EEL207 Tutorial 9 Solutions: 2015-16, Sem II

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1) a) The equation of motion for an electron using Drude model is given by

$$m\frac{d^2x}{dt^2} = F_{restoring} + F_{damping} + F_{driving} \tag{1}$$

where $F_{damping} = -m\gamma\frac{dx}{dt}$, $F_{driving} = -eE$ and $F_{restoring} = -\frac{dV}{dx}$. Given the restoring potential as $V(x) = \frac{1}{2}mw_0^2x^2 + \frac{1}{4}m\alpha x^4$. So, $F_{restoring} = -mw_0^2x - m\alpha x^3$

Final expression for the equation of motion of an electron is as follows

$$m\frac{d^2x}{dt^2} + mw_0^2x + m\alpha x^3 + m\gamma\frac{dx}{dt} = -eE \tag{2}$$

b) Expanding $x(t)$ upto first order in $\alpha : x(t) \approx x^{(0)}(t) + \alpha x^{(1)}(t)$, and substituting in equation(2), we get

$$m\frac{d^2x^{(0)}}{dt^2} + m\alpha\frac{d^2x^{(1)}}{dt^2} + mw_0^2x^{(0)} + m\alpha w_0^2x^{(1)} + m\alpha(x^{(0)} + \alpha x^{(1)})^3 + m\gamma\frac{dx^{(0)}}{dt} = -eE \tag{3}$$

Neglecting all the higher order terms of α greater than 1 and it is given that $x^{(0)}$ and $x^{(1)}$ are independent of α. We can seperate the resultant equation into terms with and without α as follows

$$\frac{d^2x^{(0)}}{dt^2} + w_0^2x^{(0)} + \gamma\frac{dx^{(0)}}{dt} \approx \frac{-eE}{m} \tag{4}$$

and

$$\frac{d^2x^{(1)}}{dt^2} + w_0^2x^{(1)} + \gamma\frac{dx^{(1)}}{dt} \approx -(x^{(0)})^3 \tag{5}$$

c) Fourier transform of polarization $\tilde{P}(\omega)$ is given as

$$\tilde{P}(\omega) = -eN_0\tilde{X}(\omega) = -eN_0(\tilde{X}^{(0)}(\omega) + \alpha\tilde{X}^{(1)}(\omega) \tag{6}$$

Applying Fourier transform in equations (4) and (5)

$$(-i\omega)^2\tilde{X}^{(0)} + w_0^2\tilde{X}^{(0)} - i\omega\gamma\tilde{X}^{(0)} = -e\tilde{E}(\omega)/m$$

$$(-i\omega)^2\tilde{X}^{(1)} + w_0^2\tilde{X}^{(1)} - i\omega\gamma\tilde{X}^{(1)} = -\tilde{X}^{(0)}(\omega) * (\tilde{X}^{(0)}(\omega) * \tilde{X}^{(0)}(\omega))$$

$$\tilde{X}^{(0)}(\omega) = \frac{-e\tilde{E}(\omega)/m}{w_0^2 - \omega^2 - i\gamma\omega} \tag{7}$$

$$\tilde{X}^{(1)} = -\frac{\int_{-\infty}^{\infty}\tilde{X}^{(0)}(\omega_1)\int_{-\infty}^{\infty}\tilde{X}^{(0)}(\omega_2)\tilde{X}^{(0)}(\omega-\omega_1-\omega_2)d\omega_2 d\omega_1}{w_0^2 - \omega^2 - i\gamma\omega} \tag{8}$$

Substituting (7) in (8) and (7),(8) in (6), we get

$$\tilde{P}(\omega) = -eN_0(\frac{-e\tilde{E}(\omega)/m}{w_0^2 - \omega^2 - i\gamma\omega} - \alpha\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{-e\tilde{E}(\omega_1)/m}{w_0^2-\omega_1^2-i\gamma\omega_1}\frac{-e\tilde{E}(\omega_2)/m}{w_0^2-\omega_2^2-i\gamma\omega_2}\frac{-e\tilde{E}(\omega-\omega_1-\omega_2)/m}{w_0^2-(\omega-\omega_1-\omega_2)^2-i\gamma(\omega-\omega_1-\omega_2)}d\omega_2 d\omega_1}{w_0^2 - \omega^2 - i\gamma\omega}) \tag{9}$$

