Computational Electromagnetics : Review of Vector Calculus

Uday Khankhoje

Electrical Engineering, IIT Madras

Topics in this module

1. Chain rule of differentiation and the gradient
2. Gradient, Divergence, and Curl operators
3. Common theorems in vector calculus
4. Corollaries of these theorems; miscellaneous results

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Chain rule of differentiation

- Consider a scalar function of several variables, $f(x, y, z)$

  $f = \frac{-q}{4\pi\varepsilon_0 |\vec{r}|}$ $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

- Want to calculate a small change in f, i.e. df
  Say each variable has changed, e.g. $x \to x + dx$ . . .

  $y \to y + dy$
  $z \to z + dz$

- Chain rule tells us: $\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz}$

  $\vec{v} \cdot \vec{u}$

  $\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$

- Dot product between $\boxed{(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})}$ and $\boxed{(dx, dy, dz)}$

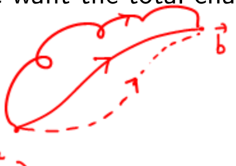
  ∇f $\vec{dl}$

Working with the gradient

- Compact way to write change $\boxed{df = \nabla f \cdot d\vec{l}}$

$$\int_{\vec{a}}^{\vec{b}} df = \underbrace{f(\vec{b})}_{\text{final}} - \underbrace{f(\vec{a})}_{\text{initial.}}$$

- Now we want the total change going from $\vec{a}$ to $\vec{b}$



$\int_{\vec{a}}^{\vec{b}} df$

- $\int_{\vec{a}}^{\vec{b}} \nabla f \cdot d\vec{l} = f(\vec{b}) - f(\vec{a})$ is path *independent*.

Corollary: $\oint \nabla f \cdot d\vec{l} = 0$ Conservative

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Gradient as the 'Del' operator

- Saw that $\boxed{\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)} = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$

- Generalize a 'Del' operator as $\boxed{\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$ $\rightarrow \nabla f$

- Acts in three ways (like an ordinary vector)

∇f (gradient) vector

$\vec{\nabla} \cdot \vec{A}$ dot (divergence) scalar

$\nabla \times \vec{A}$ (curl) vector

Divergence: $\nabla \cdot \vec{A} = \frac{x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

(A_x, A_y, A_z)

- Geometrically: measures how much a vector 'diverges' at a pt

- Examples

$\vec{A} = (x, y, z)$

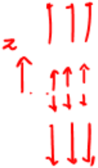
$= \vec{r}$

$\nabla \cdot \vec{A} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z$

$= 3$

$\vec{A} = (0, 0, 1)$

$\nabla \cdot \vec{A} = \frac{\partial}{\partial z} 1 = 0$



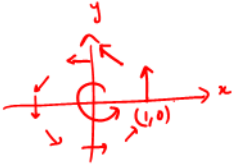
$\vec{A} = (0, 0, z)$

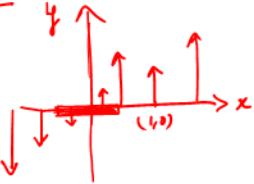
$\nabla \cdot \vec{A} = \frac{\partial z}{\partial z} = 1.$

$$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Curl: $\nabla \times \vec{A} = \begin{vmatrix} \boxed{\hat{x}} & \boxed{\hat{y}} & \boxed{\hat{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

- Geometrically: measures how much a vector 'swirls' around a pt
- Examples





$\vec{A} = (x, y, z)$ $\quad$ $\vec{A} = (-y, x, 0)$ $\quad$ $\vec{A} = (0, x, 0)$

$\nabla \times \vec{A} = (0, 0, 0)$ $\quad$ $\nabla \times \vec{A} = (0, 0, 2)$ $\quad$ $\nabla \times \vec{A} = (0, 0, 1)$

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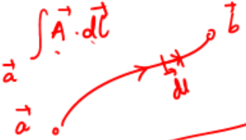
Integrals in vector calculus

- Line integrals: $\int \vec{A} \cdot \vec{dl}$

$\longrightarrow \int_{\vec{a}}^{\vec{b}} \vec{A} \cdot \vec{dl}$



$\oint_{\Gamma} \vec{A} \cdot \vec{dl}$

$\vec{ds} = |\vec{ds}|\,\hat{n}$

- Surface integrals: $\int \vec{A} \cdot \vec{ds}$

open $\int_{S} \vec{A} \cdot \vec{ds}$

$\longrightarrow$ volume

∞, undefined.

closed $\oint_{S} \vec{A} \cdot \vec{ds}$

$\iint$

- Volume integrals: $\int f\, dv$

$\iiint$

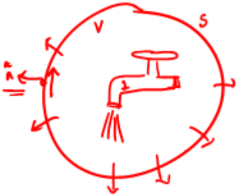
$$\int \vec{A}\, dv = \hat{x}\int A_x\, dv + \hat{y}\int A_y\, dv + \hat{z}\int A_z\, dv$$

$\longrightarrow$ vector

Divergence (a.k.a. Gauss's / Green's) Theorem

Geometrically:

steady state

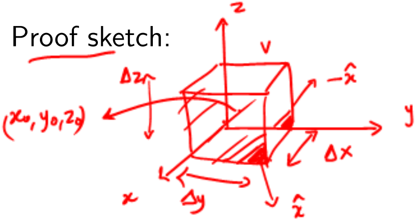


$$\int_V \nabla \cdot \vec{A}\, dv = \oint_S \vec{A} \cdot d\vec{s} \quad \hat n$$

how much it diverges

outward flux

Proof sketch:



$$\left(\oint \vec{A} \cdot d\vec{s}\right)_{\hat x \text{ surfaces}} = \left[A_x\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) - A_x\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)\right] \Delta y\, \Delta z$$

$$= \underbrace{\Delta x\, \Delta y\, \Delta z}\left[\frac{A_x\left(x_0 + \frac{\Delta x}{2}, ,\right) - A_x\left(x_0 - \frac{\Delta x}{2}, ,\right)}{\Delta x}\right]$$

