Constrained Optimization (CO)

Unconstrained " (UO)

↳ Ch 12 of NW.

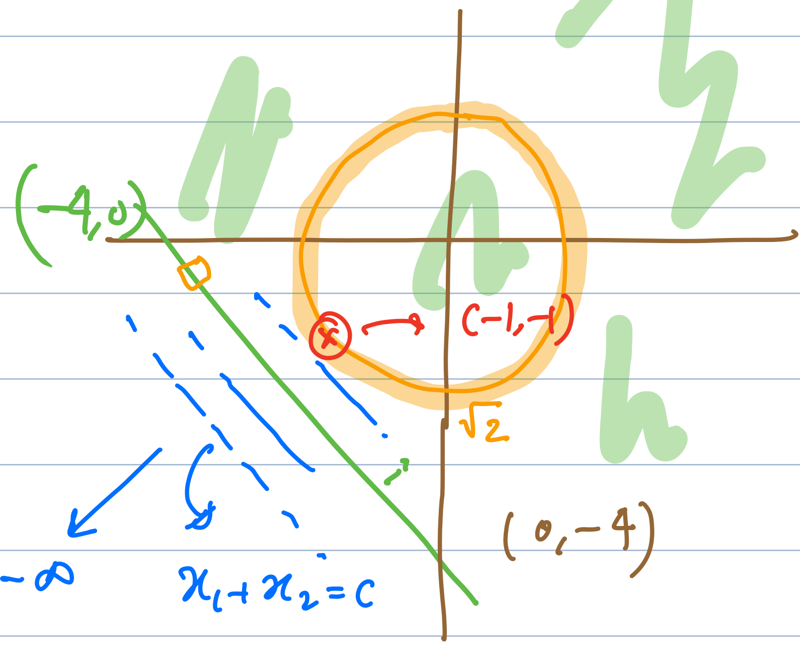
Terminology. · e.g. $f(x) = x_1 + x_2$.

a) In UO, soln $\min_x f(x)$?

b) In CO world.

↳ Say $c_1(x) = x_1^2 + x_2^2 - 2 = 0$

& $c_2(x) = x_1 + x_2 + 4 \geq 0$



$A(\hat{x}) = \{1\}$

Equality

CO can be written as:

$$\begin{cases} \min\limits_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \begin{cases} c_i(x) = 0, & i \in \mathcal{E} \\ c_i(x) \geq 0, & i \in \mathcal{I} \end{cases} \end{cases}$$

Inequality

↳ Feasible set is $\Omega = \{ x \mid c_i(x) = 0, \ i \in \mathcal{E}; \ c_i(x) \geq 0, \ i \in \mathcal{I} \}$

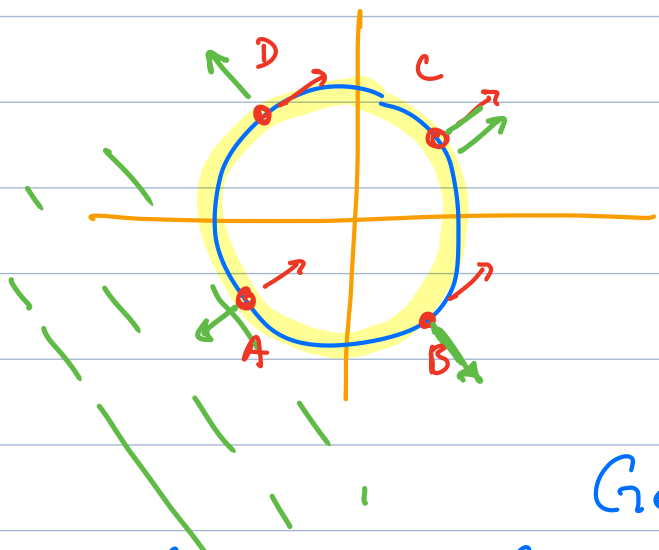
Compactly: $\boxed{\min_{x \in \Omega} f(x)}$.

↳ Any pt in feasible set Ω inequality constraint is said to be active if $c_i(x) = 0$, inactive if $c_i(x) > 0$.

↳ Active Set, $\mathcal{A}(x) = \mathcal{E} \cup \{ i \in \mathcal{I} \mid c_i(x) = 0 \}$.

Equality constraint.

