ELL212 - Tutorial 9, Sem II 2015-16

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1) **Frequency dependance of the nonlinear susceptibility:** In this problem, we will attempt to derive a simple model to compute the frequency dependance of the third order nonlinear susceptibility. The starting point of the analysis is the Drude model (covered in class, lecture 30). However, instead of assuming a linear restoring force, we consider a nonlinear restoring force with potential $V(x)$:

$$V(x) = \frac{1}{2}m\omega_0^2 x^2 + \frac{1}{4}m\alpha x^4 \tag{1}$$

In this problem, assume the number of dipoles per unit volume to be given by N_0 and the damping constant in the drude model to be given by γ. Also, α is assumed to be very small so as to consider only terms upto first order in α.

a) Write down the equation of motion for an electron (displacement $x(t)$) under the influence of an electric field $E(t)$.

b) Consider expanding $x(t)$ upto the first order in α: $x(t) \approx x^{(0)}(t) + \alpha x^{(1)}(t)$ where $x^{(0)}(t)$ and $x^{(1)}(t)$ are independent of α. Starting from the equation of motion derived in part (a), show that:

$$\ddot{x}^{(0)} + \gamma\dot{x}^{(0)} + \omega_0^2 x^{(0)} \approx -\frac{eE}{m} \tag{2a}$$

$$\ddot{x}^{(1)} + \gamma\dot{x}^{(1)} + \omega_0^2 x^{(1)} \approx -(x^{(0)})^3 \tag{2b}$$

c) Denoting the fourier transform of $E(t)$ by $\tilde{E}(\omega)$, show that the fourier transform of the polarization $\tilde{P}(\omega)$ can be expressed as:

$$\tilde{P}(\omega) = \epsilon_0\chi^{(1)}(\omega)\tilde{E}(\omega) + \epsilon_0\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi^{(3)}(\omega_1,\omega_2;\omega)\tilde{E}(\omega-\omega_1-\omega_2)\tilde{E}(\omega_1)\tilde{E}(\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2 \tag{3}$$

Obtain expressions for $\chi^{(1)}(\omega)$ and $\chi^{(3)}(\omega_1,\omega_2;\omega)$ in terms of N_0, α, ω_0 and γ.

2) **Wave propagation in a plane parallel waveguide:** Consider a metallic waveguide constructed using two perfectly conducting metal plates at $x = 0$ and $x = L$. Throughout this problem, assume that all the electromagnetic fields are independent of y i.e. there is perfect symmetry in the y direction. Also, assume that the direction of propagation is the z direction.

a) Starting from the source-free curl equations, show that as a consequence of the y independence of the electromagnetic field, the problem of solving for the 6-field components (namely E_i, H_i with $i \in \{x, y, z\}$) reduces to the problem of solving decoupled equations for the set $\{H_x, E_y, H_z\}$ (henceforth referred to as the TE fields) and $\{E_x, H_y, E_z\}$ (henceforth referred to as the TM fields). By decoupled, it is meant that a field component in one group does not affect the solution for a field component in the other group.

b) Obtain expressions for the electric and magnetic fields for the various TE and TM modes and their propagation constants β.

c) The waveguide is now excited with a TE field at $z = 0$ such that:

$$\mathbf{E}(x, z = 0, t) = \hat{y}E_0\sin(\pi x/L)\exp(j\pi ct/2L) \tag{4}$$

Compute the average power flowing in the waveguide in the z direction (per unit length along the y direction).

d) Consider the excitation of a TE field inside the waveguide such that the electric field at $z = 0$ is given by:

$$\mathbf{E}(x, z = 0, t) = \hat{y}E_0\exp(j\omega t) \tag{5}$$

Compute the electric field at $z = z_0$.