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} = dv\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]$$

CEM : Helps reduce dimensionality of problem

Curl (a.k.a Stoke's) Theorem

Geometrically:

$$\int_S (\nabla \times \vec{A}) \cdot d\vec{s} = \oint_\Gamma \vec{A} \cdot d\vec{l}$$

$$= \left[\Delta x\, A_x\left(x_0, y_0 - \tfrac{\Delta y}{2}\right) - \Delta x\, A_x\left(x_0, y_0 + \tfrac{\Delta y}{2}\right)\right] + \Delta y\, A_y\left(x_0 + \tfrac{\Delta x}{2}, y_0\right) - \Delta y\, A_y\left(x_0 - \tfrac{\Delta x}{2}, y_0\right)$$

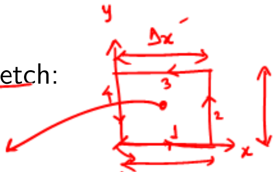
$$= (1) + (3) + (2) + (4)$$

$$= \Delta x \Delta y \frac{\left[A_x\left(x_0, y_0 - \tfrac{\Delta y}{2}\right) - A_x\left(x_0, y_0 + \tfrac{\Delta y}{2}\right)\right]}{\Delta y} + \Delta x \Delta y \frac{\left[A_y\left(x_0 + \tfrac{\Delta x}{2}, y_0\right) - A_y\left(x_0 - \tfrac{\Delta x}{2}, y_0\right)\right]}{\Delta x}$$

$\lim_{\Delta x, \Delta y \to 0}$

$$= ds\underbrace{\left[-\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}\right]}_{(\nabla \times \vec{A})_z}$$

Proof sketch:



(x_0, y_0)

$\oint \vec{A} \cdot d\vec{l}$

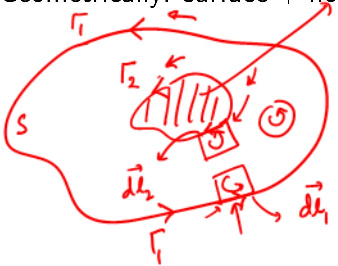
$\vec{A} = (A_x, A_y, 0)$

Corollary: $\oint_S (\nabla \times \vec{A}) \cdot d\vec{s} = 0$

Stoke's Theorem in a multiply connected region

Geometrically: surface + hole

$$\int_S (\nabla \times \vec{A}) \cdot \vec{ds} = \oint_{\Gamma_1} \vec{A} \cdot \vec{dl} - \oint_{\Gamma_2} \vec{A} \cdot \vec{dl}$$



$\int_S (\nabla \times \bar{A}) \cdot \bar{ds}$

counter clockwise

$= \oint_{\Gamma} - \left(\sum_{i=1}^{n} \oint_{\Gamma_i} \right)$

CEM : Helps reduce domain of computation

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$f(x),\ g(x)$

Corollaries: Integration by parts

- Two scalar functions, f, g. Know that: $\frac{d\,(fg)}{dx} = f\frac{d\,g}{dx} + g\frac{d\,f}{dx}$

  Rearranging, integrating: $\int_a^b f\frac{d\,g}{dx}dx = \int_a^b \frac{d\,(fg)}{dx}dx - \int_a^b \frac{d\,f}{dx}\,g\,dx$

  $$\int_a^b f g' dx = fg\Big|_a^b - \int_a^b f' g\, dx \ \checkmark$$

- Extend to vector calculus: scalar f, vector $\vec{A}$ functions

  Product rule: $\vec{\nabla}\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot\nabla f$

  S

  Volume integration: $\int_V(\quad) = \oint_S (f\vec{A})\cdot\vec{ds}$ [Divg. thm]

  Rearranging: $\int_V f(\nabla\cdot\vec{A})\,dv = \oint_S (f\vec{A})\cdot\vec{ds} - \int_V \vec{A}\cdot\nabla f\,dv$

  V S V 2

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

Miscellaneous: Some vector calculus identities

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\hat{x}\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right) = 0$$

- $\nabla \times \nabla f = \vec{0}$ for *any* scalar function f
- $\nabla \cdot (\nabla \times \vec{A}) = 0$ for *any* vector field $\vec{A}$
- $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ → Laplacian operator ∇^2

  $A \times (B \times C)$
- Vector field is specified upto a constant: if curl $(\nabla \times \vec{A})$ and divergence $(\nabla \cdot \vec{A})$ are specified

Miscellaneous: Getting the normal to a curve

A function $y = f(x)$

$g(x,y) = y - f(x)$

$\boxed{\nabla g(x,y)} = \left(-\frac{df(x)}{dx}, 1\right)$

$g(x,y) = 0$

$g(x,y) = k$

Vector along the tangent at some point: $\boxed{\vec{v} = (1, \frac{df}{dx})} \propto$

$1 \times \frac{-\partial f}{dx} + 1 \times \frac{df}{dx} = 0$

What is $\boxed{\vec{v} \cdot \nabla g}$

Thus $\hat{n}$ is along ∇g. Useful for boundary conditions in Electromagnetics.

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Reference: Chapter 1 of David Griffiths: Introduction to Electrodynamics, 4rth Ed., Pearson