Conjugate Direction Method

Seeking iterative method: $x_{k+1} = x_k + \alpha_k p_k$.

Need not be a descent direction.

2 requirements:

① The set of vectors $\{p_0, p_1, \dots, p_k\}$: conjugate w.r.t A.

② The step length α_k is an exact minimizer of the quadratic form $\phi(x)$: $\phi(x_k + \alpha p_k)$.

$$\boxed{\alpha_k = -\frac{r_k^T p_k}{p_k^T A p_k}}$$

$r_k = A x_k - b = \nabla\phi(x_k)$

Result: For any $x_0 \in \mathbb{R}^n$ the seq $\{x_k\}$ generated as per (1), (2) above converges to x^* where $Ax^* = b$ in at most n steps.

Proof: Lets' assume w.l.o.g that $k \leq n$.

We know that $\{p_0, \dots, p_{n-1}\}$ are lin. indep.

$$x^* - x^0 = \sigma_0 p_0 + \dots + \sigma_{n-1} p_{n-1}$$

$\sigma_k = ?$ left mult by $p_k^T A$

$$\sigma_k = \frac{p_k^T A (x^* - x^0)}{p_k^T A p_k}.$$

x_k in terms of $x_0, x_1, \dots, ?$

$$x_k = \underbrace{\underbrace{\underbrace{x_0 + \alpha_0 p_0}_{x_1} + \alpha_1 p_1}_{x_2} + \alpha_2 p_2 \cdots}_{x_{k-1}} + \alpha_{k-1} p_{k-1}$$

$$\Rightarrow \quad x_k - x_0 = \sum_{i=0}^{k-1} \alpha_i p_i$$

Premult by $p_k^T A \rightarrow \quad p_k^T A (x_k - x_0) = 0$

$$x^* - x^0 = (x^* - x_k + x_k - x_0)$$

$$\sigma_k = \frac{p_k^T A \left[(x^* - x_k) + (x_k - x_0) \right]}{p_k^T A p_k}$$

$$= \frac{p_k^T A (x^* - x_k)}{p_k^T A p_k} = \frac{p_k^T (\overbrace{b - A x_k}^{-r_k})}{p_k^T A p_k} = \alpha_k.$$

— x —

quadratic part $\quad x^T A x. \qquad$ CDM

<u>case 1</u> $\quad$ <u>A is diagonal:</u> what are candidate conj dirs?

<u>coordinate directions</u> $\rightarrow \quad e_i = [0 \cdots 1 \cdots 0]^T$

i^{th} pos

$$[p_1 \;\; p_2] \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 2p_1 \\ 5p_2 \end{bmatrix}$$

$$v_1 = \begin{pmatrix} p_1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ p_2 \end{pmatrix}$$

$$v_1^T A v_2 = 0$$

$$\begin{pmatrix} p_1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 5p_2 \end{pmatrix} = 0$$

<u>Case 2</u> A is not diagonal: $\underline{x^T A x}$.

$P = [p_0 \ \ldots \ p_{n-1}]$ Consider: $\underbrace{P^T A P} = D$

$$\underline{x^T A x} = x^T \underbrace{(P^T)^{-1} P^T}_{I} A \underbrace{P P^{-1}}_{I} x$$

$$= y^T D y$$

$$y = P^{-1} x.$$

$$\begin{bmatrix} - p_0^T - \\ - p_1^T - \\ \vdots \\ - p_{n-1}^T - \end{bmatrix} \begin{bmatrix} A p_0 & A p_1 & \cdots & A p_{n-1} \\ | & | & & | \end{bmatrix}$$

In y-space

— y coordinate dirs.

