EEL207 Tutorial 8 Solutions: 2015-16, Sem II

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1) Choice of coordinate axis in this problem is as follows: interface between the two media is assumed to be at $z = 0$ and normal to the x axis, and the plane of incidence is assumed to be the $x - z$ plane. The angle of incidence is denoted by θ_i and the free space propagation constant (ω/c) by k_0. A time dependance of $\exp(j\omega t)$ is assumed and suppressed throughout the solution.

   a) We know that the continuity of tangential electric and magnetic fields across the interface require k_x (component of the **k** parallel to the interface) to be continuous across the interface. Now, for the incident field, $k_x^{\text{inc}} = k_0 n_1 \sin\theta_i$ and therefore for the transmitted field $k_x^{\text{tran}} = k_x^{\text{inc}} = k_0 n_1 \sin\theta_i$. Since the magnitude of the propagation constant in the second media is $k_0 n_2$:

$$k_z^{\text{tran}} = \sqrt{k_0^2 n_2^2 - (k_x^{\text{inc}})^2} = k_0 \sqrt{\frac{n_2^2}{n_1^2} - \sin^2\theta_i} = k_0 \sqrt{\sin^2\theta_c - \sin^2\theta_i} \tag{1}$$

   Clearly, for $\theta_i > \theta_c$, k_z^{tran} is imaginary. Defining α by $k_z^{\text{tran}} = -jk_0 n_1 \alpha$, it is easy to see that the spatial dependance of the transmitted wave will be given by:

$$\mathbf{E}_{\text{tran}} \sim \exp(-j(k_x^{\text{tran}} x + k_z^{\text{tran}} z)) \sim \exp(-k_0 n_1 \alpha z)\exp(-jk_0 n_1 x \sin\theta_i) \tag{2}$$

   b) The incident, reflected and transmitted electric fields can be expressed as:

$$\mathbf{E}_{\text{inc}} = E_0 \exp(-jk_0 n_1(x\sin\theta_i + z\cos\theta_i))\hat{y} \tag{3a}$$
$$\mathbf{E}_{\text{ref}} = \Gamma E_0 \exp(-jk_0 n_1(x\sin\theta_i - z\cos\theta_i))\hat{y} \tag{3b}$$
$$\mathbf{E}_{\text{tran}} = \tau E_0 \exp(-jk_0 n_1 x\sin\theta_i)\exp(-k_0 n_1 \alpha z)\hat{y} \tag{3c}$$

   where Γ and τ are the reflection and transmission coefficients which need to be solved for. The incident, reflected and transmitted magnetic fields can be computed using the Faraday's rule:

$$\nabla \times \mathbf{E} = -j\omega\mu_0 \mathbf{H} \implies \mathbf{H} = \frac{\mathbf{k} \times \mathbf{E}}{\omega\mu_0} \tag{4}$$

   Therefore:

$$\mathbf{H}_{\text{inc}} = \frac{n_1 E_0}{\eta_0}(\hat{z}\sin\theta_i - \hat{x}\cos\theta_i)\exp(-jk_0 n_1(x\sin\theta_i + z\cos\theta_i)) \tag{5a}$$
$$\mathbf{H}_{\text{ref}} = \frac{n_1 \Gamma E_0}{\eta_0}(\hat{z}\sin\theta_i + \hat{x}\cos\theta_i)\exp(-jk_0 n_1(x\sin\theta_i - z\cos\theta_i)) \tag{5b}$$
$$\mathbf{H}_{\text{tran}} = \frac{n_1 \tau E_0}{\eta_0}(\hat{z}\sin\theta_i + j\hat{x}\alpha)\exp(-jk_0 n_1 x\sin\theta_i)\exp(-k_0 n_1 \alpha z) \tag{5c}$$

   Imposing continuity of E_y and H_x at $z = 0$:

$$1 + \Gamma = \tau \tag{6a}$$
$$1 - \Gamma = -j\alpha\tau \tag{6b}$$

   from which

$$\Gamma = \frac{1 + j\alpha}{1 - j\alpha}, \tau = \frac{2}{1 - j\alpha} \tag{7}$$

   Clearly $|\Gamma| = 1$, which implies that there is 100% reflection at the interface.

   c) Substitute Eq. 7 into Eqs. 3 and 5 to obtain the electric and magnetic fields in all space, including $z > 0$.

