EEL207 Tutorial 5 Solutions: 2015-16, Sem II

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1) The speed of a point on the disk at a distance r from the axis is $v = \omega r$, so the force per unit charge is $\vec{f}_{\text{mag}} = \vec{v} \times \vec{B} = \omega r B \hat{r}$. The EMF is therefore

$$\mathcal{E} = \int_0^a f_{\text{mag}}\, dr = \omega B a^2/2,$$

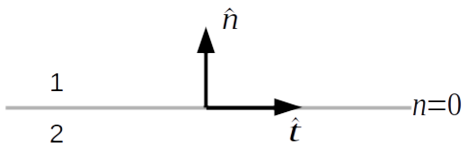
and the current is $I = \mathcal{E}/R = \omega B a^2/(2R)$. This problem illustrates a motional emf that can't be calculated directly from the flux rule, because the flux rule assumes the current to flow along a well-defined path, whereas in this example the current spreads out all over the disk.

2) **Tangential Boundary Conditions**

Two unknown vector fields $\vec{P}, \vec{Q}$ are expressed in terms of a known vector field $\vec{R}$ and some constants α, β.

$$\nabla \times \vec{P} + \beta \vec{Q} = \alpha \vec{R} \tag{1}$$

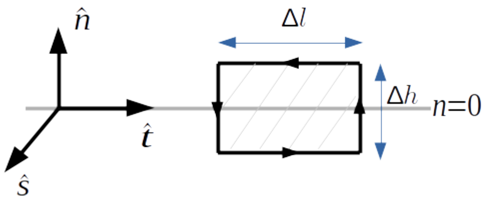
At an interface we have



$$\vec{R}(n,t) = \vec{R}_0(t)\delta(n)$$

Derive boundary condition for the tangential components of $\vec{P}$ at the interface.

(Step1) Geometry:



Take a wire frame (orientation same as right hand thumb rule

$$\hat{s} = \hat{t} + \hat{n}$$

(Step2) Take a surface integral on both sides of Eq. 1

$$\iint_s \nabla \times \vec{P}.\vec{ds} = \iint_s (\alpha\vec{R} - \beta\vec{Q}).\vec{ds}$$

(Step3) Apply Stokes theorem to LHS above

$$\oint \vec{P}.\vec{dl} = \vec{P}_2.\hat{t}\Delta l + \vec{P}_1.\hat{n}\Delta h/2 + \vec{P}_2.\hat{n}\Delta h/2 - \vec{P}_1.\hat{t}\Delta l - \vec{P}_2.\hat{n}\Delta h/2 - \vec{P}_1.\hat{n}\Delta h/2$$

(Step4) Simplify RHS (Note that $ds = dnd\tau$)

$$\alpha \iint_s \vec{R}_0(t).\hat{s}\delta(n)dnd\tau - \beta \iint_s \vec{Q}.\hat{s}\delta(n)dnd\tau = \alpha \int \vec{R}_0(t).\hat{s}dt \int \delta(n)dn - \beta\vec{Q}.\hat{s}\Delta l\Delta h$$
$$= \vec{R}_0(t).\hat{s}\Delta l - \beta\vec{Q}.\hat{s}\Delta l\Delta h$$

(Step5) Take $\Delta h \to 0$ and simplify both sides

$$\vec{P}_2.\hat{(t)} - \vec{P}_1.\hat{(t)} = \alpha \vec{R}_0(t).\hat{s}$$

Note that R_s varies as a function of t but the only component along $\hat{s}$ matters. Further $\hat{s} = \hat{t} \times \hat{n}$

$$\vec{P}.\hat{t} = \vec{P}.\hat{n} \times \hat{s} = \vec{P} \times \hat{n}.\hat{s}$$

$$\implies \vec{P}_2 \times \hat{n} - \vec{P}_1 \times \hat{n} = \alpha \vec{R}_0(t)$$

Since this must be true for all $\hat{s}$.