Say $f(x) = x_1 + x_2$, & $c_1(x) = x_1^2 + x_2^2 - 2 = 0$.



$$\nabla f = \left[\frac{\partial f}{\partial x_i} \right] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\nabla c = 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Geometry:

↳ At soln (A), $\nabla f(x^*)$ & $\nabla c(x^*)$ are $\parallel$el

$$\nabla f(x^*) = \lambda \nabla c(x^*)$$ ←

Also true at $\boxed{C}$

→ Say we are at a feasible pt; $c_1(x) = 0$.
↳ want to step to a better point
$x \longrightarrow x+s$. At this point $c_1(x+s)=0$

Taylor's thm: $c_1(x+s) = c_1(x) + \nabla c_1(x)^T s$

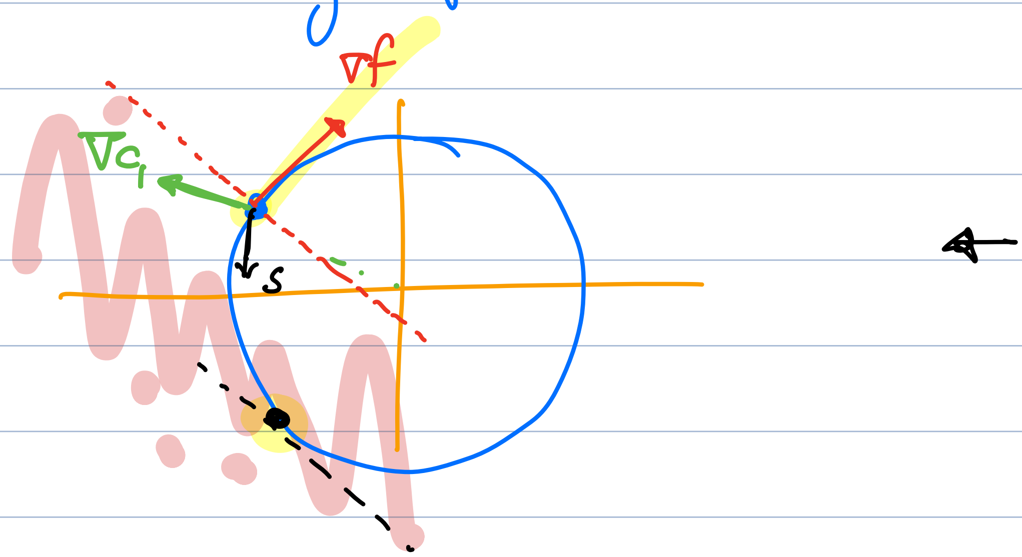
$\Rightarrow \boxed{\nabla c_1^T(x)\, s = 0}$ must be satisfied by s

↳ Also $f(x+s) < f(x)$ → By Taylor's thm
$f(x+s) = f(x) + \nabla f(x)^T s < f(x)$

$\Rightarrow \boxed{\nabla f^T(x)\, s < 0}$

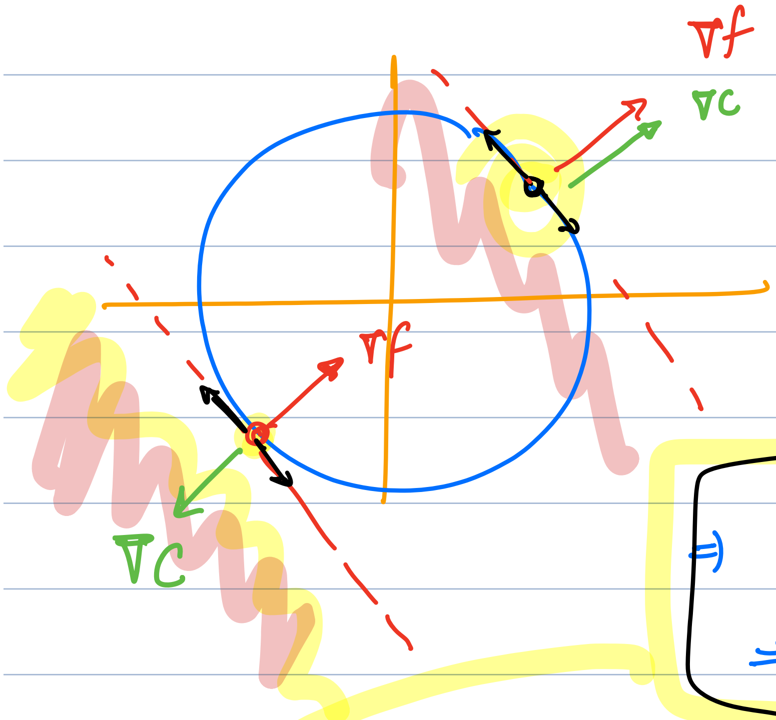
Thus: $\boxed{\nabla c_1^T(x) s = 0 \;\; \& \;\; \nabla f(x)^T s < 0}$

* As long a s can be found, J keep reducing f.



* When will it be impossible to move

further?



In both points we can't get both condns satisfied

$\Rightarrow$ $\nabla f = \lambda_1 \nabla c_1(x)$

$\Rightarrow$ possible local minima

* We introduce a Lagrangian fn

$$\mathcal{L}(x, \lambda_1) = f(x) - \underbrace{\lambda_1}_{} c_1(x)$$

Langrange multiplier for c_1.