   d) The time average poynting vector is given by:

$$\mathbf{S} = \frac{1}{2}\text{Re}(\mathbf{E} \times \mathbf{H}^*) \tag{8}$$

Thus, for the transmitted field:

$$\begin{aligned}\mathbf{S}_{\text{tran}} &= \frac{1}{2}\text{Re}(\mathbf{E}_{\text{tran}} \times \mathbf{H}^*_{\text{tran}}) \\ &= \frac{|\tau|^2|E_0|^2 n_1^2}{2\eta_0^2}\exp(-2k_0 n_1 \alpha z)\text{Re}(\hat{y} \times (\hat{z}\sin\theta_i - j\hat{x}\alpha)) \\ &= \hat{x}\frac{|\tau|^2|E_0|^2 n_1^2 \sin\theta_i}{2\eta_0^2}\exp(-2k_0 n_1 \alpha z) \qquad (9)\end{aligned}$$

Since $\mathbf{S}_{\text{tran}}$ is entirely along x axis, there is no transport of energy along the z direction on an average (there is instantaneous transport of energy even in the z direction). This is an important property of the evanescent fields set up during total internal reflection.

2) To be able to excite the surface plasmon polariton at the metal-air interface, we need an excitation which satisfies:

$$k_x = k_{\text{SP}} = k_0\sqrt{\frac{\epsilon_m}{1+\epsilon_m}} \qquad (10)$$

where ϵ_m is the relative permittivity of metal. (This is the same condition used in Prob. 1, part a). Now, at optical frequencies, $\epsilon_m < 0$, and thus ($\epsilon_m = -a$):

$$k_x = k_0\sqrt{\frac{a}{a-1}} > k_0 \qquad (11)$$

If we attempt to excite the Surface plasmon polariton from air itself by using a plane wave, we will never be able to satisfy Eq. 11 since $k_x = \sqrt{k_0^2 - k_y^2} < k_0$. Thus, we attempt to excite the surface plasmon through a medium with larger refractive index (glass in this case). A plane wave propagating in glass can potentially satisfy Eq. 11, since k_x of this wave is bounded by $k_0 n_{\text{glass}}$ and not k_0. In fact, we can compute the angle of incidence θ_i by imposing:

$$k_x = k_0 n_{\text{glass}}\sin\theta_i = k_0\sqrt{\frac{\epsilon_m}{1+\epsilon_m}} \qquad (12)$$

or, equivalently:

$$\sin\theta_i = \frac{1}{n_{\text{glass}}}\sqrt{\frac{\epsilon_m}{1+\epsilon_m}} \qquad (13)$$

3) The set up for the problem is shown in Fig. 1.

a) For this polarization $\vec{E}$ is along $\hat{y}$, and $\vec{B}$ in $x-z$ plane as in the Figure. Thus, the first boundary conditions give us $E_i + E_r = E_t$ (1), and the fourth boundary condition gives us $(E_i(-\cos\theta_1)/v_1 + E_r(\cos\theta_1)/v_1)/\mu_1 = E_t(-\cos(\theta_2))/(\mu_2 v_2)$ (2). Rearranging these two equations we get the reflection coefficient as $E_r/E_i = (1-\alpha\beta)/(1+\alpha\beta)$ where, as used in class, $\alpha = \cos(\theta_2)/\cos(\theta_1)$ and $\beta = (\mu_1 v_1)/(\mu_2 v_2)$. Thus, for the reflection to be zero at any angle, we will require $\alpha\beta = 1$. For a non-magnetic medium, this implies that (by using Snell's law)

$$\alpha = \sqrt{1-(v_2/v_1)^2\sin^2(\theta_1)}/\cos\theta_1 = 1/\beta = v_2/v_1 \Rightarrow v_1 = v_2$$

i.e. the two media must have the same refractive index, a contradiction.

b) The polarization axis should be horizontal. Since road has poor reflectivity, we won't concern ourselves too much with it – the glare is coming from the vertical sheets. Let us consider light falling on the sheet, such that the plane of incidence is horizontal*. We have derived in (a) that there is no Brewster's angle for perpendicular polarization. Thus, light that is polarized vertically (i.e. perpendicular to the plane of incidence), there is no angle at which the reflectivity goes to zero, where as for horizontally polarized light, there exists such an angle. So by aligning the axis of the polarizer horizontally, we can achieve a two fold reduction in glare: first, by eliminating vertically polarized light (by virtue of the polarizer) and two, by utilizing Brewster's angle to further reduce the horizontally polarized light that has been let through.

*: this is by no means necessary. Incoming plane waves can fall at any angle on the sheet. In such a scenario, the theoretically optimal orientation of the polarizer would depend on these angles.