So, if we write

$$\vec{R}_0(t) = R_{ot}\hat{t} + R_{on}\hat{n} + R_{os}\hat{s}$$

If we take xy plane as interface and $\vec{R}_0(t) = \hat{y}$. Then

1. x-component: Assume $\hat{t} = -\hat{x}$

$$\vec{P}_2.(-\hat{x}) - \vec{P}_1.(-\hat{x}) = \alpha$$

$$\implies P_{1x} - P_{2x} = \alpha \tag{2}$$

2. y-component: $\hat{y} = \hat{t}$

$$\vec{P}_2\hat{y} - \vec{P}_1\hat{y} = 0$$

$$\implies P_{2y} = P_{1y}$$

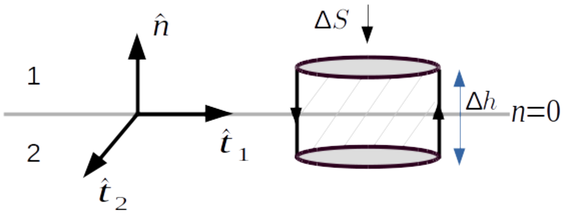
Normal Boundary Conditions

Given that $\vec{P}$ satisfies the following divergence relation:

$$\nabla.\vec{P} = \sigma_s \delta(n) + \rho_v,$$

boundary conditions on the normal components can be derived.

Take volume integral on both sides and simplify the LHS by Divergence Theorem.



$$\int_V \nabla.P dv = \oint \vec{P}.dS = \vec{P}_1.\hat{n}\Delta s - \vec{P}_2.\hat{n}\Delta s + (\quad)2\pi r \Delta h \; (curved\ sides)$$

Note that $\Delta h \to 0$. Simplify RHS

$$\iiint (\sigma_s \delta(n) + \rho_v) dn dt_1 dt_2$$

$$= \sigma_s \iint dt_1 dt_2 \int \delta(n) dn + \rho_v \Delta v$$

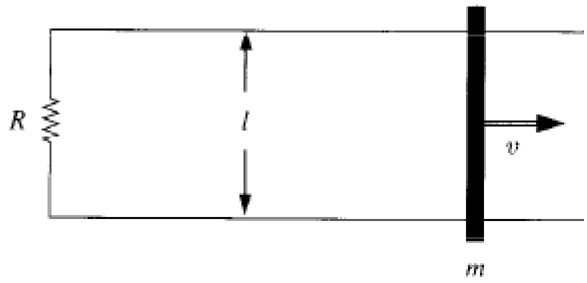
$$= \sigma_s \Delta s + \rho_v \Delta s \Delta h$$

Take $\Delta h \to 0$ and simplify

$$\vec{P}_1.\hat{n} - \vec{P}_2.\hat{n} = \sigma_s$$

If there is a surface charge density $P_{1z} - P_{2z} = \sigma_s$, else $P_{1z} - P_{2z}$.

3) A radial current will flow by symmetry: $\vec{J} = \sigma\vec{E} = \sigma Q/(4\pi\epsilon_0 r^2)\hat{r}$. This gives the conduction current as $I = -dQ/dt = \int \vec{J} \cdot d\vec{s} = \sigma Q/\epsilon_0$. By Ampere's law, the displacement current is $\vec{J}_d = \epsilon_0 d\vec{E}/dt =$



$dQ/dt\hat{r}/(4\pi r^2) = -\sigma Q/(4\pi\epsilon_0 r^2)\hat{r}$, which exactly cancels the conduction current. Since $\nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = 0$, we have $\vec{B} = 0$.

4) (a)

$$\varepsilon = -\frac{d\phi}{dt} = -Bl\frac{dx}{dt} = -Blv; \varepsilon = IR \Rightarrow I = \frac{Blv}{R}.$$

Never mind the minus sign - it just tells that the direction of flow: (v $\times$ B) is upward, in the bar so downwards in the resistor.

(b)

$$F = IlB = \frac{B^2l^2v}{R}\ (to\ the\ left)$$

(c)

$$F = ma = m\frac{dv}{dt} = -\frac{B^2l^2}{R}v \Rightarrow \frac{dv}{dt} = -(\frac{B^2l^2}{Rm})v \Rightarrow v = v_0e^{-\frac{B^2l^2}{mR}t}.$$

(d) The energy goes into heat in the resistor. The power delivered to resistor is I^2R, so

$$\frac{dW}{dt} = I^2R = \frac{B^2l^2v^2}{R^2}R = \frac{B^2l^2}{R}v_0^2e^{-2\alpha t},\ where \alpha = \frac{B^2l^2}{mR};\ \frac{dW}{dt} = \alpha m v_0^2 e^{-2\alpha t}.$$

The total energy delivered to the resistor is $W = \alpha m v_0^2 \int_0^\infty e^{-2\alpha t}dt = \alpha m v_0^2 \frac{e^{-2\alpha t}}{-2\alpha}|_0^\infty = \alpha m v_0^2 \frac{1}{2\alpha} = \frac{1}{2}mv_0^2$