Given

$$\tilde{P}(\omega) = \epsilon_0\chi^{(1)}(\omega)\tilde{E}(\omega) + \epsilon_0\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi^{(3)}(\omega_1,\omega_2;\omega)\tilde{E}(\omega-\omega_1-\omega_2)\tilde{E}(\omega_1)\tilde{E}(\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2 \tag{10}$$

Equating equation (9) and (10)

$$\chi^{(1)}(\omega) = \frac{e^2 N_0/(m\epsilon_0)}{w_0^2 - \omega^2 - i\gamma\omega} \tag{11}$$

$$\chi^{(3)}(\omega_1, \omega_2; \omega) = \frac{-\alpha e^4/(m^3\epsilon_0)}{(w_0^2 - \omega^2 - i\gamma\omega)(w_0^2 - \omega_1^2 - i\gamma\omega_1)(w_0^2 - \omega_2^2 - i\gamma\omega_2)(w_0^2 - (\omega - \omega_1 - \omega_2)^2 - i\gamma(\omega - \omega_1 -} \tag{12}$$

2) a) Start with source-free curl equations $\bigtriangledown \times \mathbf{E} = -i\omega\mu\mathbf{H}$ and $\bigtriangledown \times \mathbf{H} = i\omega\epsilon\mathbf{E}$
Taking time function as $\exp^{i\omega t}$, wave propagating in +ve Zdirection will have $\exp^{-i\beta z}$ and given fields are independent of y i.e $\frac{d}{dy} = 0$, we get

$$i\beta E_y = -i\omega\mu H_x \tag{13}$$

$$-i\beta E_x - \frac{\partial E_z}{\partial x} = -i\omega\mu H_y \tag{14}$$

$$\frac{\partial E_y}{\partial x} = -i\omega\mu H_z \tag{15}$$

$$i\beta H_y = i\omega\epsilon E_x \tag{16}$$

$$-i\beta H_x - \frac{\partial H_z}{\partial x} = i\omega\epsilon E_y \tag{17}$$

$$\frac{\partial H_y}{\partial x} = i\omega\epsilon E_z \tag{18}$$

Clearly equations (13),(15) and (17) are in terms of E_y, H_x and H_z. Equation (14),(16) and (18) are in terms of H_y, E_x and E_z. So, they are decoupled.

b) $\mathbf{TE\ mode}$ (i.e. $E_z = 0$)
From (13) and (17)

$$H_x = \frac{-i\beta}{k_c^2}\frac{\partial H_z}{\partial x} \tag{19}$$

$$E_y = \frac{i\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} \tag{20}$$

where $k = \omega\sqrt{\mu\epsilon}$ and $k_c^2 = k^2 - \beta^2$.
Once we know H_z, we can calculate other fields using (19) and (20)
$\bigtriangledown \times \bigtriangledown \times \mathbf{H} = i\omega\epsilon \bigtriangledown \times\mathbf{E} = \omega^2\mu\epsilon\mathbf{H} = -\bigtriangledown^2 \mathbf{H}$
Taking the z-component, we will get the wave equation

$$\frac{\partial H_z}{\partial x} + (\omega^2\mu\epsilon - \beta^2)H_z = 0 \tag{21}$$

Solution of equation (21) is

$$H_z(x) = A\cos(k_c x) + B\sin(k_c x) \tag{22}$$

Substituting (22) in (20)

$$E_y = \frac{i\omega\mu}{k_c}(-A\sin(k_c x) + B\cos(k_c x)) \tag{23}$$

Applying boundary conditions (Tangential Electric fields are zero at metal surfaces) i.e. $E_y = 0$ at $x = 0, x = L$
$B = 0$ and $k_c = \frac{n\pi}{L}$ where $n = 1, 2, 3...$
The final field equations comesout to be

$$H_z = A\cos(\frac{n\pi}{L}x)\exp^{i(\omega t - \beta z)} \tag{24}$$

$$E_y = \frac{-i\omega\mu}{k_c}A\sin(\frac{n\pi}{L}x)\exp^{i(\omega t - \beta z)} \tag{25}$$

$$H_x = \frac{i\beta}{k_c} A \sin(\frac{n\pi}{L}x) \exp^{i(\omega t - \beta z)} \tag{26}$$