Consider $\nabla_x \mathcal{L}(x, \lambda_1) = \nabla f(x) - \lambda_1 \nabla c_1(x)$

stationary point $\Rightarrow \nabla_x \mathcal{L} = 0$

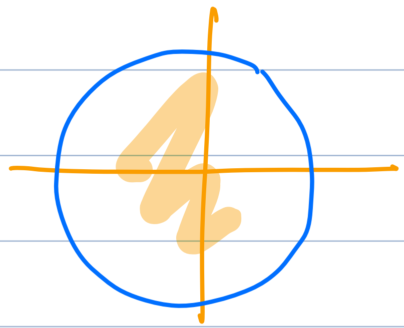
Necessary condn $\nabla_x \mathcal{L}(x^*, \lambda_1^*) = 0$ for some λ_1^*.

Nec. condn for unconstrained opt?
Local unconstrained minimizers have
$\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is PSD.

★ Inequality Constraints.

Similar problem $f(x) = x_1 + x_2$

$\longrightarrow c_2(x) = 2 - x_1^2 - x_2^2 \geq 0$



↳ We are at x . $\Rightarrow c_2(x) \geq 0$

I want to goto $x \to x+s$

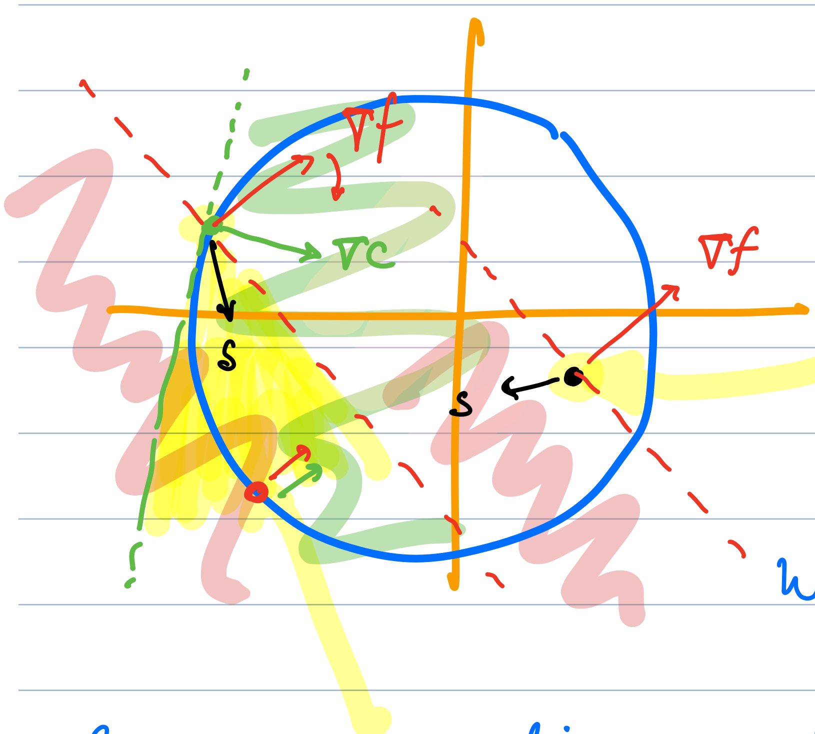
$\Rightarrow c_2(x+s) \geq 0$

Taylor's thm: $c_2(x) + \nabla c_2(x)^T s \geq 0$ ①

w.r.t f: $\nabla f(x)^T s < 0$ ② (descent dir)

$\Rightarrow$ 2 cases : x lies inside, or x on bdry.

Case 1 x lies inside $\Rightarrow c_2(x) > 0 \Rightarrow$ Inactive.



$\underbrace{\underbrace{c_2}_{>0} + \underbrace{\nabla c_2^T s}} \geq 0$

any small s will work.

or eg. $s = -\alpha \nabla f$

works as long as $\nabla f \neq 0$

Case 2 x lies on the circle: $c_2(x) = 0$

$$c_2 + \nabla c_2^T s \geq 0 \qquad \text{(active)}$$

$$\Rightarrow \left[\begin{array}{c} \nabla c_2(x)^T s \geq 0 \\ \hline \nabla f(x)^T s < 0 \end{array} \right] \begin{array}{l} \rightarrow \text{Closed H.S.} \\ \rightarrow \text{open H.S.} \end{array}$$

possible in the intersection of 2 half space

= When can I find no s?

When ∇f ||lel to ∇c

$$\nabla f(x) = \lambda_2 \nabla c_2(x), \quad \lambda_2 \geq 0$$

↳ Combine into something compact.