min(n, r) rank.

$$\hookrightarrow \quad x^* = \sum_{i=1}^{n} \frac{1}{\sigma_i} (u_i^T y) v_i$$

matrix

data

$\sigma_n \rightarrow$ smallest sing val

Any errors in $y \longrightarrow$ amplified by $\frac{1}{\sigma_n}$.

$$K = \frac{\sigma_1}{\sigma_n} \gg 1$$

$$= \sum_{i=1}^{n-1} \frac{1}{\sigma_i} (\quad) + \frac{1}{\sigma_n} (\quad)$$

Equivalent to truncating the SVD.

Trade off

accuracy $\longleftrightarrow$ stability

$\hookrightarrow$ Aside outer product form SVD

$$Ax = b$$

$\hookrightarrow$ rank r

$$A = USV^T$$

$$\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}_{m \times n.}$$

$$A = \sum_{i=1}^{r} \sigma_i \underbrace{u_i v_i^T}_{m \times n \,:\, \text{rank } 1.}$$

$$\left(\sum_{i=1}^{r} \sigma_i u_i v_i^T x = b \right)$$

$$u_j^T(\quad) \to \sigma_j (v_j^T x) = u_j^T b$$

$$(v_j^T x) = \frac{1}{\sigma_j} u_j^T b$$

In general any $x = \sum_{i=1}^{n} \alpha_i v_i$ → basis $\mathbb{R}^n$

$$\underline{v_j^T x} = \alpha_j$$

$$\Rightarrow \quad x = \sum_{i=1}^{n} \left(\frac{1}{\sigma_j} u_j^T b \right) v_i$$

if full col rank.

if no full col rank.

$$\boxed{x = \sum_{i=1}^{r} \frac{1}{\sigma_j} (u_j^T b) v_i + \sum_{i=r+1}^{n} \alpha_i v_i}$$

↳ ask which soln has min norm?

$$\|x\|_2^2 = \sum |\alpha_i|^2$$

$$\left(\sum_{i=1}^{r} \frac{1}{\sigma_j} u_j^T b \, v_i \right)$$

— x —

Dis advantge → expensive.

Alt Cholesky decomp.

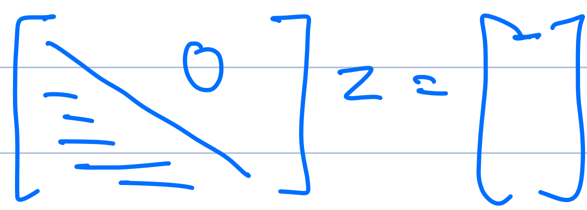
Normal eqns: $J^T J \hat{x} = J^T y$

$J^T J \to$ sym PD

① $\Rightarrow$ It has a decomp: $J^T J = W^T W$

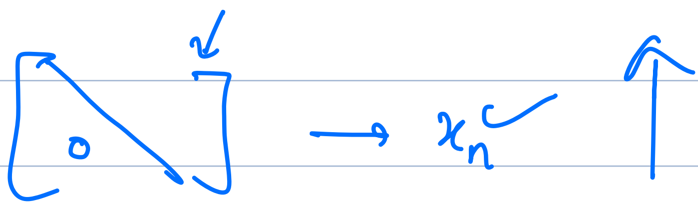
W: $n \times n$ upper tri, +ve diagonal entries.

② $W^T \underbrace{W x^*}_{z} = J^T y$ $\qquad$ $W^T z = J^T y$



back substitution.

$W x^* = z$



$J \to \kappa = \dfrac{\sigma_1}{\sigma_n}$ $\qquad$ $J^T J \to \kappa^2$

disadv.

[QR decomp. $\to$ κ $\quad$ 10.2 NW]

— x —

Nonlinear Least Squares

Gauss-Newton method.

Newton recap

① Calc ∇f & $\nabla^2 f$

② $(\nabla^2 f)\,p = -\nabla f$: solve for p

③ Calc step length α → linesearch
subject to ? Wolfe condns.

→ $$f(x) = \frac{1}{2}\|r(x)\|^2$$

$$\nabla f(x) = J(x)^T r(x)$$

$$\nabla^2 f(x) = J^T(x)J(x) + \sum_j r_j(x)\nabla^2 r_j(x)$$

low

→ $$\nabla^2 f(x) \approx J^T(x)J(x).$$

How does it help? → H is PSD

$$y^T \nabla^2 f(x)\, y \geq 0$$

makes descent direction legal.
H should be PD → J has full col rank.

→ $$J_k^T J_k p_k = -J_k^T r_k$$ to solve for p

→ once done, get α_k.

$\rightarrow$ get to x_{k+1}.

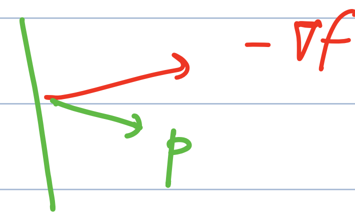
(Method converges to $\lim_{k \to \infty} \nabla f_k \to 0$
need $J_k(x)$ to be full col rank.)

Notes:

1) Cheaper $\rightarrow$ all I need is J

2) p is a legit descent dir if J is full col rank.

$\Rightarrow \quad p_k^T \nabla f_k < 0$



$$p_k^T \nabla f_k = p_k^T J_k^T r_k$$

$$= -p_k^T J_k^T J_k p_k = -\| J_k p_k \|^2$$

$$\leq 0$$

Assume J_k

if $\| \ \|^2 = 0 \Rightarrow p_k = 0$

$$\Rightarrow J_k^T r_k = 0 = \nabla f_k$$

3) $J_k^T J_k p = -J_k^T r_k$

$$\underbrace{J_k^T (J_k p + r_k) = 0}$$

$\Leftrightarrow$ Eqv to
$$\min_p \| J_k p_k + r_k \|^2$$

$\rightarrow$ To solve NLSQ prob. $\Rightarrow$ solve LSQ at each stage!

— x —

$\mathrm{eval}(J^T J p) \rightarrow (Jp)$ MV ✓

$J^T(Jp)$ $M^T V$

$\rightarrow (J^T J)$ MM ✗

$(J^T J)p$ MV

$\rightarrow y = (Jp)$ MV ✓

y^T

$y^T J = z$

z^T

— x —