{st} order feasible direction exists:

Then $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$ with $\lambda_2^* \geq 0$

i.e. $\nabla f(x^*) = \lambda_2^* \nabla c_2(x^*)$

We also needed $c_2(x^*) = 0$ active ineq

case 2 so far

In case 1: $c_2(x) > 0$ & no feasible dir

if $\nabla f(x) = 0$.

→ $\nabla_x \mathcal{L}(x^*, \lambda^*) = \nabla f - \lambda_2 \nabla c_2 = 0$ if $\lambda_2 = 0$.

These 2 cases are thus combined:

→ $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$ with $\lambda_2^* \geq 0$

→ and $\underbrace{\lambda_2 c_2(x^*)}_{} = 0$

case 1	case 2
$\lambda_2^* = 0$	$c_2(x^*) = 0$

Known as the complementarity condn.

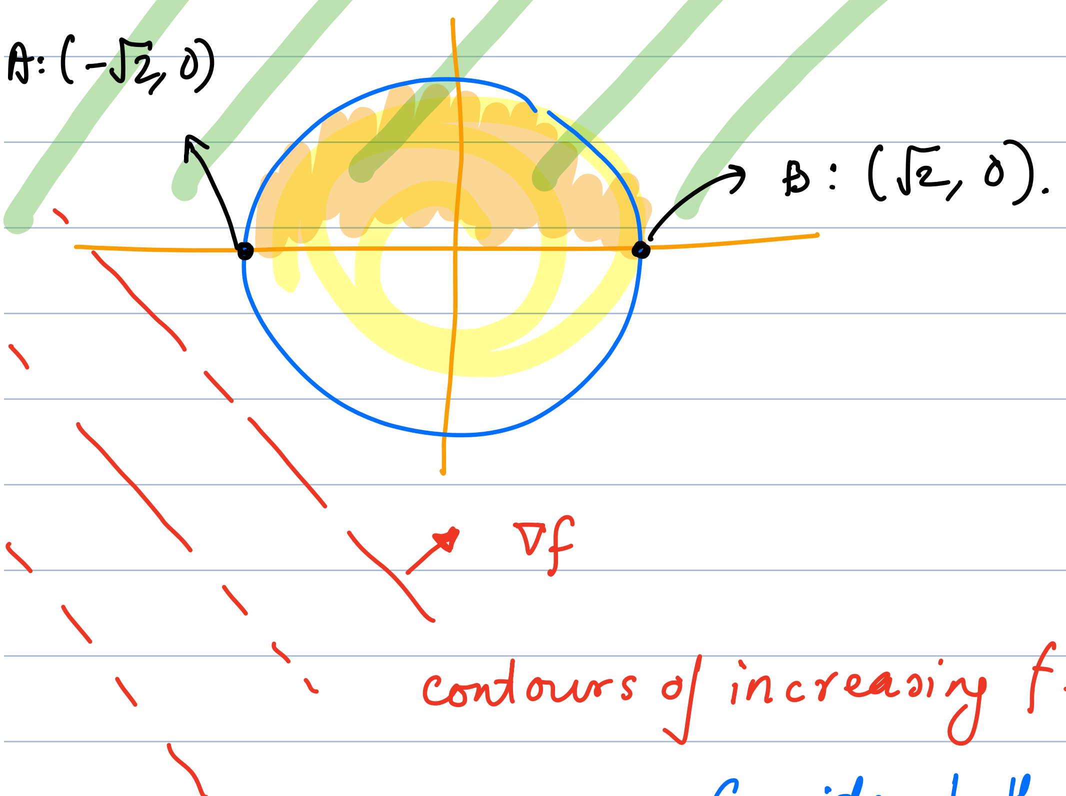
↳ central to CO.

<u>Summary</u>: derived necessary cond^{n}s for x^* to be a local minimizer of a C.O.

→ Take an e.g. $f(x) = x_1 + x_2$ s.t. $\begin{cases} x_1^2 + x_2^2 \leq 2 \\ x_2 \geq 0 \end{cases}$

↳ Put into standard form

$$\min_x f(x), \quad s.t. \quad c_1(x) = 2 - x_1^2 - x_2^2 \geq 0$$
$$c_2(x) = x_2 \geq 0$$



Consider both points A, B.

↳ both constraint are active.