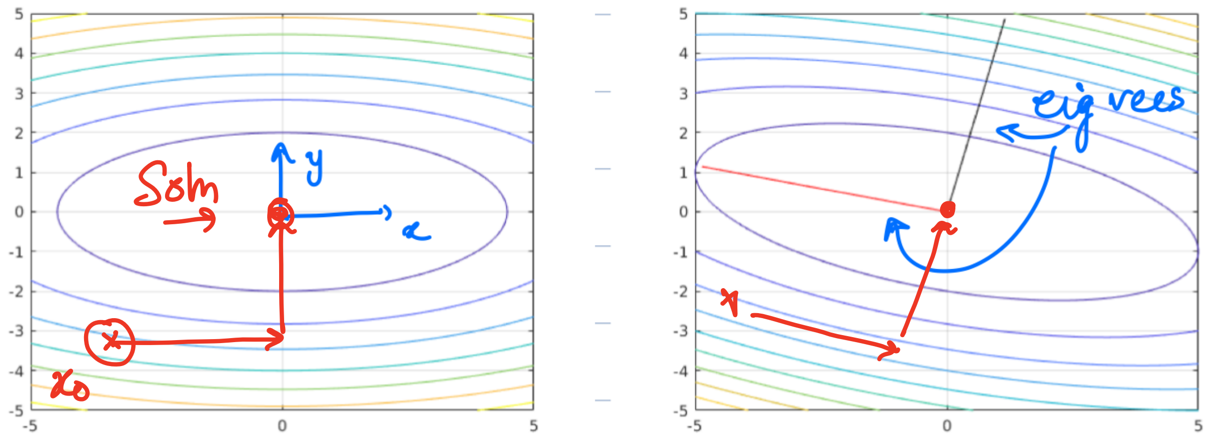
In x-space.

$$x = Py$$

$$x = [\,p_0 \;\; p_i \;\; p_{n-1}\,]\begin{bmatrix} y_0 \\ y_i \\ y_{n-1}\end{bmatrix} \begin{matrix} \to 0 \\ \to 1 \\ \to 0 \end{matrix}$$

$$= y_i \, p_i$$

$\Rightarrow$ We walk along the conjugate dirs.



diag   non diag.

conjugacy $\Rightarrow$ $p_i^T A p_j = 0 \qquad i \neq j$

① eigen $\quad A p_j = \lambda p_j \Longrightarrow p_i^T p_j = 0 \quad i \neq j$

$O(n^3)$ ✓

$$\left.\begin{matrix} \text{Conjugacy} \not\Rightarrow \text{eigenvecs.} \\ \text{eigenvecs} \Rightarrow \text{conjugacy} \end{matrix}\right\} \quad A \text{ sym pos def.}$$

② Gram-Schmidt (modified) $\quad O(n^3)$

Result: Say CDM $\to \{x_k\}$ starting from x_0. (minimize ϕ) then

① $r_k^T p_i = 0$ for $i \in [0, k-1]$.

② In an affine space $\{x \mid x_0 + \text{span}\{p_0, \dots, p_{k-1}\}\}$
x_k is the minimizer of $\phi(x)$.

↪ Proof: $\quad r(x) = Ax - b = \nabla\phi(x)$.

$\hat{x}$ belonging to the affine space.

$$\hat{x} = x_0 + \sum_{i=0}^{k-1} \sigma_i p_i .$$ Find minimizer of $\phi(\hat{x})$?

$$\frac{\partial \phi(x_0 + \sum_i \sigma_i p_i)}{\partial \sigma_i}$$

$$= \nabla\phi(\hat{x})^T p_i = 0 \quad \text{for } i \in [0, k-1]$$

$$\Rightarrow \quad r(\hat{x})^T p_i = 0 \quad \text{for } i \in [0, k-1].$$

→ A is not sym pos. def. Then?

$$Ax = b$$

"normal" eqns:
$$A^T A x = A^T b$$
$$A' x = b'$$

$\mathrm{Cond}(A) = \kappa$

$\mathrm{Cond}(A^T A) = \kappa^2$

$$\|x_{k+1} - x^*\| \leq \underbrace{\left(\frac{\kappa - 1}{\kappa + 1}\right)}_{\to 1 \text{ when } \kappa \text{ large}} \underbrace{\|x_k - x^*\|}_{\text{steepest descent.}}$$

—— x ——