Similarly,
TM mode (i.e. $H_z = 0$)
From (14) and (16)

$$E_x = \frac{-i\beta}{k_c^2} \frac{\partial E_z}{\partial x} \tag{27}$$

$$H_y = \frac{-i\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial x} \tag{28}$$

Once we know $\mathbf{E}_z$, we can calculate other fields using (27) and (28)
$\bigtriangledown \times \bigtriangledown \times \mathbf{E} = -i\omega\mu \bigtriangledown \times \mathbf{H} = \omega^2\mu\epsilon\mathbf{E} = - \bigtriangledown^2 \mathbf{E}$ Taking the z-component, we will get the wave equation

$$\frac{\partial E_z}{\partial x} + (\omega^2\mu\epsilon - \beta^2)E_z = 0 \tag{29}$$

Solution of equation (29) is

$$E_z(x) = C\cos(k_c x) + D\sin(k_c x) \tag{30}$$

Applying boundary conditions (Tangential Electric fields are zero at metal surfaces) i.e. $E_z = 0$ at $x = 0, x = L$
$C = 0$ and $k_c = \frac{n\pi}{L}$ where $n = 1, 2, 3...$
The final field equations comesout to be

$$E_z = D\sin(\frac{n\pi}{L}x)\exp(i(\omega t - \beta z)) \tag{31}$$

$$H_y = \frac{-i\omega\epsilon}{k_c} D\cos(\frac{n\pi}{L}x)\exp(i(\omega t - \beta z)) \tag{32}$$

$$E_x = \frac{-i\beta}{k_c} D\cos(\frac{n\pi}{L}x)\exp(i(\omega t - \beta z)) \tag{33}$$

where their propagation constant is given as follows, $\beta = \sqrt{\omega^2\mu\epsilon - (\frac{n\pi}{L})^2}$

c) Given $\mathbf{E}(x, z = 0, t) = \widehat{y}E_0 \sin(\pi x/L)\exp(i\pi ct/2L)$ Comparing this with (25) by substituting $z = 0$, we get
$k_c = \frac{\pi}{L}$, $\omega = \frac{\pi c}{2L}$ which implies $k = \frac{\pi}{2L}$ and $\beta = \frac{-i\sqrt{3}\pi}{2L}$ (i.e. wave is attenating)

$$\mathbf{E}(x, z = z_0, t) = \widehat{y}E_0 \sin(\pi x/L)\exp(i\pi ct/2L)\exp(-\frac{\sqrt{3}\pi z_0}{2L}) \tag{34}$$

d) Given $\mathbf{E}(x, z = 0, t) = \widehat{y}E_0 \exp(i\omega t)$
Here, this electric field is formed using multiple TE modes, i.e.

$$E_y(x, z, t) = \Sigma_{n=1}^{\infty} \frac{-i\omega\mu L}{n\pi} A_n \sin(\frac{n\pi}{L}x) \exp^{i(\omega t - \beta z)} \tag{35}$$

$$\Rightarrow \Sigma_{n=1}^{\infty} \frac{-i\omega\mu L}{n\pi} A_n \sin(\frac{n\pi}{L}x) \exp i\omega t = E_0 \exp(i\omega t) \tag{36}$$

multipliying both sides by $\sin(\frac{m\pi}{L}x)$ and integrating from 0 to L, we get

$$\Sigma_{n=1}^{\infty} \int_0^L \frac{-i\omega\mu L}{n\pi} A_n \sin(\frac{n\pi}{L}x)\sin(\frac{m\pi}{L}x)dx = \int_0^L E_0 \sin(\frac{m\pi}{L}x)dx \tag{37}$$

$$= \int_0^L E_0 \sin(\frac{m\pi}{L}x)dx \tag{38}$$

$$\frac{i\omega\mu L^2}{2m\pi} A_m = \begin{cases} \frac{2E_0 L}{m\pi} & m \textit{ is odd} \\ 0 & m \textit{ is even} \end{cases} \tag{39}$$

$$\Rightarrow \mathbf{E}(x, z = z_0, t) = \Sigma_{n \textit{ is even}}(\widehat{y}\frac{-4E_0}{n\pi}\sin(\frac{n\pi}{L}x)\exp^{i(\omega t - \beta z_0)}) \tag{40}$$