↳ For there to be a 1st order feasible dir; d should satisfy: $\nabla f^T(x) d < 0$,

→ $\nabla c_i(x)^T d \geq 0$ (This is like case 2 earlier)

Define $\mathcal{L}(x, \lambda) = f(x) - \lambda_1 c_1(x) - \lambda_2 c_2(x)$

$\lambda = [\lambda_1 \; \lambda_2]^T$

$\therefore$ We will want $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$ for $\underbrace{\lambda^* \geq 0}_{\lambda_1^* \geq 0,\; \lambda_2^* \geq 0.}$

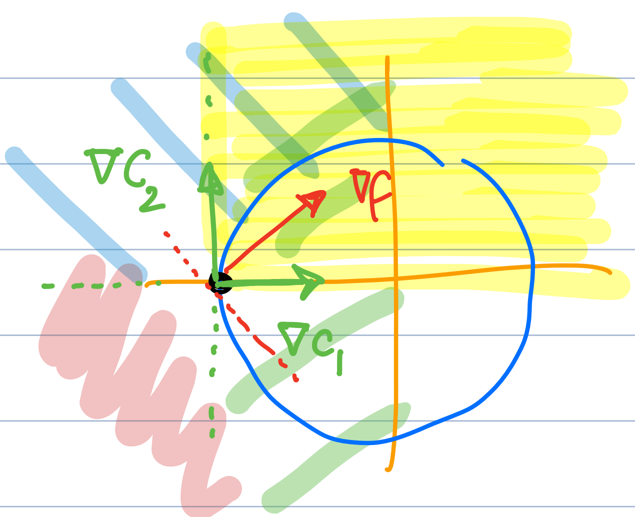
complimentary condns. $\lambda_1^* c_1(x) = 0$

$\lambda_2^* c_2(x^*) = 0$

$$\underline{\nabla_x \mathcal{L}(x, \lambda)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} - \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\nabla_x \mathcal{L} = \begin{pmatrix} 1 + 2x_1 \lambda_1 \\ 1 + 2x_2 \lambda_1 - \lambda_2 \end{pmatrix} = 0$$

<u>At A</u> $(-\sqrt{2}, 0)$

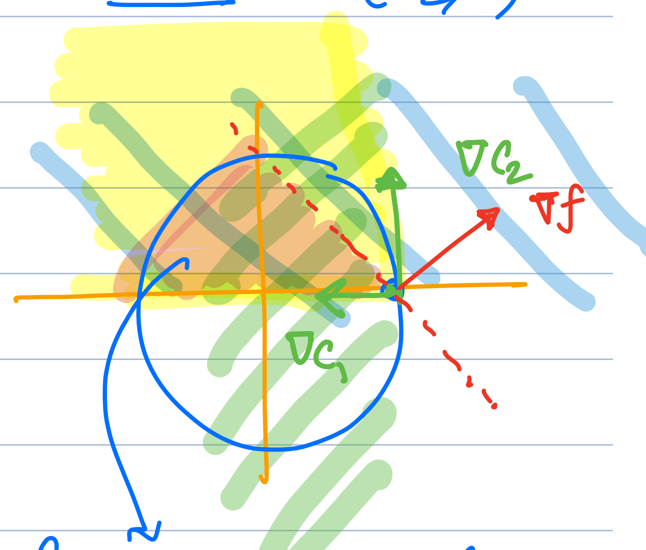


No feasible dir.

Soln to $\nabla_x \mathcal{L} = 0$

$\Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1/2\sqrt{2} \\ 1 \end{pmatrix}$

<u>At B</u> $(\sqrt{2}, 0)$



feasible dir exists

$\Rightarrow$ B is not a stationary point.

$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1/2\sqrt{2} \\ 1 \end{pmatrix}$

A satisfies first order cond^{n}s | violates $\hat{x} \geq 0$ $\Rightarrow$ B is not stat. pt.

—x—

A set of linearized feasible directions:

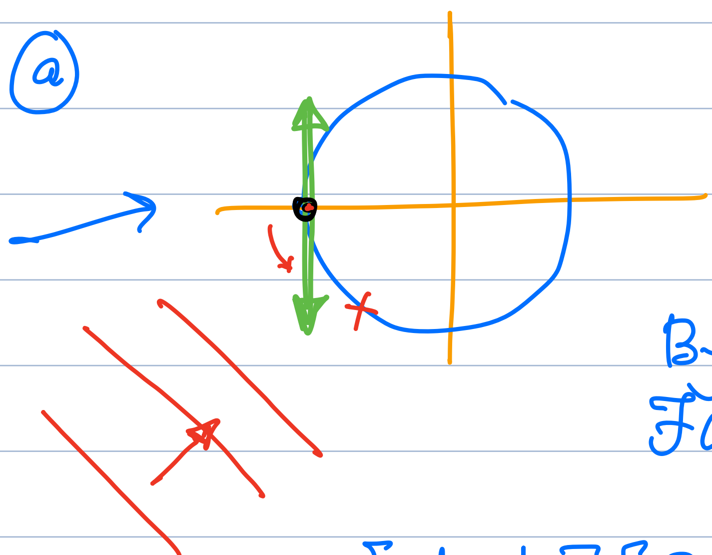
$$\mathcal{F}(x) = \left\{ d \;\middle|\; \begin{array}{ll} d^T \nabla c_i(x) = 0 & \forall i \in \mathcal{E} \\ d^T \nabla c_i(x) \geq 0 & \forall i \in \mathcal{A}(x) \cap \mathcal{I} \end{array} \right\}$$