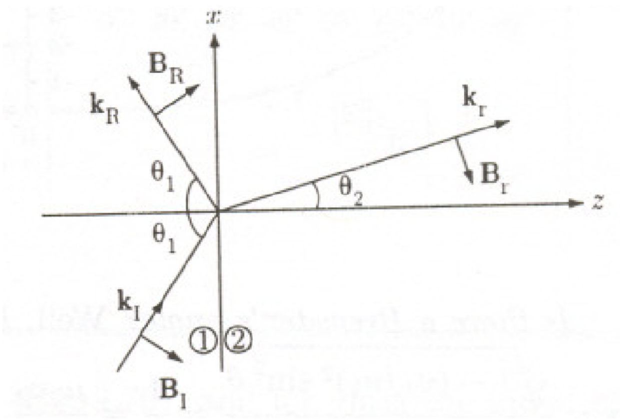

Fig. 1. Schematic for problem 3

But, since the question asked you to choose from horizontal or vertical alignment, it is clear that horizontal wins, based on the above reasoning.

4) We start with source free Maxwell's equations;

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \tag{14}$$
$$\nabla \times \vec{H} = j\omega\epsilon\vec{E}. \tag{15}$$

Then do some algebra;

$$\vec{E} \cdot (\nabla \times \vec{H}^*) - \vec{H}^* \cdot (\nabla \times \vec{E}) = -j\omega(\epsilon^* \vec{E} \cdot \vec{E}^* - \mu\vec{H}^* \cdot \vec{H}) \tag{16}$$
$$\text{also:} \quad \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}). \tag{17}$$

Next, we derive the energy conservation theorem.
Take volume integral and apply divg. thm. $\int_v \nabla \cdot \vec{A}\, dV = \oint_S \vec{A} \cdot \hat{n}\, dS$

$$\oint_S (\vec{E} \times \vec{H}^*) \cdot \hat{n}\, dS = j\omega \int_V \left(\epsilon^* |\vec{E}|^2 - \mu |\vec{H}|^2\right) dV \tag{18}$$

Simplify into real and imag. parts: $\epsilon = \epsilon' + j\epsilon''$, $\mu = \mu' + j\mu''$

$$\oint_S (\vec{E} \times \vec{H}^*) \cdot \hat{n} dS = j\omega \int_V (\epsilon' |\vec{E}|^2 - \mu' |\vec{H}|^2) dV + \omega \int_V (\epsilon'' |\vec{E}|^2 + \mu'' |\vec{H}|^2) dV \tag{19}$$

Say that there are two solutions: $\vec{E}_1, \vec{E}_2$ and denote $\delta\vec{E} = \vec{E}_1 - \vec{E}_2$. In the presence of sources, we have:

$$\nabla \times \vec{E} = \vec{M}_i - j\omega\mu\vec{H} \tag{20}$$
$$\nabla \times \vec{H} = \vec{J}_i + j\omega\epsilon\vec{E} \tag{21}$$

Subtract each of these equations for the two solutions

$$\nabla \times \delta\vec{E} = -j\omega\mu\delta\vec{H} \tag{22}$$
$$\nabla \times \delta\vec{H} = j\omega\epsilon\delta\vec{E} \tag{23}$$

In other words, $\delta\vec{E}, \delta\vec{H}$ satisfy the source *free* equations.
Apply the energy conservation theorem to $\delta\vec{E}, \delta\vec{H}$

$$\oint_S (\delta\vec{E} \times \delta\vec{H}^*) \cdot \hat{n} dS = j\omega \int_V (\epsilon' |\delta\vec{E}|^2 - \mu' |\delta\vec{H}|^2) dV + \omega \int_V (\epsilon'' |\delta\vec{E}|^2 + \mu'' |\delta\vec{H}|^2) dV \tag{24}$$

Re-examine LHS

$$\oint_S (\delta\vec{E} \times \delta\vec{H}^*) \cdot \hat{n} dS = \oint_S (\hat{n} \times \delta\vec{E}) \cdot \delta\vec{H}^* dS = \oint_S (\delta\vec{H}^* \times \hat{n}) \cdot \delta\vec{E} dS \tag{25}$$

So, if $\hat{n} \times \vec{E}$ and/or $\hat{n} \times \vec{H}$ is specified over S, it implies that: $\oint_S (\delta\vec{E} \times \delta\vec{H}^*) \cdot \hat{n} dS = 0$.
Real and imag. parts must go to zero, then:

$$0 = j\omega \int_V (\epsilon' |\delta\vec{E}|^2 - \mu' |\delta\vec{H}|^2) dV + \omega \int_V (\epsilon'' |\delta\vec{E}|^2 + \mu'' |\delta\vec{H}|^2) dV \tag{26}$$

$\epsilon', \epsilon'', \mu', \mu''$ are all > 0 in a lossy-medium
When $\epsilon'' > 0, \mu'' > 0$, **throughout the medium**, the real part of the above term can be zero only if both $\delta\vec{E} = 0 = \delta\vec{H}$ **throughout** V. This proves uniqueness.