(fancy f)

i.e. only active ineq. constraints. (like case 2)

↳ Revisit older e.g.s.

ⓐ



$f(x) = x_1 + x_2$. s.t.

$x_1^2 + x_2^2 - 2 = 0$

consider $(-\sqrt{2}, 0)$

By def^n:

$$\mathcal{F}(x) = \{ d \mid d^T \nabla c_1 = 0 \}$$

$$[d_1 \; d_2] \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = -2\sqrt{2}\, d_1 = 0$$

$$\Rightarrow \quad \mathcal{F}(x) = \left\{ \begin{pmatrix} 0 \\ d_2 \end{pmatrix} \;\middle|\; d_2 \in \mathbb{R} \right\} \checkmark$$

(b) However, if C_1: $G(x) = (x_1^2 + x_2^2 - 2)^2 = 0$

Has feasible set changed?

$\mathcal{F}(x)$: $d^T \nabla C_1 = 0$

$$\begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 4(x_1^2 + x_2^2 - 2)x_1 \\ 4(x_1^2 + x_2^2 - 2)x_2 \end{bmatrix} = 0$$

$$\downarrow$$

$$\begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \quad \forall\, d_1, d_2 \in \mathbb{R}$$

$\mathcal{F}(x) = \mathbb{R}^2$ ✗

Geometry is the same, but algebraic spec changed ($\Rightarrow$ changed set of linearized feasible dirs)

$\Rightarrow$ Geometry is not being captured.

$\downarrow$ Come up with something else.

Constraint Qualifications: $\rightarrow$ Assumptions that ensure similarity of the constraint set Ω & it's linearized approx at x

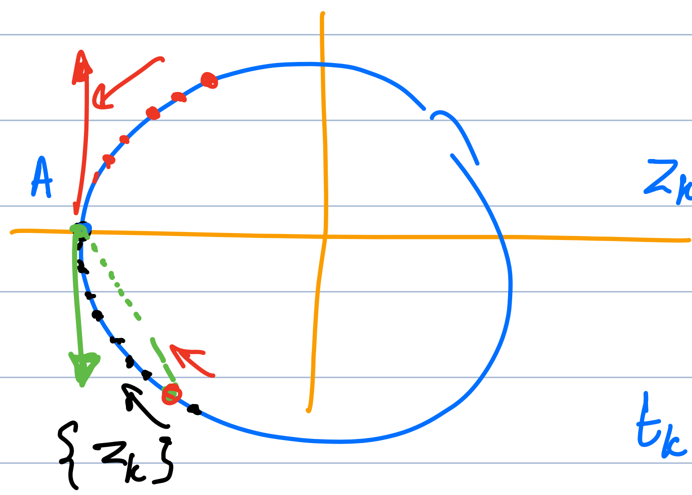
Need new tech tools

① Given $x \in \Omega$, we call $\{z_k\}$ a feasible sequence if $z_k \in \Omega$ $\forall$ large k, $z_k \to x$.

② (a) A tangent is a limiting direction of a feasible sequence. In addition to $\{z_k\}$, we find a sequence of +ve scalars $\{t_k\}$ with $t_k \to 0$, then 'd' is said to be a tangent:

$$d = \lim_{k \to \infty} \frac{z_k - x}{t_k}$$

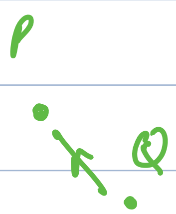
(b) Set of all tangents to Ω at x is called a tangent cone, $T_\Omega(x)$.



$$z_k = \begin{pmatrix} -\sqrt{2 - 1/k^2} \\ -1/k \end{pmatrix}$$

$$t_k = \| z_k - x_A \|$$

$$\begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix}$$



$$d = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$v_{PQ} = Q - P$

↳ Tangent cone depends on geometry, not on algebraic spec.